Structure Analysis II

✗ Types of structure:-

(a) Determinate structure:-

→ Can be analysed by using eq^{ns} of static equilibrium.

→ Total no. of unknown = total no. of available eq^n.

(b) Indeterminate structure:-

↳ Cannot be analysed by using equations of static eq^b.

↳ Total no. of unknown rxn > Total no. of available eq^n

✗ Degree of static & kinematic indeterminancy:-

1. Degree of static indeterminancy:- (D_s)

It is the number of exceed unknown reaction with respect to available eq^n of static eq^b. There are two types of degree of static indeterminancy.

(a) Degree of static indeterminancy due to external (D_{se})

↳ due to ~~exceep~~ exceed of support rxn.

(b) Degree of static indeterminancy due to internal (D_{si})

↳ due to exceed of member (internal forces)

$$D_s = D_{se} + D_{si}$$

2. Degree of kinematic indeterminancy:-

→ If slope & deflection can be find out by applying compability ~~equati~~ conditions, the structure is called kinematically determinate.

→ However, if slope & deflections can not be find out by applying compability condition, such structure is called kinematically indeterminate structure.

→ The degree of kinematic indeterminancy is defined as the number of unknowns (deflection/slope) which cannot be calculated by compability eqns.

structure		static indeterminancy		kinematic indeterminancy
		D_{se}	D_{si}	
Beams	2D	$r_e - 3$	$-r_r$	$3j - r_e - m' + r_r$
Frames	2D	$r_e - 3$	$3C - r_r$	$3j - r_e - m' + r_r$
	3D	$r_e - 6$	$6C - r_r$	$6j - r_e - m' + r_r$
Truss	2D	$r_e - 3$	$m - (2j - 3)$	$2j - r_e$
	3D	$r_e - 6$	$m - (2j - 6)$	$3j - r_e$

$$r_r = \begin{cases} \Sigma (m'' - 1), & 2D \\ \Sigma 3(m'' - 1), & 3D \end{cases}$$

where,

r_e = external reactions
C = number of closed loops
m = number of member
j = number of joints
m' = number of axially rigid members.
m'' = number of members connected at internal hinge location.

Note: In case of extensible / Inextensible

(i) If beams alone are mentioned axially inextensible, then m' will be equal to number of beams in the structure.

(ii) If columns alone are mentioned axially inextensible, then m' will be equal to no. of columns in the structure

(iii) If mentioned all members are inextensible, then m' will be sum of all beam and column.

(iv) If nothing is mentioned, m' will be takes as zero.

Note:-
In case of Internal Hinge
1) Internal hinge contribution is deducted in internal static indeterminancy (D_{si})
2) Internal hinge contribution is added in kinematic indeterm

①

$D_{se} = r_e - 3 = 7 - 3 = 4$ $\qquad D_s = D_{se} + D_{si} = 4$

$D_{si} = 0$

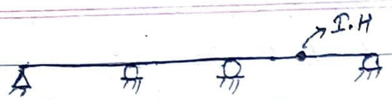
$D_K = 3j - r_e = 3 \times 3 - 7 = 2$

②

$D_{se} = r_e - 3 = 5 - 3 = 2$

$D_{si} = 0$, $D_s = 2$ & $D_K = 3j - r_e = 3 \times 4 - 5 = 7$

③



Internal hinge is considered as joint & divide the member into separate part.

$r_e = 5$

$D_{se} = 5 - 3 = 2$

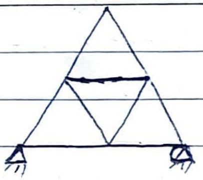
$D_{si} = -r_r = ~~0~~ - (2-1) = -1$

$D_s = 2 ~~+~~ 1 = 1$ & $D_K = 3j - r_e - m' + r_r$

$= 3*5 - 5 - 0 + 1$

$= 11$

④



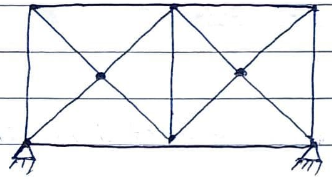
$D_{se} = r_e - 3 = 3 - 3 = 0$

$D_{si} = m - (2j - 3)$

$= 9 - (2*6 - 3) = 0$

$D_s = 0$

⑤



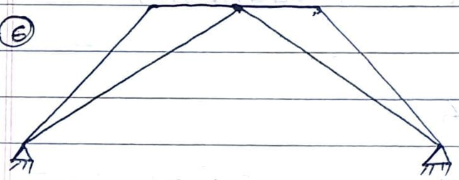
$D_{se} = 4 - 3 = 1$

$D_{si} = m - (2j - 3)$

$= 15 - (2*8 - 3) = 2$

$D_s = 3$

⑥



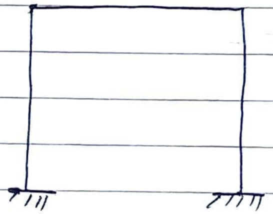
$D_{se} = 4 - 3 = 1$

$D_{si} = 6 - (2*5 - 3)$

$= -1$

$D_s = 0$

⑦

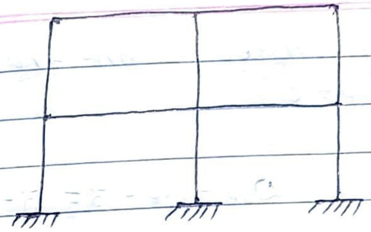


$D_{se} = r_e - 3 = 6 - 3 = 3$

$D_{si} = 3C = 0$

$D_s = 3$

⑧

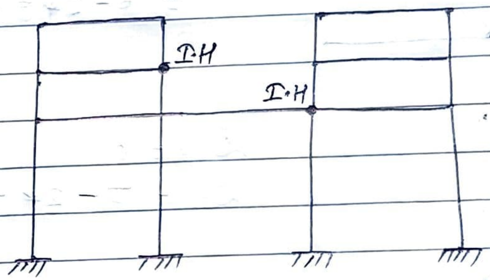


$D_{se} = r_e - 3 = 9 - 3 = 6$

$D_{si} = 3C = 3*2 = 6$

$D_s = 12$

⑨



$D_{se} = 12 - 3 = 9$

$D_{si} = 3C - \Sigma(m'' - 1)$

$= 3*4 - [(3-1) + (4-1)]$

$= 7$

$D_s = 16$

⑩

$r_e = 6$

$D_{se} = r_e - 3 = 3$

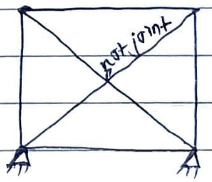
$D_{si} = 3C - r_r$

$= 3 \times 2 = 6.$

$D_s = 9$

Pinjointed – Truss
Otherwise – Frame

⑪



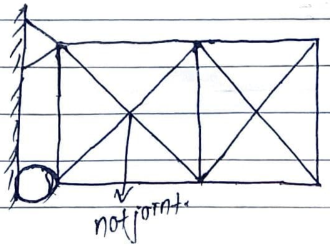
$r_e = 4$, $D_{se} = 4 - 3 = 1$

$D_{si} = m - (2j - 3)$

$= 6 - (2*4 - 3) = 1$

$D_s = 2$

⑫



$D_{se} = r_e - 3 = 3 - 3 = 0$

$D_{si} = m - (2j - 3)$

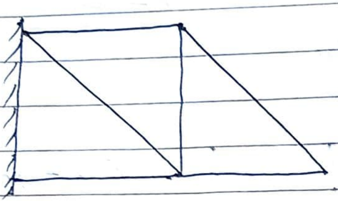
$= 11 - (2*6 - 3) = 2$

$D_s = 2$

Wall or roof attached pin jointed plane truss

[$D_{si} = m - 2j$ & $D_{se} = 0$]

⑬

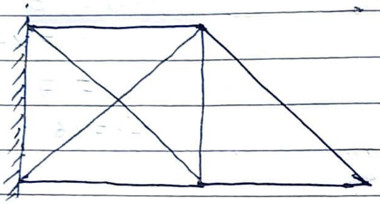


$D_{se} = r_e - 3 = 3 - 3 = 0$

$D_{si} = m - 2j$

$= 6 - 2*3 = 0$

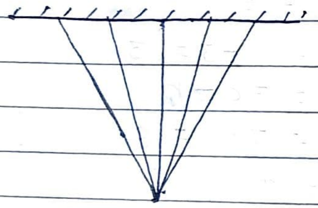
⑭



$D_{se} = 0$

$D_{si} = 7 - 2*3 = 1$

⑮



$D_{se} = 0$

$D_{si} = 5 - 2*1 = 3$

For 3D structure:-

①

$F_x, F_y, F_z, M_x, M_y, M_z$

$n = 6$

②

, f_x, f_y, f_z

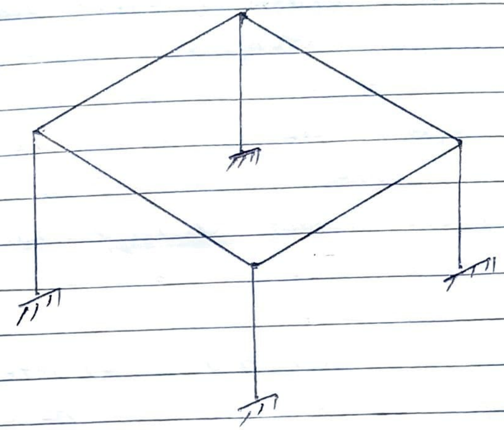
$n = 3$

③

, f_y

$n = 1$

⑯



$D_{se} = r_e - 6 = 24 - 6 = 18$

$D_{si} = 6c - r_r = 6*1 = 6$

$D_s = 18 + 6 = 24$

$D_K = 6j - r_e - m' + r_r$

$= 6*8 - 24$

$= 24$

⑰ $r_e = 9$

$D_{se} = r_e - 6 = 9 - 6 = 3$

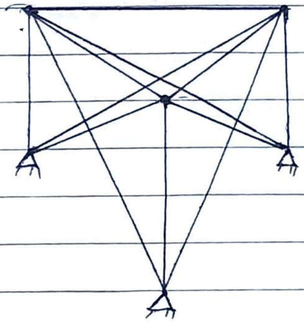
$D_{si} = m - (3j - 6)$

$= 12 - (3*6 - 6)$

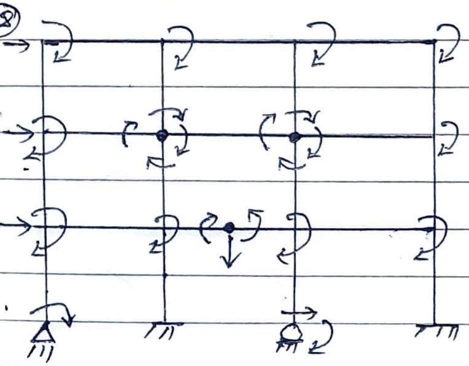
$= 0$

$D_s = 3 + 0 = 3$

$D_K = 3j - r_e = 3*6 - 9 = 9$



⑱



$D_{se} = r_e - 3 = 9 - 3 = 6$

$D_{si} = 3c - r_r$

$= 3*6 - [(4-1)+(4-1)+(2-$

$= 11$

$D_s = 11 + 6 = 17$

$D_K = 3j - r_e - m' + r_r$

$= 3*17 - 9 - 22 + [(4-1)+(4$

$+ (2-1)$

$\doteq 27$

Assuming all beams & columns are inextensible ($m' = 22$)

✻ Physical Meaning of static indeterminancy:-

It is the minimum of independent forces/stress as that are required for finding all the stresses developed in the structure. That means, 'n' degree of static indeterminancy means, there is 'n' unknown stresses to findout during complete stress analysis subjected to load.

✻ physical Meaning of kinematic Indeterminancy:-

Actually, kinematic Indeterminancy is the minimum number of displacement Quantity which makes enable to define the complete displace geometry of structure which subjected to loads.

~~Ch-8~~ :- MATRIX METHOD OF ANALYSIS:

Matrix method is method of analysis of Multistorey building having high degree of static indeterminancy. For calculation of unknown forces in different computer & structural software, this method of analysis is used.
There are two ways of finding unknown by matrix method

Matrix method

↓ Flexibility matrix method (Force Method / Compability method)
→ In this method, basic unknowns are redundant force

↓ stiffness matrix method (Displacement method / Equilibrium method)
→ In this method, basic unknowns are displacement of joints.

✻ Flexibility Matrix Method:-

This method is one of the basic methods of analysis of indeterminate structure. This method involves the removing of restraints from the indeterminate structure to make it statically determinate. This determinate structure, which must be statically stable is referred as primary structure. The excess restraints removed from the given indeterminate structure to convert it into the determinate primary structure are called redundant restraints, & the reactions or internal forces associated with these restraints are called redundant. These redundant are applied as unknown loads on primary structure & their values are determined by solving compability equations.

→ Note:- Flexibility matrix method is also known as method of consistent deformation.

* Flexibility matrix:-

→ The matrix of size 'n'*'n' which denotes the displacement due to the force applied & can be represented as,

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} & ---- & \delta_{1n} \\ \delta_{21} & \delta_{22} & ---- & \delta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{n1} & \delta_{n2} & ---- & \delta_{nn} \end{bmatrix}$$

The element, δ_{ij} of flexibility matrix is the displacement at 'i' due to unit force at 'j'.

→ Flexibity matrix has diagonal symmetry $[\delta_{ij} = \delta_{ji}]$

* Stiffness Matrix:

→ The square matrix of order 'n' which gives the force corresponding to the displacement & represented as,

$$k = \begin{bmatrix} k_{11} & k_{12} & ---- & k_{1n} \\ k_{21} & k_{22} & ---- & k_{2n} \\ k_{31} & k_{32} & --- & k_{3n} \end{bmatrix}$$

→ The element k_{ij} of stiffness matrix is the force at coordinate 'i' due to unit displacement at 'j'.

→ The stiffness matrix is symmetrical matrix

$$k_{ij} = k_{ji}$$

* Flexibility & stiffness:-

Flexibility of structure is defined as the displacement due to applied force. whereas stiffness of structure is defined as the force required to produce the given deflection.

* Compatibility equation of force method:-

From the principle of superposition, the deformation of primary structure due to combined effect of redundant & given loading must be the same as the deformation of the original indeterminate structure. This condition is known as compatibility condition.

The compatibity equation of the force method for statically indeterminate structure with 'n' redundant P_1, P_2, P_3 --- P_n at coordinate direction 1, 2, 3 --- n are written as follows,

$$\Delta_1 = \Delta_{1L} + \delta_{11}P_1 + \delta_{12}P_2 + --- + \delta_{1n}P_n$$
$$\Delta_2 = \Delta_{2L} + \delta_{21}P_1 + \delta_{22}P_2 + --- + \delta_{2n}P_n$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$\Delta_n = \Delta_{nL} + \delta_{n1}P_1 + \delta_{n2}P_2 + --- + \delta_{nn}P_n$$

where,

Δ_i = displacement of ith coordinate in indeterminate structure.

δ_{ij} = displacement of 'i' due to unit force at j.

Δ_{iL} = displacement along ~~dis~~ direction of unknown P_i due to action of actual load in primary structure.

Then,

In matrix form,

$$[\Delta] = [\Delta_L] + [\delta][P]$$

$$[\Delta - \Delta_L] = [\delta][P]$$

$$[P] = [\delta]^{-1}[\Delta - \Delta_L]$$

$$\Rightarrow \begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix} = \begin{bmatrix}\delta_{11}&\delta_{12}&\delta_{13}\\ \delta_{21}&\delta_{22}&\delta_{23}\\ \delta_{31}&\delta_{32}&\delta_{33}\end{bmatrix}^{-1}\left\{\begin{bmatrix}\Delta_1\\\Delta_2\\\Delta_3\end{bmatrix} - \begin{bmatrix}\Delta_{1L}\\\Delta_{2L}\\\Delta_{3L}\end{bmatrix}\right\}$$

* Compability eqn for yielding & Unyielding Supports:-

The compabity eqn is,

$$[\Delta] = [\Delta_L] + [\delta][P] \quad --- ①$$

→ When supports are unyielding (no settlement at support)

$$[\Delta] = 0$$

$$\begin{bmatrix}\Delta_1\\\Delta_2\\\Delta_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

So, from eqn ①,

$$\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix} = -\begin{bmatrix}\delta_{11}&\delta_{12}&\delta_{13}\\ \delta_{21}&\delta_{22}&\delta_{23}\\ \delta_{31}&\delta_{32}&\delta_{33}\end{bmatrix}^{-1}\begin{bmatrix}\Delta_{1L}\\\Delta_{2L}\\\Delta_{3L}\end{bmatrix}$$

→ When supports are yielding, this means the supports have linear & angular displacement $\delta_1, \delta_2, \theta_1$ at redundant coordinate 1, 2 & 3, then,

$$[\Delta] = \begin{bmatrix}\Delta_1\\\Delta_2\\\Delta_3\end{bmatrix} = \begin{bmatrix}\delta_1\\\delta_2\\\theta_1\end{bmatrix}$$

* Calculation of flexibility coefficient.

For beam & frame,

$$\Delta_{iL} = \int_0^L \frac{M m_i\,dx}{EI} \quad \& \quad \delta_{ij} = \int_0^L \frac{m_i \times m_j\,dx}{EI}$$

For truss,

$$\Delta_{iL} = \sum \frac{P * P_i L}{AE} \quad \& \quad \delta_{ij} = \sum \frac{P_i P_j L}{AE}$$

M = Bending moment due to given loading in primary system

m_i = Bending moment due to unit force applied at 'i'

m_j = Bending moment due to unit force applied at 'j' in primary system.

P = Member force due to applied load

P_i = Force in member due to unit force at 'i'.

P_j = Force in member due to unit force at 'j'.

* Relation between stiffness matrix & flexibility matrix.

Consider the system in which forces $P_1, P_2, P_3 ---- P_n$ produces displacement $\Delta_1, \Delta_2, --- \Delta_n$ at coordinate 1, 2, --n then flexibility matrix can be written as,

$$\Delta_1 = \delta_{11}P_1 + \delta_{12}P_2 + ---- \delta_{1n}P_n$$

$$\Delta_2 = \delta_{21}P_1 + \delta_{22}P_2 + --- \delta_{2n}P_n$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\Delta_n = \delta_{n1}P_1 + \delta_{n2}P_2 + --- \delta_{nn}P_n$$

In matrix form,

$$\begin{bmatrix}\Delta_1\\\Delta_2\\\vdots\\\Delta_n\end{bmatrix} = \begin{bmatrix}\delta_{11}&\delta_{12}&---&\delta_{1n}\\ \delta_{21}&\delta_{22}&---&\delta_{2n}\\ --&--&&---\\ \delta_{n1}&\delta_{n2}&---&\delta_{nn}\end{bmatrix}\begin{bmatrix}P_1\\P_2\\\vdots\\P_n\end{bmatrix}$$

$$\Rightarrow [\Delta] = [\delta][P] \quad -- ①$$

From the defn of stiffness matrix,

$$P_1 = K_{11}\Delta_1 + K_{12}\Delta_2 + \cdots + K_{1n}\Delta_n$$
$$P_2 = K_{21}\Delta_1 + K_{22}\Delta_2 + \cdots + K_{2n}\Delta_n$$
$$\vdots$$
$$P_n = K_{n1}\Delta_1 + K_{n2}\Delta_2 + \cdots + K_{nn}\Delta_n$$

In matrix form,

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix}$$

$$\Rightarrow [P] = [K][\Delta] \quad \text{-----} (II)$$

Using value of $[\Delta]$ from (I) in eqn (II),

$$[p] = [K][\delta][p]$$
$$\Rightarrow [K][\delta] = [p][p]^{-1}$$
$$\Rightarrow [K][\delta] = [I] \quad \text{--} (III) \quad \therefore [[A][A]^{-1} = I]$$

Where, 'I' is identity matrix. $[[p][p]^{-1} = I]$

Eqn (III) shows the required relationship between flexibility matrix & stiffness matrix. This shows that, flexibity & stiffness matrices are inverse of each other.

---

* Steps of Flexibility Matrix Method:

→ Determine degree of static indeterminancy 'n'.

→ Choose the redundants.

→ Assign the coordinate to the redundant force directions.

→ Remove restraints to redundant forces and get basic determinate structure.

→ Determine deflection in coordinate direntions due to givenloading condition in the basic determinate structure.

→ Determine flexibility matrix.

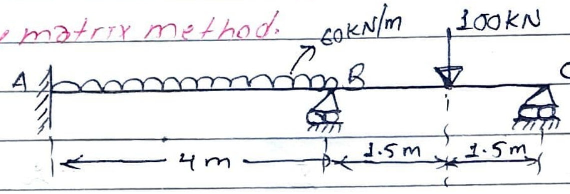
→ Apply the compability eqn,

$$[p] = [\delta]^{-1}[(\Delta - \Delta_L)]$$

→ knowing the redundant forces, find other unknown force using eqn of static equilibrium.

→ We can use unit load method or conjugate beam method to calculate Δ_L & δ.

* Analyse the continuous beam shown in figure by flexibility matrix method.



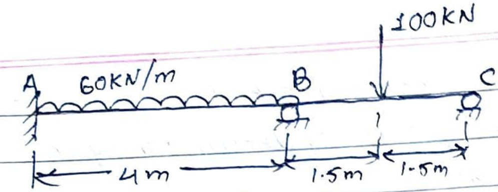
Solution →

Number of unknown rxn = 5

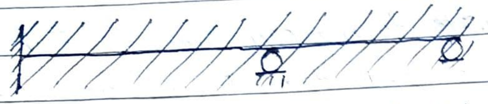
Degree of static indeterminancy = r − 3 = 5 − 3 = 2

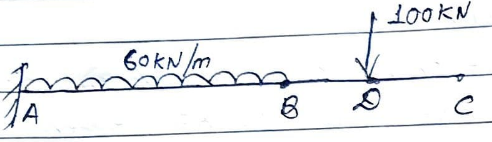
∴ No. of redundant = 2.

Let, R_1 & R_2 be the redundant.

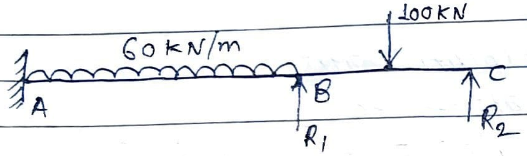


Given indeterminate structure.

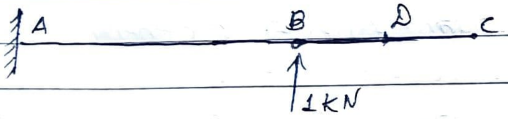




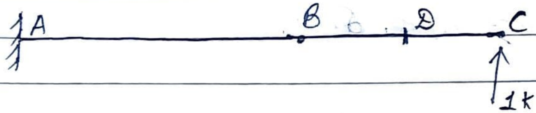
Primary structure without redundant (M)



Primary structure with redundant.



Applying 1KN load at 'B'. (m_1)



Applying 1k load at C (m_2)

Table

Segment	CD	BD	AB
Origin	C	D	B
Limit	0 – 1.5	0 – 1.5	0 – 4
M	0	$-100x$	$-100(1.5+x)$ $-30x^2$
m_1	0	0	x
m_2	x	$1.5+x$	$3+x$
EI	EI	EI	EI

$$\Delta_{1L} = \int_0^L \frac{Mm_1\,dx}{EI} = \frac{-5253.33}{EI}$$

$$\Delta_{2L} = \int_0^L \frac{Mm_2\,dx}{EI} = \frac{-11654.58}{EI}$$

$$\delta_{11} = \int_6^L \frac{m_1 * m_1\,dx}{EI} = \frac{21.33}{EI}$$

$$\delta_{12} = \delta_{21} = \int_0^L \frac{m_1 * m_2\,dx}{EI} = \frac{45.33}{EI}$$

$$\delta_{22} = \int_6^L \frac{m_2 * m_2\,dx}{EI} = \frac{114.33}{EI}$$

The flexibility matrix is, $\delta = \frac{1}{EI}\begin{bmatrix} 21.33 & 45.33 \\ 45.33 & 114.33 \end{bmatrix}$

From compatibility eqn for force method,

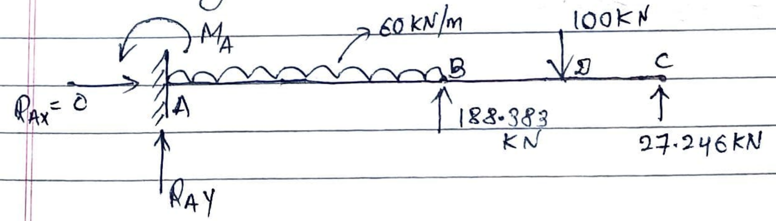
$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} - \begin{bmatrix} \Delta_{1L} \\ \Delta_{2L} \end{bmatrix} \right\}$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \cancel{EI} \begin{bmatrix} 21.33 & 45.33 \\ 45.33 & 114.33 \end{bmatrix}^{-1} * \begin{bmatrix} 5253.33 \\ 11654.58 \end{bmatrix} * \frac{1}{\cancel{EI}}$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 188.383 \\ 27.246 \end{bmatrix}$$

∴ Redundants are, $R_1 = 188.383$ KN & $R_2 = 27.246$ KN.

Then, the given indeterminate structure is converted into,



Now, using equations of static equilibrium.

$+\circlearrowleft\ \Sigma M_A = 0$

$\Rightarrow -M_A + 60\times4\times2 + 100\times5.5 - 188.383\times4 - 27.246\times7 = 0$

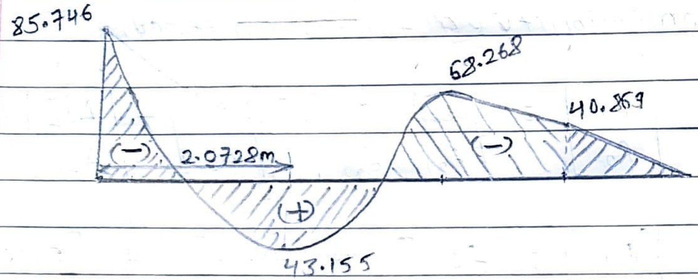
$\Rightarrow M_A = 85.746$ kNm

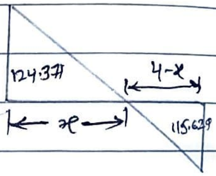
$(+\uparrow)\ \Sigma F_y = 0$

$\Rightarrow R_{Ay} - 60\times4 - 100 + 188.383 + 27.246 = 0$

$\Rightarrow R_{Ay} = 124.371$ kN

Bending Moment Diagram.





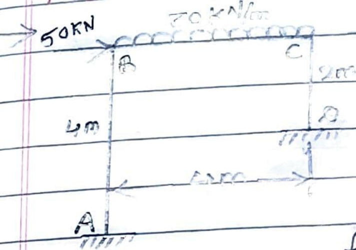
$$\frac{124.371}{x} = \frac{115.629}{4-x}$$

$\Rightarrow x = 2.0728$ m

$$(BM)_{max} = -85.746 + 124.371\times2.0728 - 60\times\frac{2.0728^2}{2}$$

$$= 43.155 \text{ kNm}$$

11.8 Analyse the portal frame shown in figure by flexibility matrix method. (EI−constant).



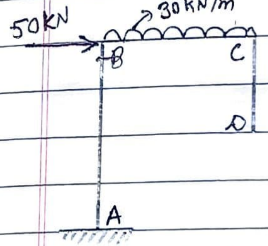
→ Given Indeterminate structure.

Here, total no. of unknown reactions = 6

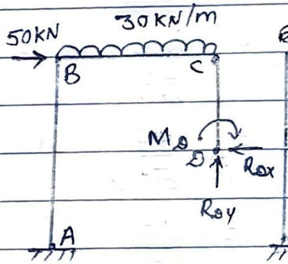
Degree of static indeterminancy = 6−3 = 3

Number of Redundant, = 3

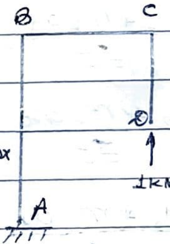
Let, R_{Dy}, R_{Dx} & M_D be three redundant. Then,



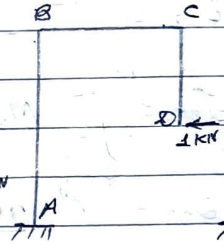
(M) Primary structure without redundant



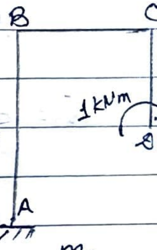
Primary structure with redundant



m_1 diagram



m_2 diagram



m_3 diagram

Portion	DC	CB	BA
Origin	D	C	B
Limit	0 − 2	0 − 4	0 − 4
M	0	$-15x^2$	$-240-50x$
m_1	0	x	4
m_2	$-x$	-2	$-(2-x)$
m_3	-1	-1	-1
EI	EI	EI	EI

$$\Delta_{1L} = \int_0^L \frac{Mm_1 dx}{EI} = \frac{-6400}{EI}$$

$$\Delta_{2L} = \int_0^L \frac{Mm_2 dx}{EI} = \frac{373.33}{EI}$$

$$\Delta_{3L} = \int_0^L \frac{Mm_3 dx}{EI} = \frac{1680}{EI}$$

$$\delta_{11} = \int_0^L \frac{m_1 * m_1 dx}{EI} = \frac{85.33}{EI}$$

$$\delta_{12} = \delta_{21} = \int_0^L \frac{m_1 m_2 dx}{EI} = \frac{-16}{EI}$$

$$\delta_{13} = \delta_{31} = \int_0^L \frac{m_1 m_3 dx}{EI} = \frac{-24}{EI}$$

$$\delta_{22} = \int_0^L \frac{m_2 * m_2 dx}{EI} = \frac{24}{EI}$$

$$\delta_{23} = \delta_{32} = \int_0^L \frac{M_2 * m_3 dx}{EI} = \frac{10}{EI}$$

$$\delta_{33} = \int_0^L \frac{m_3 * m_3 dx}{EI} = \frac{10}{EI}$$

Flexibility matrix is, $\delta = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}$

$$\delta = \frac{1}{EI} \begin{bmatrix} 85.33 & -16 & -24 \\ -16 & 24 & 10 \\ -24 & 10 & 10 \end{bmatrix}$$

From compatibility equation for force method,

$$[P] = [\delta]^{-1} \{[\Delta - \Delta_D]\}$$

In other question, If Δ is given, then put value of Δ.

Since, $\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ no settlement [unyielding condition]

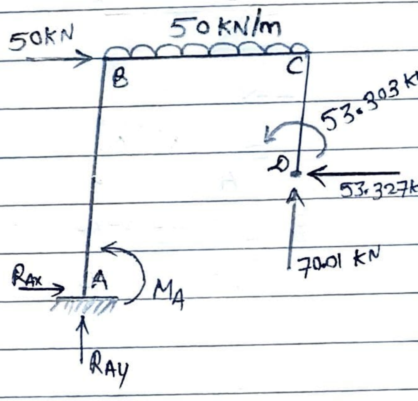
Then,

$$[P] = -[\delta]^{-1} [\Delta_L]$$

$$\begin{bmatrix} R_{Dy} \\ R_{Dx} \\ M_D \end{bmatrix} = -EI \begin{bmatrix} 85.33 & -16 & -24 \\ -16 & 24 & 10 \\ -24 & 10 & 10 \end{bmatrix}^{-1} * \begin{bmatrix} -6400 \\ 373.33 \\ 1680 \end{bmatrix} * \frac{1}{EI}$$

$$\begin{bmatrix} R_{Dy} \\ R_{Dx} \\ M_D \end{bmatrix} = \begin{bmatrix} 70.01 \\ 53.327 \\ -53.303 \end{bmatrix}$$

$\therefore R_{Dy} = 70.01$ KN, $R_{Dx} = 53.327$ KN, $M_D = 53.303$ KN (↺)



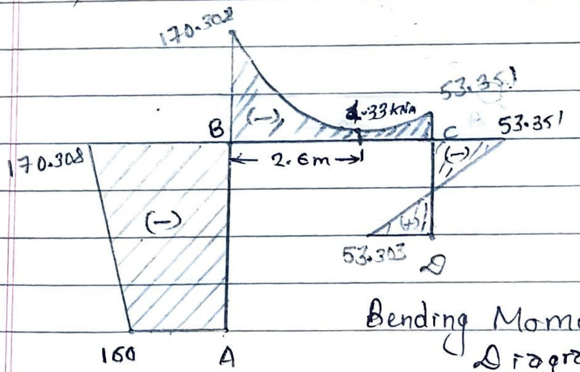
Now using eqns of static eqb,

$+\circlearrowright \Sigma M_A = 0$

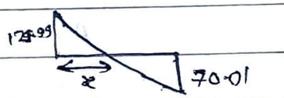
$\Rightarrow -M_A + 50 \times 4 + 50 \times 4 \times 2 - 53.303 - 70.01 \times 4 - 53.327 \times 2 = 0$

$\Rightarrow M_A = 160$ KNm

$R_{Ay} = 129.99$ KN & $R_{Ax} = 3.327$ KN



Bending Moment Diagram.



$$\frac{129.99}{x} = \frac{70.01}{4-x}$$

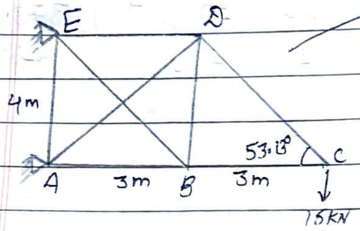
$x = 2.599$m

BM = −1.33 KNm

2018 Spring

2. Use consistent deformation method, analyze the loaded truss shown below, if all inclined bars are ~~[illegible]~~ found to be 2mm short and the temperature of vertical bars is raised by 15°C. Take, $E = 2 \times 10^5\ N/mm^2$, $\alpha = 10.8 \times 10^{-6}/°C$ Cross-sectional area of all members $= 40cm^2$.

→



→ Given indeterminate structure.

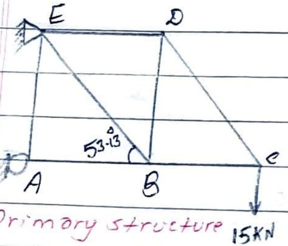
Here, $D_{se} = r - 3 = 4 - 3 = 1$

$D_{si} = m - (2j - 3)$

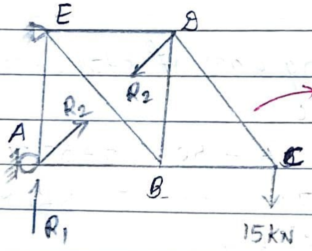
$= 8 - (2 \times 5 - 3) = 1$

Hence, one rxn is due to exceed of support & another due to exceed of Member.

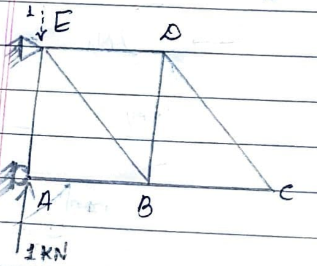
→ Consider R_1 & R_2 as redundant (shown in fig below)



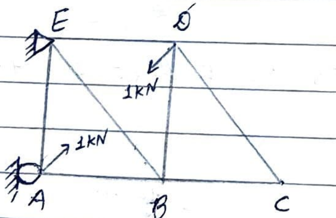
(p) Primary structure



→ primary structure with redundant



P_1 diagram



P_2 diagram

Calculation of member force.

(a) Member force due to external loading (p).

$F_{CD} \sin 53.13° - 15 = 0,\quad F_{CD} = 18.75KN$

$-F_{BC} - 18.75 \cos 53.13° = 0 \Rightarrow F_{BC} = -11.25KN$

$-F_{BD} - 18.75 \sin 53.13° = 0 \Rightarrow F_{BD} = -15KN$

$-F_{DE} + 18.75 \cos 53.13° = 0 \Rightarrow F_{DE} = 11.25KN$

$F_{BE} \sin 53.13° + (-15) = 0 \Rightarrow F_{BE} = 18.75KN$

$-F_{AB} - 18.75 \cos 53.13° - 11.25 = 0 \Rightarrow F_{AB} = -22.5KN$

$15 - F_{EA} - 18.75 \sin 53.13° = 0 \Rightarrow F_{EA} = 0$

(b) Member force (P_1) due to unit load at Roller support.

$F_{EA} = -1\ KN$ & other member force = 0.

(c) Member force (P_2) due to ~~[illegible]~~ unit load at A & D (shown in figure

$F_{CD} = 0,\quad F_{BC} = 0,\quad F_{AB} = -0.6KN.$

$-F_{BD} - 1 \sin 53.13° = 0 \Rightarrow F_{BD} = -0.8KN$

$-F_{DE} - 1 \cos 53.13° = 0 \Rightarrow F_{DE} = -0.6KN$

$F_{BE} \sin 53.13° - 0.8 = 0 \Rightarrow F_{BE} = 1\ KN$

Rxn at support

$+\circlearrowleft \Sigma M_A = 0 \Rightarrow R_{EX} * 4 - \cos 53.13° \times 4 + \sin 53.13° \times 3 = 0$

$\Rightarrow R_{EX} = 0$

$\therefore F_{EA} + \sin 53.13° = 0 \Rightarrow F_{EA} = -0.8KN$

Due to misfit $(\Delta_{CD})_m = (\Delta_{BE})_m = -2mm$

Due to tempr changes, $(\Delta_{EA})_t = (\Delta_{BD})_t = 4000 \times 10.8 \times 10^{-6} \times 15$

$= 0.648mm.$

$E = 2 \times 10^5\ N/mm^2 = 200\ KN/mm^2$

$A = 4000\ mm^2$

Member	SN	P	P_1	P_2	L/AE	PP_1L/AE	PP_2L/AE	P_1^2L/AE	P_1P_2L/AE	P_2^2L/AE	$P_1(\Delta)_m$	$(P_2)\Delta_m$	$P_1(\Delta)_t$	$P_2(\Delta_t)$	P^* (final force) (KN)
AE	1	0	-1	-0.8	1/200	0	0	0.005	0.004	0.0032	0	0	-0.648	-0.518	-129.5
BD	2	-15	0	-0.8	1/200	0	0.06	0	0	0.0032	0	0	0	-0.518	138.09
BE	3	18.75	0	1	1/160	0	0.117	0	0	0.0062	0	-2	0	0	210.12
CD	4	18.75	0	0	1/160	0	0	0	0	0	0	0	0	0	18.75
AB	5	-22.5	0	-0.6	3/800	0	0.05	0	0	0.00135	0	0	0	0	-137.3
BC	6	-11.25	0	0	3/800	0	0	0	0	0	0	0	0	0	-11.25
DE	7	11.25	0	-0.6	3/800	0	-0.025	0	0	0.00135	0	0	0	0	-103.5
Sum		→	→	→	→	0	0.202	0.005	0.004	0.0153	0	-2	-0.648	-1.036	

$$\Delta_{1L} = \Sigma \frac{PP_1L}{AE} = 0$$

$$\Delta_{2L} = \Sigma \frac{PP_2L}{AE} = 0.202 \text{ mm}$$

$$\delta_{11} = \Sigma \frac{P_1^2L}{AE} = 0.005 \text{ mm}$$

$$\delta_{21} = \delta_{12} = \Sigma \frac{P_1P_2L}{AE} = 0.004 \text{ mm}$$

$$\delta_{22} = \Sigma \frac{P_2^2L}{AE} = 0.0153 \text{ mm}$$

Flexibility matrix $\delta = \begin{bmatrix} 0.005 & 0.004 \\ 0.004 & 0.0153 \end{bmatrix}$

The compability eqn can be written as,

$\Delta_{1L} + \Delta_{1T} + \Delta_{1f} + \delta_{11}R_1 + \delta_{12}R_2 = 0$

$\Delta_{2L} + \Delta_{2T} + \Delta_{2f} + \delta_{21}R_1 + \delta_{22}R_2 = 0$

Substituting value,

$0 + (-0.648) + 0 + 0.005R_1 + 0.004R_2 = 0$

$0.005R_1 + 0.004R_2 = 0.648$ —(i)

$0.202 - 1.036 - 2 + 0.004R_1 + 0.0153R_2 = 0$

$0.004R_1 + 0.0153R_2 = 2.834$ —(ii)

On solving (i) & (ii),

$R_1 = -23.5$ KN

$R_2 = 191.37$ KN

Final forces can be calculated as,

$P^* = P + P_1R_1 + P_2R_2$

Ch-3.

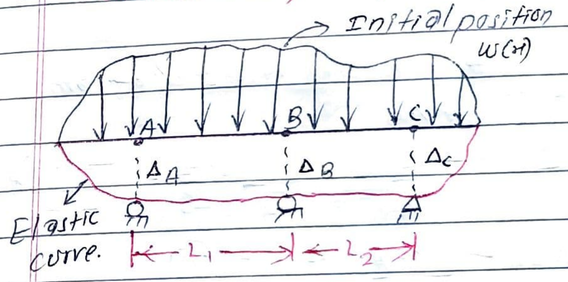
* Slope ~~and~~ deflection method:-

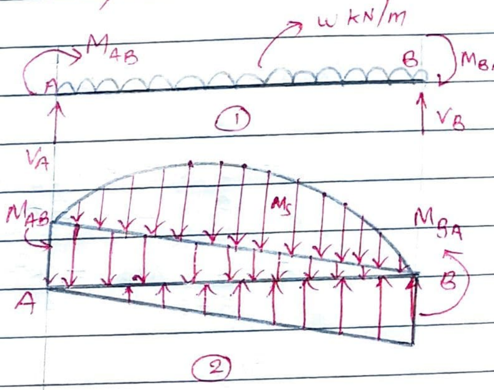
→ This method is ideally suited for the analysis of continuous beams & rigid jointed frames.

→ The assumption of this method is that all deformation is due to the effect of bending moment only.

→ Using this method, basic unknowns written in terms of slope & deflection of the end joints. Considering the joint equilibrium condition, a set of equations are formed & solution of this eqns gives unknown slope & deflections.

* Derivation of Slope deflection eqn:-





Here, A, B, C = Continuous beam with constant EI.

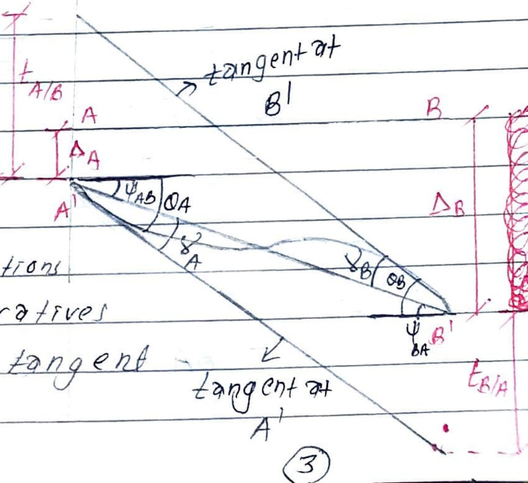
w(x) = arbitrary loading function.

$\Delta_A, \Delta_B, \Delta_C$ → Support settlement of A, B & C.

ψ_{AB} = Chord rotation

$\theta_A = \theta_B$ = Member end rotations

$t_{A/B}, t_{B/A}$ = Tangential derivatives (Vertical distance from tangent line to elastic curve).



For small deformation theory,

$$\gamma_A = \frac{t_{B/A}}{L} \quad \& \quad \gamma_B = \frac{t_{A/B}}{L} \qquad [\tan\gamma_A = \gamma_A]$$

From Geometry,

$$\gamma_A = \theta_A - \psi_{AB}$$

Then,

$$\frac{t_{B/A}}{L} = \theta_A - \psi_{AB} \text{ ----- (i)}$$

$$\frac{t_{A/B}}{L} = \theta_B - \psi_{BA} \text{ --- (ii)}$$

From figure, using second moment theorem,

$t_{A/B}$ = Sum of moment of area under M/EI diagram about end of member AB

$$= \frac{1}{2} \times L \times \frac{M_{BA}}{EI} \times \frac{2}{3} \times L - \frac{1}{2} \times L \times \frac{M_{AB}}{EI} \times \frac{1}{3} L - \frac{(M_s)_A}{EI}$$

$$= \frac{L^2}{3EI} M_{BA} - \frac{L^2}{6EI} M_{AB} - \frac{(M_s)_A}{EI} \text{ --- (iii)}$$

Similarly,

$$t_{B/A} = \frac{L^2}{3EI} M_{AB} - \frac{L^2}{6EI} M_{BA} + \frac{(M_s)_B}{EI} \text{ --- (iv)}$$

From (i) & (iv)

$$\theta_A - \psi_{AB} = \frac{1}{L}\left[\frac{L^2}{3EI} M_{AB} - \frac{L^2}{6EI} M_{BA} + \frac{(M_s)_B}{EI}\right] \text{ --- (v)}$$

Similarly,

From (ii) & (iii)

$$\theta_B - \psi_{BA} = \frac{1}{L}\left[\frac{L^2}{3EI} M_{BA} - \frac{L^2}{6EI} M_{AB} - \frac{(M_s)_A}{EI}\right] \text{ --- (vi)}$$

From (v) & (vi),

$$M_{AB} = \frac{2EI}{L}[2\theta_A + \theta_B - 3\psi_{AB}] + \frac{2(M_s)_A}{L^2} - \frac{4(M_s)_B}{L^2} \text{ --- (vii)}$$

$$M_{BA} = \frac{2EI}{L}[\theta_A + 2\theta_B - 3\psi_{BA}] + \frac{4(M_s)_A}{L^2} - \frac{2(M_s)_B}{L^2} \text{ --- (viii)}$$

If both supports are fixed,

$\theta_A = 0, \quad \theta_B = 0, \quad \psi_{AB} = 0, \quad \psi_{BA} = 0,$

$M_{AB} = F.E.M_{AB}$ & $M_{BA} = F.E.M_{BA}$

Now,

$$M_{AB} = \frac{2EI}{L}\left[2\theta_A + \theta_B - \frac{3\Delta_{AB}}{L}\right] + FEM_{AB}$$

$$M_{BA} = \frac{2EI}{L}\left[\theta_A + 2\theta_B - \frac{3\Delta_{BA}}{L}\right] + FEM_{BA}$$

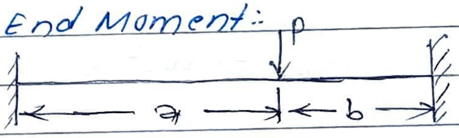
Here,

$$(FEM)_{AB} = \frac{2(M_s)_A}{L^2} - \frac{4(M_s)_B}{L^2} \text{ --- (ix)}$$

$$(FEM)_{BA} = \frac{4(M_s)_A}{L^2} - \frac{2(M_s)_B}{L^2} \text{ --- (x)}$$

✻ Fixed End Moment:

a)

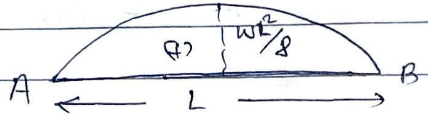


$$M_{AB} = -\frac{Pab^2}{L^2} \quad \& \quad M_{BA} = \frac{Pa^2b}{L^2}$$

b)

$$M_{AB} = -\frac{wL^2}{12} \quad \& \quad M_{BA} = \frac{wL^2}{12}$$

Proof



From eqn (ix) & (x),

$$FEM_{AB} = \frac{2(M_s)_A}{l^2} - \frac{4(M_s)_B}{l^2}$$ &

$$FEM_{BA} = \frac{4(M_s)_A}{l^2} - \frac{2(M_s)_B}{l^2}$$

In case of simply supported beam with udl along its span, then,

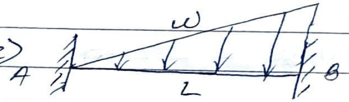
$$(M_s)_A = \frac{2}{3} \times L \times \frac{wL^2}{8} \times \frac{L}{2} = \frac{wL^4}{24}$$

$$(M_s)_B = \frac{2}{3} \times L \times \frac{wL^2}{8} \times \frac{L}{2} = \frac{wL^4}{24}$$

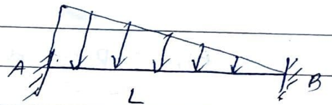
Then,

$$(FEM)_{AB} = \frac{2}{l^2} \times \frac{wL^4}{24} - \frac{4}{l^2} \times \frac{wL^4}{24} = -\frac{wL^2}{12}$$

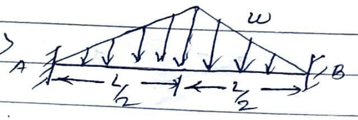
$$(FEM)_{BA} = \frac{4(M_s)_A}{l^2} - \frac{2(M_s)_B}{l^2} = \frac{wL^2}{12}$$

c)


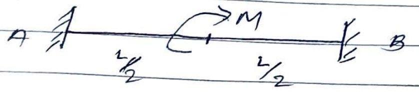
$M_{AB} = -\frac{wL^2}{30}$ & $M_{BA} = \frac{wL^2}{20}$

d)


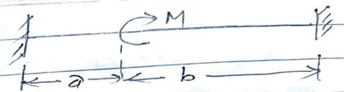
$M_{AB} = -\frac{wL^2}{20}$ & $M_{BA} = \frac{wL^2}{30}$

e)


$M_{AB} = -\frac{5wL^2}{96}$ & $M_{BA} = \frac{5wL^2}{96}$

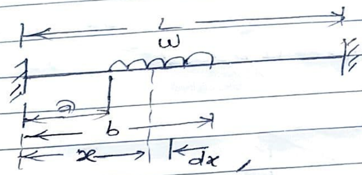
f)


$M_{AB} = \frac{M}{4}$ & $M_{BA} = M/4$

(g)


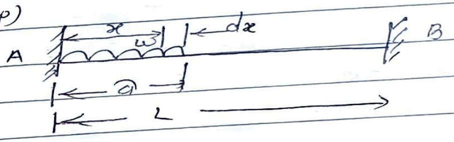
$$M_{AB} = \frac{Mb(3a-L)}{L^2}$$

$$M_{BA} = \frac{Ma(3b-L)}{L^2}$$

(h)


$$M_{AB} = -\int_a^b \frac{wx(l-x)^2 dx}{l^2}$$

$$M_{BA} = \int_a^b \frac{wx^2(l-x)\,dx}{l^2}$$

(i)


$$M_{AB} = -\int_0^a \frac{wx(l-x)^2\,dx}{l^2}$$

$$M_{BA} = \int_0^a \frac{wx^2(l-x)\,dx}{l^2}$$

※ Sign convention:-

clockwise moment → +ve

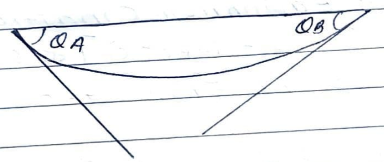
Anticlockwise moment → -ve

Rotation:-

If 'θ' be the rotations, the angle 'θ' is positive if the elastic curve has clockwise when the tangent to the elastic curve from its original direction.

$\theta_A = +ve$

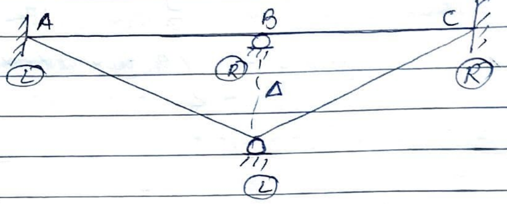
$\theta_B = -ve$.



Settlement:-

If Δ is +ve → Right side support is below left side support.

If Δ is -ve → Left side support is below the right side support.



Δ is positive for beam AB

Δ is negative for beam BC.

Steps for analysis of beam by slope deflection method:-

1. Calculate fixed end Moment (FEM). In case of cantilever beam (i.e no support at end), the FEM will be equal to moment at support of cantilever beam.

. Apply slope deflection eqn,

$$M_{AB} = \frac{2EI}{L}\left[2\theta_A + \theta_B - \frac{3\Delta_{AB}}{L}\right] + FEM_{AB}$$

$$M_{BA} = \frac{2EI}{L}\left[\theta_A + 2\theta_B - \frac{3\Delta_{BA}}{L}\right] + FEM_{BA}$$

Applying joint equilibrium condition,
[$\Sigma M_B = 0$ ($M_{BA} + M_{BC} = 0$)] & so on.

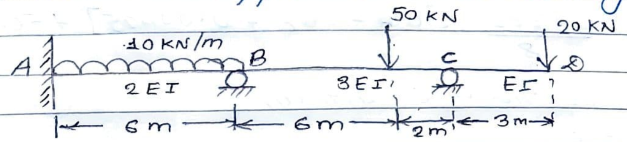
Solve the eqns coming from step '3' & find unknown oint rotation.

Back substitution of 'θ' in eqns of step '2' gives FEM.

raw FBD, & calculate rxn force.

* Analyse the beam in the given figure by slope-deflection method. Draw BM diagram considering given external loading & rotation of support 'A' by $(1/15)^c$ clockwise, support 'B' settles down by 3mm.

→



(i) Calculation of FEM,

$$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{10\times 6^2}{12} = -30\ KNm$$

$$(FEM)_{BA} = 30\ KNm$$

$$(FEM)_{BC} = -\frac{Pab^2}{L^2} = -\frac{50\times 6\times 2^2}{8^2} = -18.75\ KNm$$

$$(FEM)_{CB} = \frac{Pa^2b}{L^2} = \frac{50\times 6^2\times 2}{8^2} = 56.25\ KNm$$

$$M_{CD} = -20\times 3 = -60\ KNm$$

ii) Applying slope Deflection eqn,

$$M_{AB} = \frac{2\times 2EI}{6}\left[2\times 1/15 + \theta_B - \frac{3\times 0.003}{6}\right] - 30$$

$$M_{AB} = \frac{2}{3}EI\left[\theta_B + 0.1318\right] - 30 \text{ ----- (i)}$$

$$M_{BA} = \frac{2\times 2EI}{6}\left[1/15 + 2\theta_B - \frac{3\Delta}{L}\right] + 30$$

$$M_{BA} = \frac{2}{3}EI\left[2\theta_B + 0.0651\right] + 30 \text{ ----- (ii)}$$

$$M_{BC} = \frac{2 \times 3EI}{8}\left[2\theta_B + \theta_C - \frac{3 * (-0.003)}{8}\right] - 18.75$$

$$M_{BC} = \frac{3EI}{4}\left[2\theta_B + \theta_C + 0.001125\right] - 18.75 \text{ --- (iii)}$$

$$M_{CB} = \frac{2 \times 3EI}{8}\left[\theta_B + 2\theta_C + 0.001125\right] + 56.25 \text{ --- (v)}$$

(iii) Joint equilibrium condition,

$$\Sigma M_B = 0$$

$$\Rightarrow M_{BA} + M_{BC} = 0$$

$$\Rightarrow \frac{2}{3}EI(2\theta_B + 0.0651) + 30 + \frac{3EI}{4}\left[2\theta_B + \theta_C + 0.001125\right] - 18.75 = 0$$

$$\Rightarrow \frac{17EI}{6}\theta_B + \frac{3EI}{4}\theta_C + 0.04424EI + 11.25 = 0 \text{ --- (vi)}$$

$$\Sigma M_C = 0$$

$$\Rightarrow M_{CB} + M_{CD} = 0$$

$$\Rightarrow \frac{3EI}{4}\left[\theta_B + 2\theta_C + 0.001125\right] + 56.25 - 60 = 0$$

$$\Rightarrow \frac{3EI}{4}\theta_B + \frac{3EI}{2}\theta_C + 0.0008437EI - 3.75 = 0 \text{ --- (vii)}$$

Assume, $EI = 1000 \text{ kNm}^2$

From (vi) & (vii),

$$2833.33\theta_B + 750\theta_C = -55.49 \text{ --- Ⓐ}$$

&

$$750\theta_B + 1500\theta_C = 2.9063 \text{ --- Ⓑ}$$

Solving Ⓐ & Ⓑ, we get,

$$\theta_B = -0.02316$$

$$\theta_C = 0.01352$$

(iv) Substituting value of θ_B and θ_C in (i), (ii), (iii) & (iv)

$M_{AB} = 42.42 \text{ kNm}$

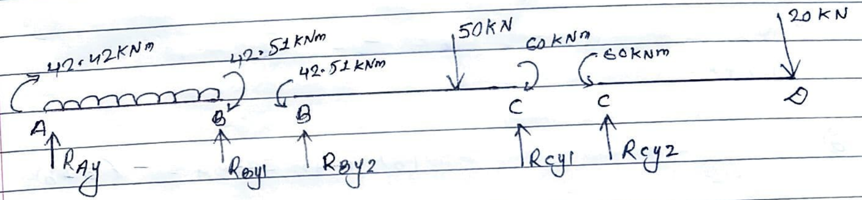
$M_{BA} = 42.51 \text{ kNm}$

$M_{BC} = -42.58 \text{ kNm}$

$M_{CB} = 60 \text{ kNm}$

$M_{CD} = -60 \text{ kNm}$

(v) Draw FBD,



(vi) Applying equilibrium condition for calculation of reaction force in each member.

For AB,

$$+\circlearrowleft \Sigma M_A = 0 \Rightarrow 42.42 + 10 \times 6 \times 3 + 42.51 - R_{By1} \times 6 = 0$$

$$\Rightarrow R_{By1} = 44.155 \text{ kN}$$

$$(+\uparrow)\ \Sigma F_y = 0 \Rightarrow R_{Ay} + 44.155 - 10 \times 6 = 0 \Rightarrow R_{Ay} = 15.845 \text{ kN.}$$

For BC, $+\circlearrowleft \Sigma M_B = 0 \Rightarrow -42.51 + 50 \times 6 + 60 - R_{cy1} \times 8 = 0$

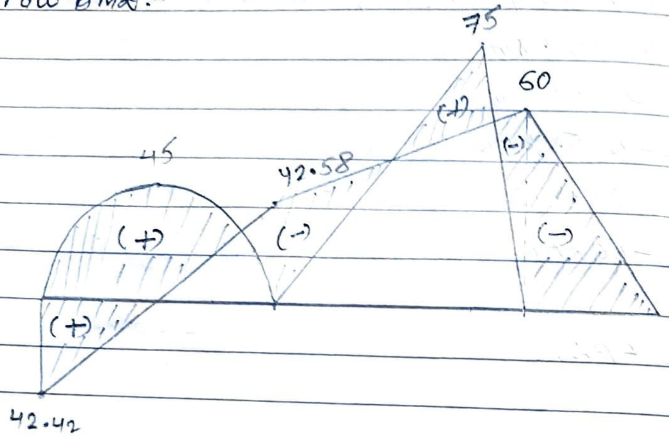
$$\Rightarrow R_{cy1} = 39.686 \text{ KN}$$

$$R_{By2} = 10.31 \text{ KN}$$

For CD, $R_{cy2} = 20 \text{ kN}$

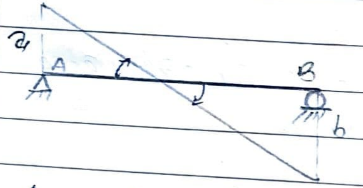
$R_{Ay} = 15.84 \text{ kN}$, $R_{By} = 54.40 \text{ kN}$ $R_{cy} = 59.68 \text{ kN}$

(v) Draw BMD:-



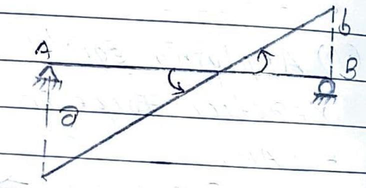
☆ Note:-

a)

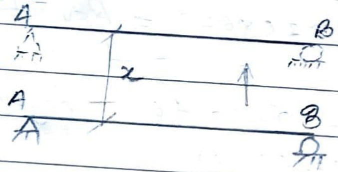


$\Delta_{AB} = \Delta_{BA} = (a+b)$

(b) $\Delta_{AB} = \Delta_{BA} = -(a+b)$

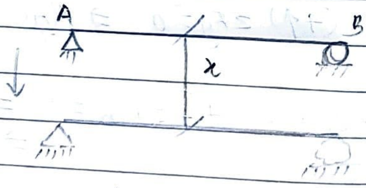


c)



$\Delta_{AB} = \Delta_{BA} = 0.$

d)



$\Delta_{AB} = \Delta_{BA} = 0.$

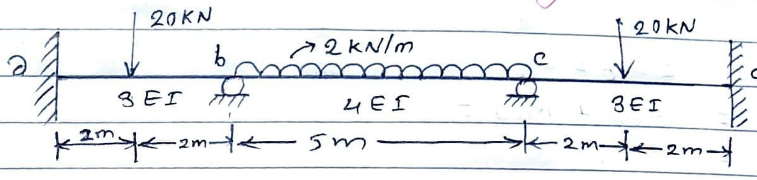
e)

$\Delta_{AB} = +(b-a)$

f)

$\Delta_{AB} = -(b-a)$

☆ Use slope deflection method to analyse the continuous beam shown in figure. Draw free body diagram, B.M.D & SFD. The support 'a' rotates by 0.001 rad. clockwise & support 'b' rotates by 0.001 rad anticlockwise. Support 'b' and 'c' both settle down by 10mm.



→ (a) Fixed end moment calculation,

$FEM_{ab} = -\frac{pab^2}{L^2} = -\frac{20 \times 2 \times 2^2}{4^2} = -10$ kNm

$FEM_{ba} = 10$ kNm

$FEM_{bc} = -\frac{wL^2}{12} = -\frac{2 \times 5^2}{12} = -4.167$ kNm

(b) Applying slope deflection equation,

$M_{ab} = \frac{2 \times 3EI}{4}\left[2 \times 0.001 + \theta_b - \frac{3 * 10 \times 10^{-3}}{4}\right] - 10$

$M_{ab} = -0.0055EI + 1.5\theta_b EI - 10$ ~~0.1125~~ ------ (i)

$M_{ba} = \frac{2 \times 3EI}{4}\left[0.001 + 2\theta_b - \frac{3 \times 10 \times 10^{-3}}{4}\right] + 10$

$M_{ba} = -0.0065EI + 3\theta_b EI +$ ~~9.98875~~ 10 --- (ii)

$M_{bc} = \frac{2 \times 4EI}{5}\left[2\theta_b + \theta_c - 3 \times 0\right] - 4.167$

$M_{bc} = 3.2EI\theta_b + 1.6EI\theta_c - 4.167$ --- (iii)

(c) Joint equilibrium,

$\Sigma M_b = 0$

$\Rightarrow M_{ba} + M_{bc} = 0$

$\Rightarrow -0.0065EI + 3EI\theta_b + 10 + 3.2EI\theta_b + 1.6EI\theta_c - 4.167=$

$\Rightarrow 6.2EI\theta_b + 1.6EI\theta_c - 0.0065EI + 5.833 = 0$ ---- (iv)

$\Sigma M_c = 0$

$\Rightarrow M_{cb} + M_{cd} = 0$

$\Rightarrow$

→ θ_b is also given, $\theta_b = -0.001$ rad,

From (iv), put θ_b & $EI = 1500\ kNm^2$

$$6.2 \times 1500 \times (-0.001) + 1.6 \times 1500 \times \theta_c - 0.0065 \times 1500 + 5.833 = 0$$

$$\theta_c = 0.005507 \text{ rad}$$

Then,

$M_{ab} = -20.5$ kNm

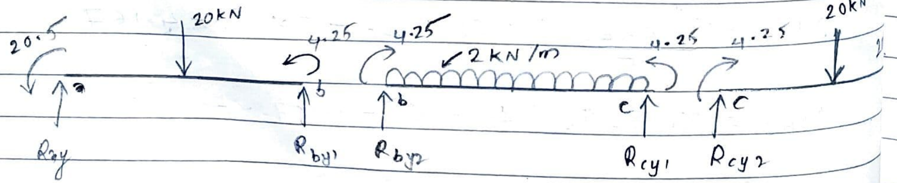
$M_{ba} = -4.25$ kNm

$M_{bc} = 4.25$ kNm

$M_{cb} = -4.25$ kNm

$M_{cd} = 4.25$ kNm

$M_{dc} = 20.5$ kNm



$+\circlearrowright \Sigma M_a = 0;\quad 20 \times 2 - 20.5 - 4.25 - R_{by1} \times 4 = 0 \Rightarrow R_{by1} = 3.812kN$

$R_{ay} - 20 + 3.812 = 0 \Rightarrow R_{ay} = 16.188kN$

$+\circlearrowright \Sigma M_c = 0 \Rightarrow R_{by2} \times 5 + 4.25 - 4.25 - 2 \times 5 \times 2.5 = 0$

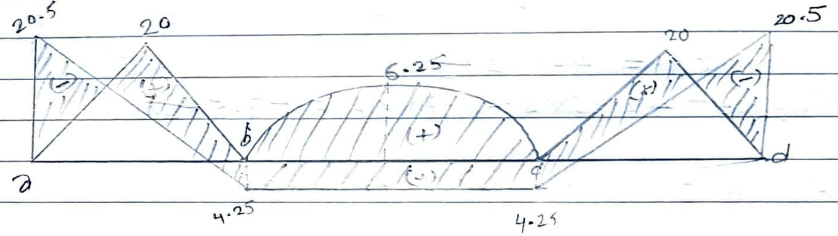
$R_{by2} = 5\ kN$

∴ Reaction are,

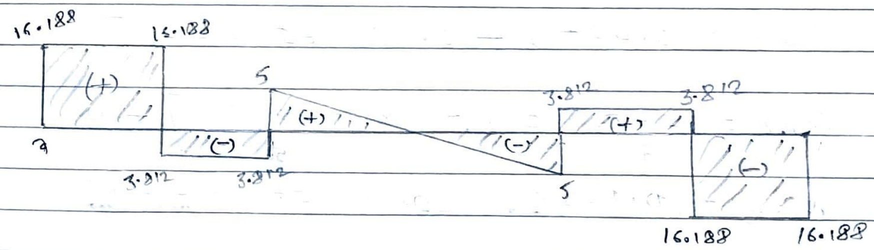
$R_{ay} = 16.188$ kN, $\quad R_{by} = 8.812$ kN $\quad R_{cy} = 8.812$ kN

$R_{dy} = 16.188$ kN

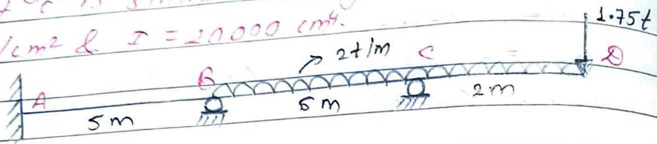
Bending Moment Diagram :- (wrong)



Shear Force diagram,



Q. Determine the member end moments using slope deflection method for the given loaded beam as shown in figure. Support 'B' is lifted up by 10mm & Support 'C' is sinks down by 5mm. Take, E = 2000 t/cm² & I = 10000 cm⁴.



→

(i) Calculation of FEM,

$(FEM)_{AB} = (FEB)_{BA} = 0$

$(FEM)_{BC} = -\frac{2\times 6^2}{12} = -$ ~~4.167~~ 6 kt/m

$(FEM)_{CB} =$ ~~4.167~~ 6 kt/m

~~EI~~ $(M)_{CD} = -2\times 2\times 1 - 1.75\times 2 = -7.5$ kt/m

(ii) Applying slope deflection eqn

$$M_{AB} = \frac{2EI}{5}\left[2\theta_A + \theta_B - \frac{3\Delta_{AB}}{5}\right] + 0$$

$$M_{AB} = \frac{2EI}{5}\left[2\times 0 + \theta_B - \frac{3\times(-10\times 10^{-3})}{5}\right]$$

$$M_{AB} = 0.4EI\theta_B + 0.0024EI \quad \text{---} (1)$$

$$M_{BA} = \frac{2EI}{5}\left[2\theta_B - \frac{3\times(-10\times 10^{-3})}{5}\right] + 0$$

$$M_{BA} = 0.8EI\theta_B + 0.0024EI \quad \text{---} (11)$$

$$M_{BC} = \frac{2EI}{6}\left[2\theta_B + \theta_C - \frac{3\times\Delta_{BC}}{6}\right] - \cancel{4.167}\,6$$

$$M_{BC} = \frac{EI}{3}\left[2\theta_B + \theta_C - \frac{3\times 15\times 10^{-3}}{6}\right] - \cancel{4.167}\,6$$

$$M_{BC} = 0.67EI\theta_B + 0.33EI\theta_C - 0.0025EI - \cancel{4.167}\,6 \quad \text{--} (iii)$$

$$M_{CB} = \frac{2EI}{6}\left[2\theta_C + \theta_B - \frac{3\times 15\times 10^{-3}}{6}\right] + \cancel{4.167}\,6$$

$$M_{CB} = 0.33EI\theta_B + 0.67EI\theta_C - 0.0025EI + \overset{6}{\cancel{4.167}} \quad - (iv)$$

(iii) Condition of joint equilibrium,

$\Sigma M_B = 0;$

$\Rightarrow M_{BA} + M_{BC} = 0$

$\Rightarrow 0.8EI\theta_B + 0.0024EI + 0.67EI\theta_B + 0.33EI\theta_C - 0.0025EI - 6$ ~~4.167~~ $= 0$

$\Rightarrow 1.47EI\theta_B + 0.33EI\theta_C - 0.0001EI - 6$ ~~4.167~~ $= 0$

But,

$EI = 2000 \times 10000$

$= 2\times 10^{7}\ tcm^2$

$= 2\times 10^{3}\ tm^2$

$\Rightarrow 2940\theta_B + 660\theta_C = \overset{6.2}{\cancel{4.367}} \quad \text{---} (v)$

$\Sigma M_C = 0;$

$\Rightarrow M_{CB} + M_{CD} = 0$

$\Rightarrow 0.33EI\theta_B + 0.67EI\theta_C - 0.0025EI + \overset{6}{\cancel{4.167}} - 7.5 = 0$

$\Rightarrow 660\theta_B + 1340\theta_C = 8.58 \quad \text{---} (vi)$

On solving (v) & (vi),

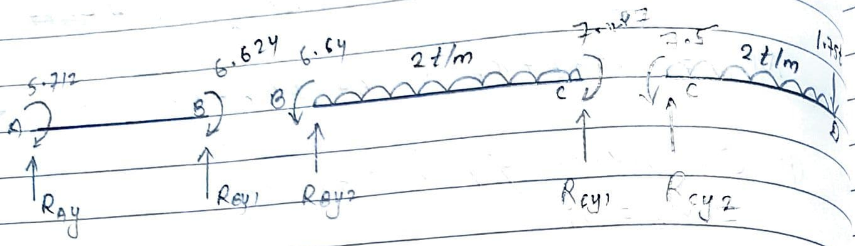
$\theta_C = 4.28\times 10^{-3}$ rad

$\theta_B = 1.44\times 10^{-3}$ rad

Then,

$M_{AB} =$ ~~4.88 tm~~ 5.712 tm	$M_{BC} = -6.64$ tm
$M_{BA} = 6.624$ tm	$M_{CB} = 7.487$ tm

Free body diagram,



Rxn calculation,

$5.712 + 6.624 - R_{By1} \times 5 = 0 \Rightarrow R_{By1} = 2.467t$

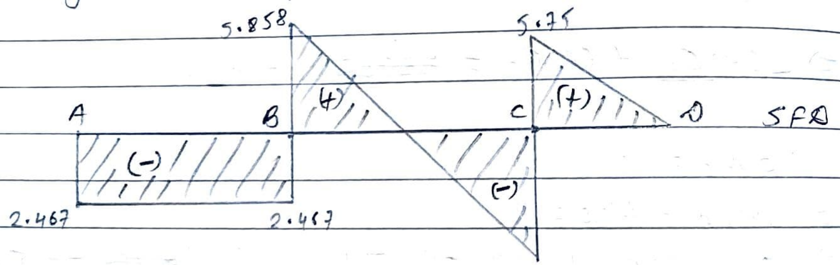
$R_{Ay} = -2.467t$

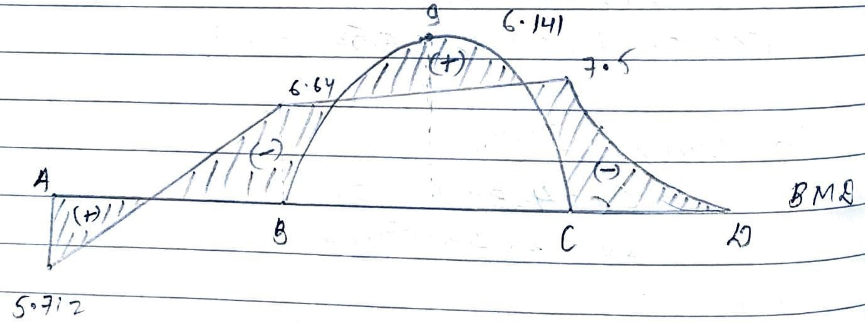
$-6.64 + 7.487 + 2 \times 6 \times 3 - R_{cy1} \times 6 = 0 \Rightarrow R_{cy1} = 6.141t$

$R_{By2} - 12 + 6.141 = 0 \Rightarrow R_{By2} = 5.858t$

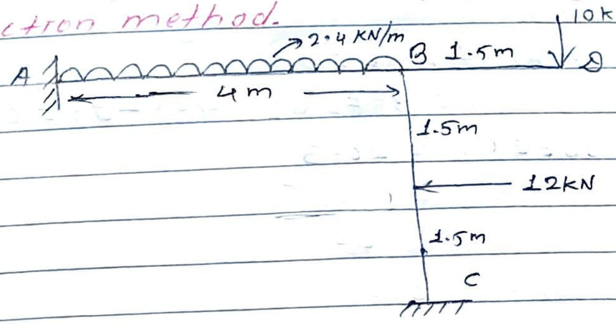
$R_{cy2} - 2 \times 2 - 1.75 = 0 \Rightarrow R_{cy2} = 5.75t$

$\therefore R_{Ay} = -2.467t, \quad R_{By} = 8.325t, \quad R_{cy} = 11.891t$





** Analyse the frame shown in figure by using slope deflection method.



→

(i) Calculation of Fixed End Moments,

$(FEM)_{AB} = -\frac{wL^2}{12} = -\frac{2.4 \times 4^2}{12} = -3.2\,kNm$

$(FEM)_{BA} = 3.2\,kNm$

$(FEM)_{BD} = -10 \times 1.5 = -15\,kNm$

$(FEM)_{BC} = -\frac{PL}{8} = -\frac{12 \times 3}{8} = -4.5\,kNm$

$(FEM)_{CB} = 4.5\,kNm$

ii) Applying slope deflection equation,

$M_{AB} = \frac{2EI}{4}[0 + \theta_B - 0] - 3.2 = 0.5EI\theta_B - 3.2$

$M_{BA} = \frac{2EI}{4}[2\theta_B + 0 - 0] + 3.2 = 0.5EI\theta_B + 3.2$

$M_{BC} = \frac{2EI}{3}[2\theta_B + 0 - 0] - 4.5 = 1.33EI\theta_B - 4.5$

$M_{CB} = \frac{2EI}{3}[\theta_B] + 4.5 = 0.67EI\theta_B + 4.5$

(iii) Applying joint equilibrium condition,

$$\Sigma M_B = 0$$

$$M_{BA} + M_{BC} + M_{BD} = 0$$

$$EI\theta_B + 3.2 + 1.33EI\theta_B - 4.5 - 15 = 0$$

$$2.33EI\theta_B = 16.3$$

$$\theta_B = \frac{6.995}{EI}$$

(iv) Final Moments are,

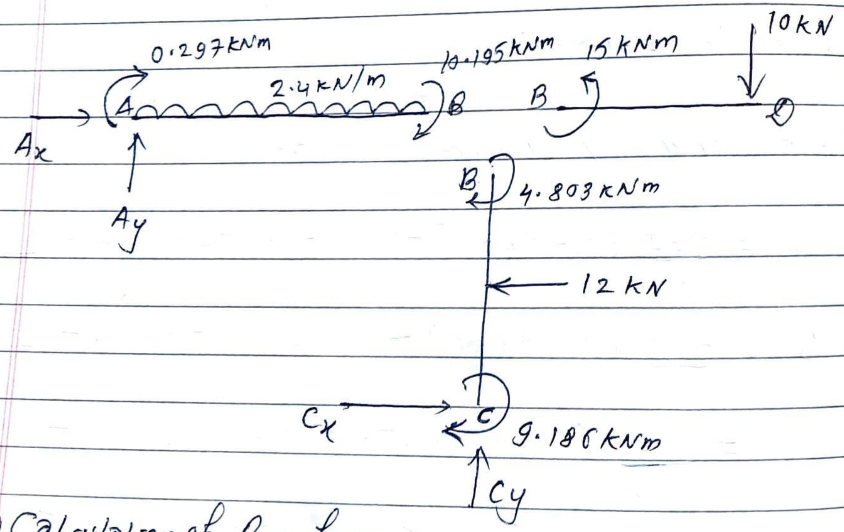
$$M_{AB} = EI \times 0.5 \times \frac{6.995}{EI} - 3.2 = 0.297\ KNm$$

$$M_{BA} = 10.195\ KNm$$

$$M_{BC} = 4.803\ KNm$$

$$M_{CB} = 9.186\ KNm$$

(v) Free body diagram,



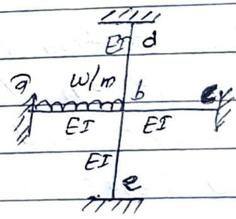
vi) Calculation of Rxn force,

$A_x = 0$, $A_y = 9.6KN$, $C_y = 0$, $C_x = 12KN$

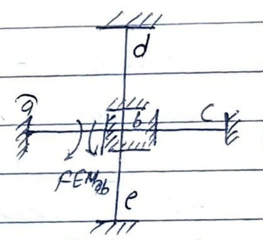
Ch-4

Moment Distribution Method:

This method is quite similar to slope deflection method. The problem of solving large number of simultaneous eqns was simplified by professor Hardy Cross in 1990. As, this method was presented by professor Hardy Cross, this method is also called Hardy cross method. In Moment distribution method, every joint of the structure to be analysed is fixed as to develop fixed end moment. Then, each fixed joint is released & the fixed end moments (by the time of release) are distributed to adjacent member until equilibrio[...] is balanced. The MDM in mathematical terms can be demonstrated as the process of solving a set of simultaneous equation by means of iteration.



Frg ①



Frg ② Locked at joint b

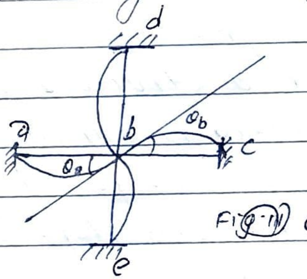


Fig (iii) Unlucked at joint B.

A moment distribution method is applied to a Structural joint to produce rotation without translation gets distributive among the connective member at the joint in the same proportion as their stiffness. The unbalanced moment applied at the joint is approximately shared by the members at the joints, which is called distributed moments.

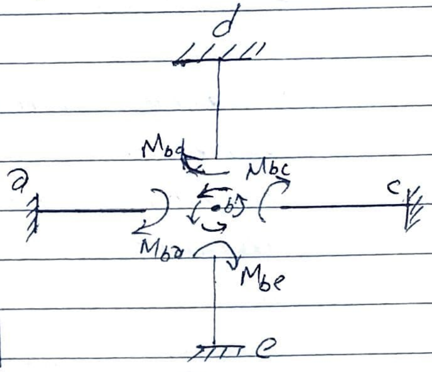


Fig: (iv)

unbalanced moment, $M = FEM_{ba}$

Distributed moments = M_{ba}, M_{bc}, M_{bd}, M_{be}

* Terminology:-

a) **Fixed end moment:-**

FEM are the moments produce at member due to external loads when the joints are fixed.

b) **Unbalanced moment:**

The algebric sum of the FEM acting on the joint.

from figure,

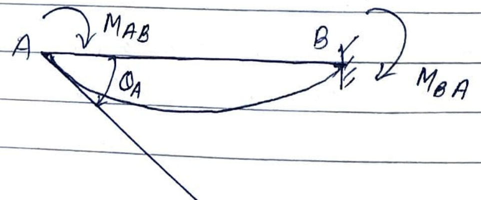
Unbalanced Moment = FEM_{ba}.

c) **Stiffness:**

When a structural member of uniform section is subjected to a moments at one end only, then the moment required so, as to rotate that end to produce unit slope (1 rad) is called stiffness of the member.

d) **Carry over moment:-**

When moment is applied at one end of member, rotation of that end & fixing the far end, some moment is developed at the far end ~~of a moment~~ also, this moment is called COM.



from slope deflection eqn,

$$M_{AB} = \frac{2EI}{L}\left[2\theta_A + \overset{\nearrow 0}{\theta_B} - \overset{\nearrow 0}{\frac{3\Delta_{AB}}{L}}\right] + FEM_{AB}$$

$$M_{AB} = \frac{4EI}{L}\theta_A$$

$$M_{BA} = \frac{2EI}{L}\left[\theta_A + 2\theta_B - \frac{3\Delta_{AB}}{L}\right] + (FEM)_{BA}$$

$$M_{BA} = \frac{2EI}{L}\theta_A = \frac{M_{AB}}{2}$$

∴ Carry over moment = 1/2 of distributed moment with same sign.

e) **Stiffness or stiffness factor:-**

For prismatic member, stiffness or stiffness factor is defined as the ratio of MOI to its span.

$$k = I/L$$

f) **Distribution factor:-**

The ratio of effective stiffness of a member to the sum of relative stiffness meeting at the joint.

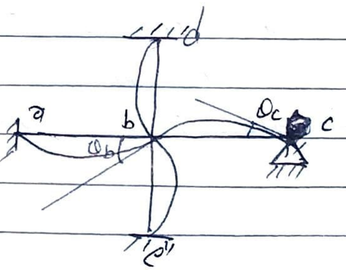
$$D.F = \frac{K_i}{\Sigma K_i} = \frac{I/L}{\Sigma I/L}$$

When the far end is hinged or roller,

$M_{cb} = 0$ [Support 'c' is hinged]

$$\frac{2EI}{L}\left(2\theta_c + \theta_b - \frac{3\Delta_{ab}}{L}\right) + FEM_{cb} = 0$$

$$\theta_c = -\frac{\theta_b}{2}$$

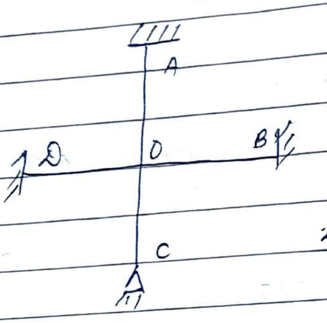


Also,

$$M_{bc} = \frac{2EI}{L}\left[2\theta_b + \theta_c - \frac{3\Delta_{bc}}{L}\right] + FEM_{bc}$$

$$= \frac{3}{4}\left[\frac{4EI}{L}\theta_b\right]$$

i.e 75% of that for end with fixed support.



*

$$DF_{OA} = \frac{M_{OA}}{M}$$

$$= \frac{K_{OA}}{K_{OA} + K_{OB} + 3/4\,K_{OC} + K_{OD}}$$

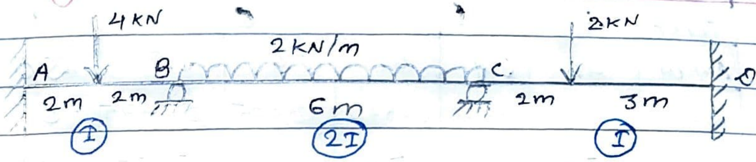
$$DF_{OC} = \frac{M_{OC}}{M}$$

$$= \frac{3/4\,K_{OC}}{K_{OA} + K_{OB} + 3/4\,K_{OC} + K_{OD}}$$

* Steps of Moment distribution method:-

a) Firstly, findout FEM for all members.
b) Calculate distribution factors for members meeting at joints.
c) Tabulate FEM for joints. Distribution factor also.
d) Consider joint equilibrium to balance the moment & also take carry over moment is negligible.
e) Take the sum of moments to get the total moments in the joint.
f) Draw SFD, BMD and elastic Curve also.

Q. Analyse the given beam and draw SFD and BMD:-



(i) Calculation of FEM,

$$(FEM)_{AB} = -\frac{PL}{8} = -\frac{4\times 4}{8} = -2\ kNm$$

$$(FEM)_{BA} = 2\ kNm$$

$$(FEM)_{BC} = -\frac{wL^2}{12} = \frac{-2\times 6^2}{12} = -6\ kNm$$

$$(FEM)_{CB} = 6\ kNm$$

$$(FEM)_{CD} = -\frac{Pab^2}{L^2} = \frac{-2\times 2\times 3^2}{5^2} = -1.44\ kNm$$

$$(FEM)_{DC} = \frac{Pa^2b}{L^2} = \frac{2\times 2^2\times 3}{5^2} = 0.96\ kNm.$$

ii) Calculation of distribution factor:-

For joint A, $\gamma_{AB} = 0$

For joint B, $\gamma_{BA} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{(I/L)_{BA}}{(I/L)_{BA}+(I/L)_{BC}} = \frac{I/4}{I/4 + 2I/6} = 0.42$

$$\gamma_{BC} = \frac{K_{BC}}{K_{BA}+K_{BC}} = \frac{2I/6}{2I/6 + I/4} = 0.572$$

For joint C,

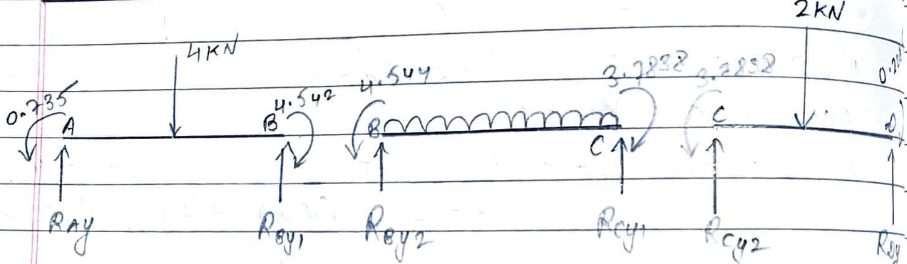
$$\gamma_{CB} = \frac{K_{CB}}{K_{CB}+K_{CD}} = \frac{2I/6}{2I/6 + I/5} = 0.625$$

$$\gamma_{CD} = 1 - 0.625 = 0.375$$

For joint D,

$$\gamma_{DC} = 0$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	0	0.428	0.572	0.625	0.375	0
FEM	-2	2	-6	6	-1.44	0.96
Balance		1.712	2.288	-2.85	-1.71	
C.O	0.856		-1.425	1.144		-0.855
Balance		0.61	0.815	-0.715	-0.429	
CO	0.305		-0.3575	0.407		-0.214
Balance		0.153	0.204	-0.254	-0.152	
C.O	0.0765		-0.127	-0.102		-0.076
Balance		0.054	0.072	-0.0637	-0.038	
C.O	0.027		-0.03185	0.636		-0.019
Balance		+0.0136	0.0182	-0.0225	-0.0135	
Final M	-0.735	4.542	-4.544	3.7838	-3.7838	-0.204



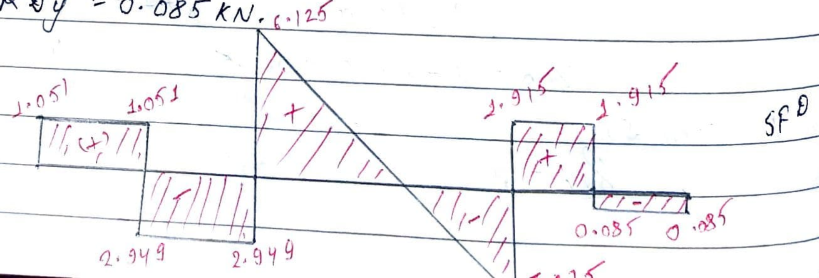
* Calculation of rxn force:-

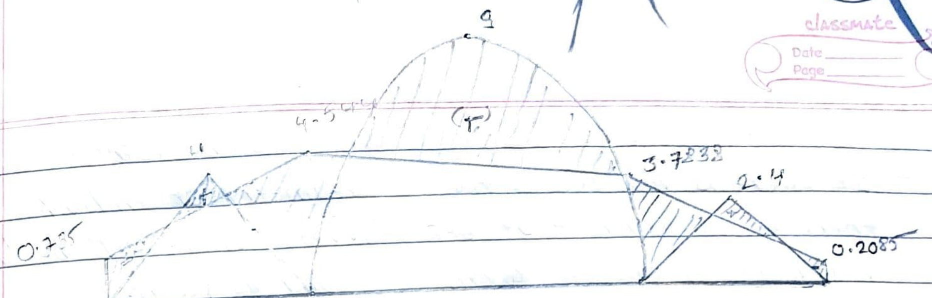
$R_{Ay} = 1.051$ KN

$R_{By} = R_{By1} + R_{By2} = 2.949 + 6.125 = 9.074$ KN

$R_{Cy} = R_{Cy1} + R_{Cy2} = 5.875 + 1.915 = 7.79$ KN

$R_{Dy} = 0.085$ KN.

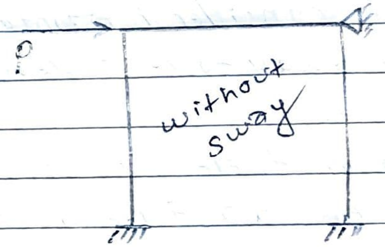




BMD.

* Analysis of frame without sway:-

There is nosway in symmetrical portal frames & analysis is similar to continuos beam. The frame is said to be nosway if, there is no movement of frame member after applying load.



* Analysis of frame with sway:-

→ The frame is said to be sway if, if there is movement of frame member/joints after applying loads.

→ The portal frames sway due to following reasons.

(i) Unsymmetrical loading

(ii) Unsymmetrical outline of portal frame

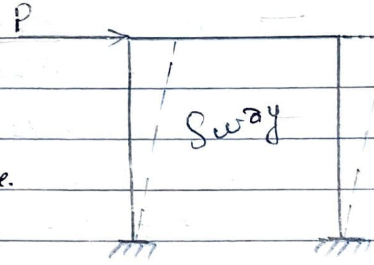
(iii) Different end conditions of column.

(iv) Non uniform section of member

v) Horizontal loading on column of frame.

vi) Settlement of support of frame.

vii) A combination of above.



☆ Method of analysis of frame with sway.

→ If there is no symmetry in geometry or in loading & if there is no support at beam level, there will be lateral movement of column, which is called as sway.

→ procedure for analysis is,

(a) Assume the sway in the frame is prevented by giving external support at beam level. Carryout analysis as in usual manner. This is called non-sway analysis.

→ Considering free body diagrams of columns, find horizontal forces developed at supports.

→ Then, consider horizontal equilibrium of entire system find sway force 's' developed at assumed additional supports.

→ Actually, there is no support at beam level & hence 's' is the sway force moving the beam laterally (fig c)

→ For the given sway force, it is difficult to find the end moments developed. Hence following procedure is adopted.

→ Assume arbitary sway Δ, Then fixed end moments developed in column AB & CD (in fig d) are,

$$M_{F1} = -\frac{6EI_1\Delta}{L_1^2} \quad \& \quad M_{F2} = -\frac{6EI_2\Delta}{L_2^2}$$

→ Then,
$$\frac{M_{F1}}{M_{F2}} = \frac{(I_1/L_1^2)}{(I_2/L_2^2)} = \frac{I_1 L_2^2}{I_2 L_1^2}$$

→ Now, arbitary but proportionate values may be assumed for M_{F1} & M_{F2}. Then, Moment distribution is carried out to get final Moments. Let M_{AB}, M_{BA}, M_{DC} & M_{CD} be the final values (fig e). Then,

→ $$H_A = \frac{M_{AB} + M_{BA}}{L_1} \quad \& \quad H_D = \frac{M_{CD} + M_{DC}}{L_2}$$

→ The sway force "s'" acting in this case is obtained by considering horizontal equilibrium of frame as,

$$S' + H_A + H_D = 0$$

→ To represent case of fig(c) by fig(e), we have to multiply case of fig(e) by constant called sway correction factor (k)

$$k = \frac{S}{S'}$$

→ Final moments = Non-sway moments + k * Sway moments.

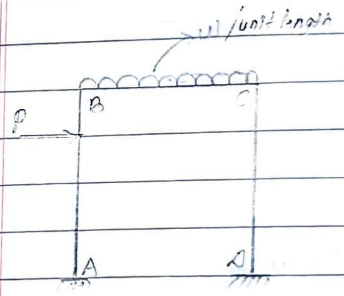


Fig 'a': Typical frame with sway

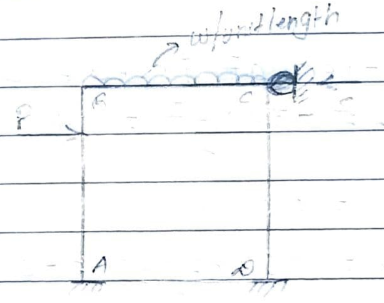


Fig 'b': Swayprevented by external support

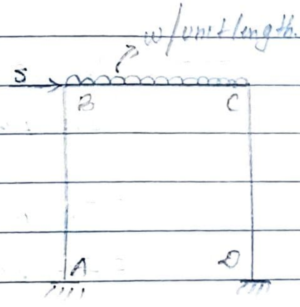


Fig c: Sway force 's'.

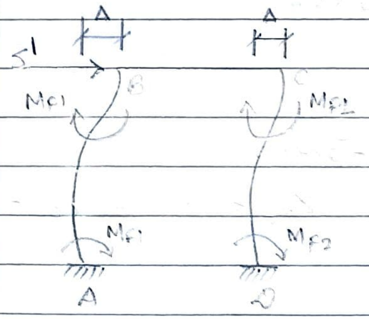
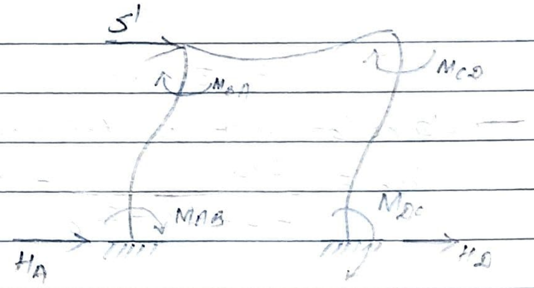
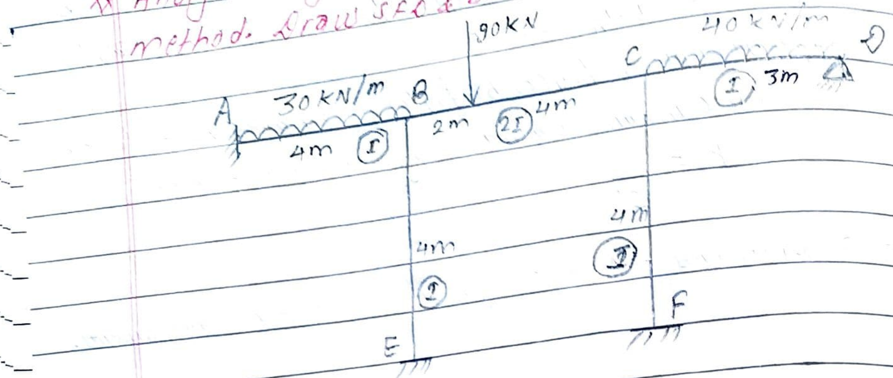


Fig 'd': Fixed end moment due to arbitary force S'.



Fig(e): Reactions & final end moments.

※ Analyse the given frame by using moment distributi method. Draw SFD & BMD.



→ Calculation of fixed end moments.

$(FEM)_{AB} = \frac{-30 \times 4^2}{12} = -40 kNm$

$(FEM)_{BA} = 40 kNm$

$(FEM)_{BC} = \frac{-90 \times 2 \times 4^2}{6^2} = -80 kNm$

$(FEM)_{CB} = \frac{90 \times 2^2 \times 4}{6^2} = 40 kNm$

$(FEM)_{CD} = \frac{-40 \times 3^2}{12} = -30 kNm$

$(FEM)_{DC} = 30 kNm$

→ Calculation of Distribution factor.

$\gamma_{BA} = \frac{I/4}{I/4 + I/4 + 2I/6} = 0.3$, $\gamma_{BE} = 0.3$ & $\gamma_{BC} = 0.4$

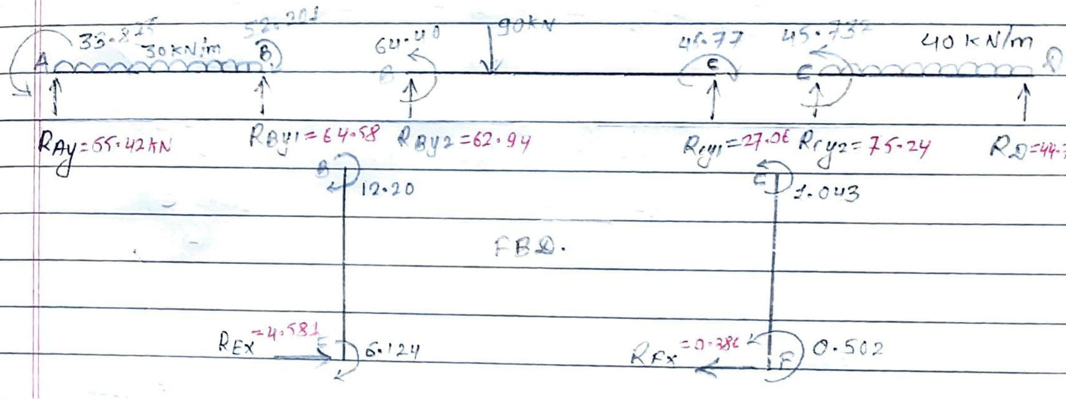
$\gamma_{CB} = \frac{2I/6}{2I/6 + I/4 + (I/3 \times 3/4)} = 0.4$

$\gamma_{CF} = \frac{I/4}{2I/6 + I/4 + 3/4(I/3)} = 0.3$

$\gamma_{CD} = 0.3$ & $\gamma_{DC} = 1$.

No moment is carred from c to D. wrong First balance D.

Joint	A	B			C			D	E	F
Member	AB	BA	BE	BC	CB	CF	CD	DC	EB	FC
D.F	0	0.3	0.3	0.4	0.4	0.3	0.3	1	0	0
FEM	-40	40	0	-80	40	0	-30	30	0	0
Balance		12	12	16	-4	-3	-3	-30		
CO	6			-2	8		-15	-1.5	6	-1.5
Balance		0.6	0.6	0.8	2.8	2.1	2.1	1.5		
CO	0.3			1.4	0.4		0.75	1.05	0.3	1.05
Balance		-0.42	-0.42	-0.56	-0.46	-0.345	-0.345	-1.05		
CO	-0.21			-0.23	-0.28		-0.525	-0.1725	-0.21	-0.1725
Balance		0.069	0.069	0.092	0.322	0.241	0.241	0.1725		
CO	0.0345			0.161	0.046		0.0862	0.1205	0.0345	0.1205
Balance		-0.048	-0.048	-0.064	-0.0528	-0.0396	-0.0396	-0.1205		
~~Final~~ Moments	-33.875	52.201	12.20	-64.40	46.77	-1.043	-45.732	0	6.124	-0.50



→ Calculation of rxn force,

For AB, $+\circlearrowright \Sigma M_B = 0 \Rightarrow 52.201 - 33.875 + R_{Ay} \times 4 - 30 \times 4 \times 2 = 0 \Rightarrow R_{Ay} = 55.42kN$

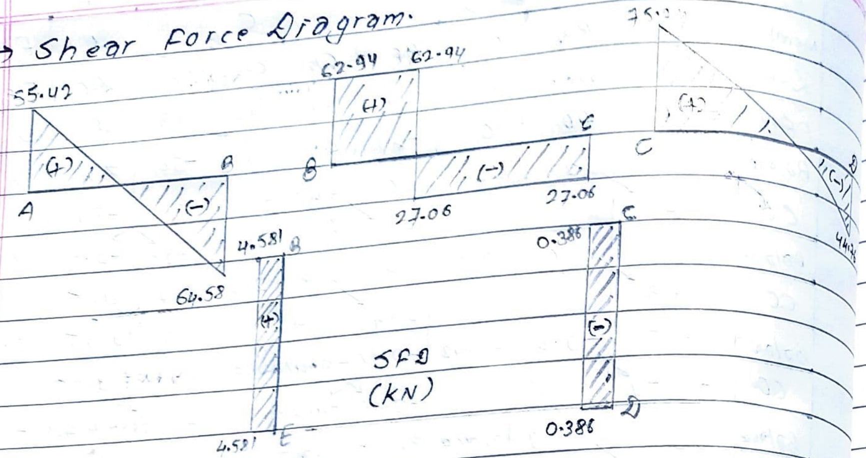
For BE, $+\circlearrowright \Sigma M_B = 0 \Rightarrow 12.20 + 6.124 - R_{Ex} * 4 = 0 \Rightarrow R_{Ex} = 4.581 KN$

For BC, $+\circlearrowright \Sigma M_B = 0 \Rightarrow -64.4 + 90 \times 2 + 46.77 - R_{cy1} \times 6 = 0 \Rightarrow R_{cy1} = 27.06 KN$

For CF, $+\circlearrowright \Sigma M_c = 0 \Rightarrow -1.043 - 0.502 + R_{Fx} \times 4 = 0 \Rightarrow R_{Fx} = 0.386 KN$

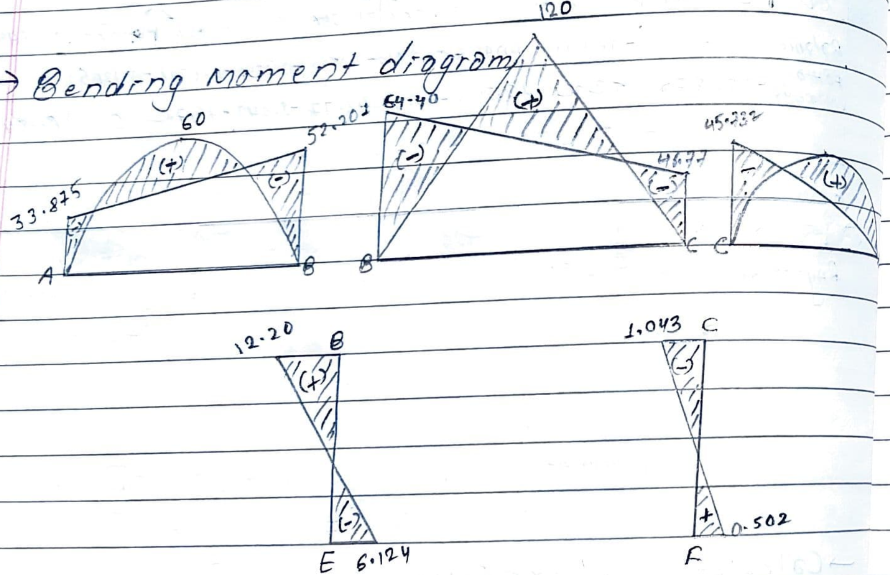
For CD, $+\circlearrowright \Sigma M_c = 0 \Rightarrow -45.732 + 40 \times 3 \times 1.5 - R_D \times 3 = 0 \Rightarrow R_D = 44.76 KN$

→ Shear Force Diagram.



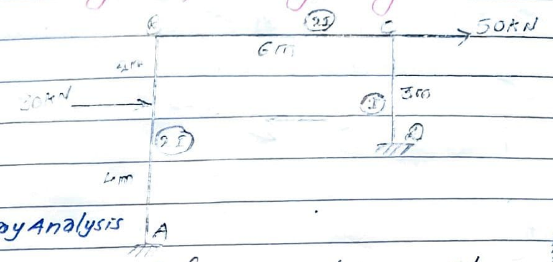
SFD (kN)

→ Bending Moment diagram



BMD (KNm)

* Analyse the given frame by using Moment Distribution Method:-



(a) Non-sway Analysis

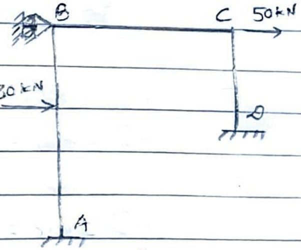


Fig: Sway prevent with external Support.

→ Calculation of fixed end moment,

$$(FEM)_{AB} = -\frac{30 \times 4 \times 2^2}{6^2} = -13.33 \text{ kNm}$$

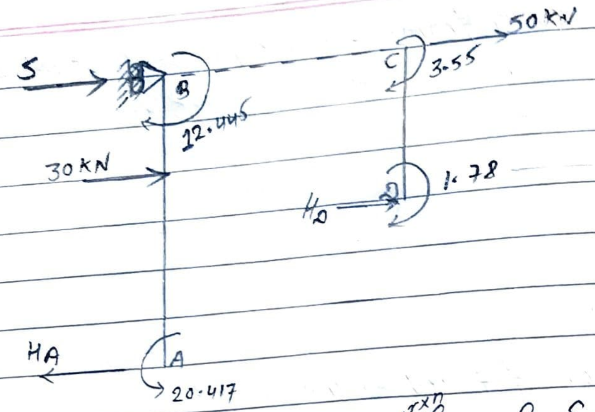
$$(FEM)_{BA} = \frac{30 \times 4^2 \times 2}{6^2} = 26.67 \text{ kNm}$$

→ Calculation of D.f,

$$\delta_{BA} = \frac{(2I/6)}{(2I/6)+(2I/6)} = 0.5, \ \delta_{BC} = 0.5$$

$$\delta_{CB} = 0.5 \ \& \ \delta_{CD} = 0.5$$

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	0	0.5	0.5	0.5	0.5	0
FEM	−13.33	26.67	0	0	0	0
Balance		−13.335	−13.335			
C.O	−6.667			−6.667		
Balance				3.34	3.34	
C.O			1.67			1.67
Balance		−0.84	−0.84			
C.O	−0.42			−0.42		
Balance				0.21	0.21	
C.O			0.11			0.11
Balance		−0.055	−0.055			
Final Moments	−20.417	12.445	−12.445	−3.537	3.55	1.78



→ Calculation of horizontal rxn force & Swayforce,

$+\circlearrowright \Sigma M_B = 0 \Rightarrow 12.445 - 20.417 + H_A \times 6 - 30 \times 2 = 0$

$\Rightarrow H_A = 11.33\ kN$

$+\circlearrowright \Sigma M_C = 0 \Rightarrow 3.55 + 1.78 - H_D \times 3 = 0$

$\Rightarrow H_D = 1.77\ kN$

⇒ Now considering eqb of whole system,

$S + 30 - 11.33 + 1.77 + 50 = 0$

$S = -70.45\ kN$

Actually, there is no horizontal resistance at B. so, swayforce 'S' moves the beam laterally.

→ (b) Sway Analysis:-

Let M_{f1} & M_{f2} be fixed end moments in AB & CD respectively. Then,

$$\frac{M_{f1}}{M_{f2}} = \frac{I_1 L_2^2}{I_2 L_1^2} = \frac{2I \times 3^2}{I \times 6^2} = \frac{1}{2}$$

→ let us assume arbitrary Moments,

$M_{f1} = -10\ kNm$ & $M_{f2} = -20\ kNm$

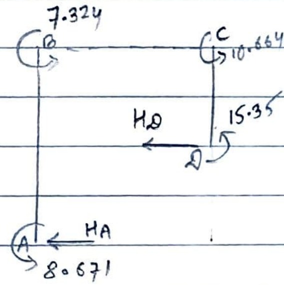
→ Again calculate final moments for M_{f1} & M_{f2}.

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
D.F	–	0.5	0.5	0.5	0.5	–
FEM	−10	−10			−20	−20
Balance		5	5	10	10	
C.O	2.5		5	2.5		5
Balance		−2.5	−2.5	−1.25	−1.25	
CO	−1.25		−0.625	−1.25		−0.625
Balance		0.3125	0.3125	0.625	0.625	
CO	0.15625		0.3125	0.15625		0.3125
Balance		−0.15625	−0.15625	−0.0781	−0.0781	
C.O	−0.0781		−0.039	−0.0781		−0.039
Balance		0.0195	0.0195	0.039	0.039	
Final Moments	−8.671	−7.324	+7.324	10.664	−10.664	−15.35

→ Horizontal Rxn,

$H_A = \frac{8.67 + 7.32}{6} = 2.65\ KN$

$H_B = \frac{10.66 + 15.67}{3} = 8.77\ KN$



Sway force (s') ⇒ $s' - 8.77 - 2.65 = 0$

$\Rightarrow s' = 11.45\ KN\ (\rightarrow)$

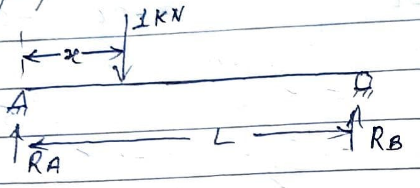
Sway correction factor(k) = $S/s' = 70.46/11.45 = 6.15$

	Member	AB	BA	BC	CB	CD	DC
	Moment (Arbitary)	−8.67	−7.32	7.32	10.67	−10.67	−15.35
K*A.M	Actual sway moment	−53.32	−45.02	45.02	65.55	−65.55	−95.55
	No sway moments	−20.42	12.44	−12.44	−3.54	3.54	1.78
	Total Moments	−73.74	−32.58	32.58	62.01	−62.01	−92.62

ch-5

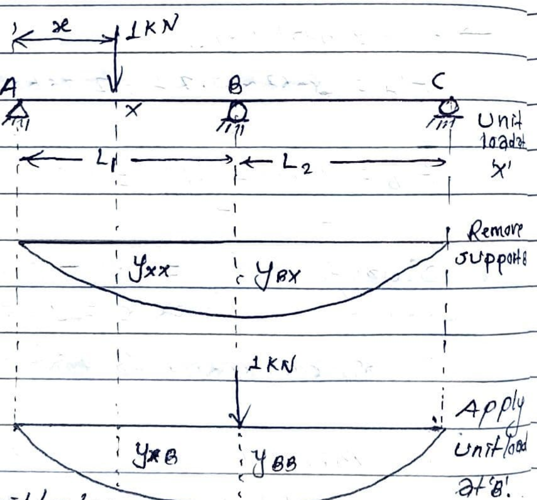
Influence Lines for Indeterminate Beams:-

→ An influence line is a curve of ordinate to which at any section equal to the value of some particular forcing function due to unit load.



✳ MULLER BRESLAU PRINCIPLE:-

It states that, "The influence line for any stressed function of a structure such as shear force, bending moment, any reactive force or Moment is given by the deflected curve of a structure obtained by imposing of unit load in the direction of stress function."



Let,

Y_{Bx} = Deflection at 'B' due to unitload at x

Y_{BB} = Deflection at 'B' due to unitload at B

Y_{xx} = Deflection at 'x' due to Unit load at 'x'.

Y_{xB} = Deflection at 'x' due to Unit load at 'B'.

(a) ILD for reaction (R_B)

From consistent deformation method,

$$R_B * y_{BB} = y_{Bx}$$

$$R_B = \frac{y_{Bx}}{y_{BB}}$$

Again, from Maxwell reciprocal theorem,

$$Y_{Bx} =. Y_{xB}$$

Then,

$$R_B = \frac{y_{xB}}{y_{BB}}$$

The value from above eqn gives the ordinate of influence line for each distance 'x'.

(b) ILD for Moment at D.

From consistent deformation method,

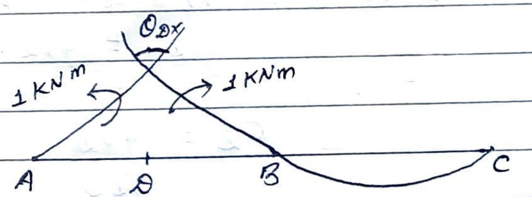
$$M_D \times \theta_{DD} = \theta_{Dx}$$

$$M_D = \frac{\theta_{Dx}}{\theta_{DD}}$$

Unit load at 'x'

Again, from Maxwell reciprocal theorem,

$$\theta_{Dx} = \theta_{xD}.$$

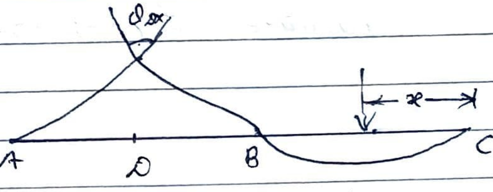


Releasing Unit Moment at D

Then,

$$M_D = \frac{\theta_{xD}}{\theta_{DD}}$$

The value from above eqn gives the ordinate of influence line for each distance 'x'.



Unit load at 'x' distance from c.

(c) ILD for shear force at 'D'.

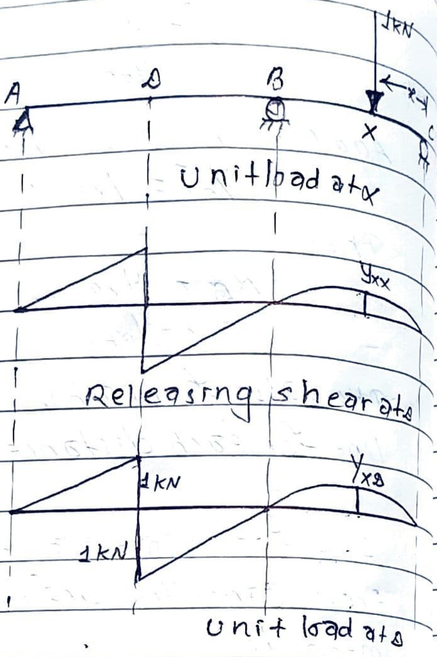
Let,

F_D = Shear force at D

Y_{Dx} = deflection at D due to unit load at 'x'.

Y_{xD} = deflection at 'x' due to unit load at D

Y_{DD} = deflection at 'D' due to unit load at D.



Now,

from consistent deformation method,

$$F_D * y_{DD} = y_{Dx}$$

$$F_D = \frac{y_{Dx}}{y_{DD}}$$

Also, from maxwell reciprocal theorem,

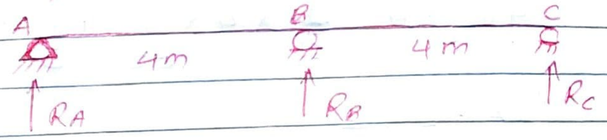
$$y_{Dx} = y_{xD}.$$

Then,

$$F_D = \frac{y_{xD}}{y_{DD}}$$

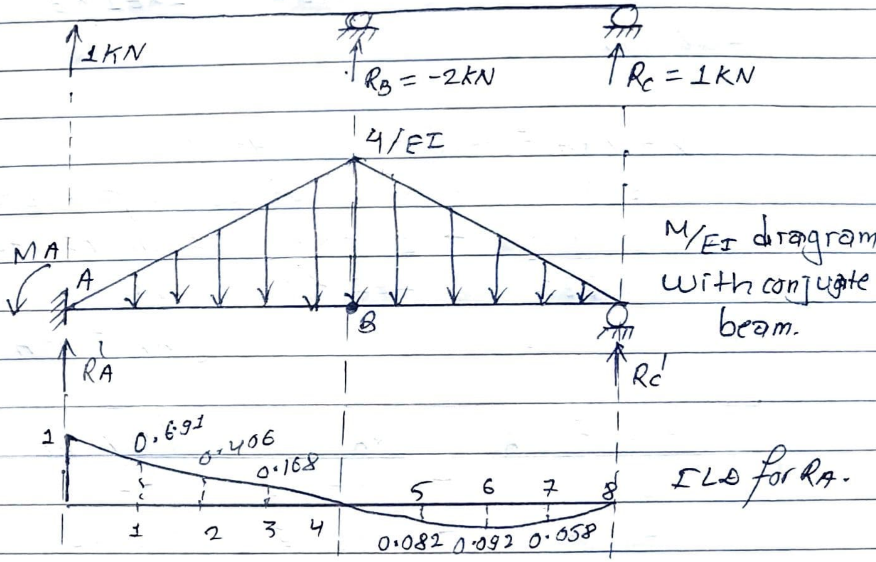
The value from the above equation gives the ordinate of conjugate line for each distance 'x'.

Q. Draw ILD for reaction R_A at an interval of 1m.



→ Step1

Removing support 'A' and applying unit load.



Step 2

Calculation of rxn in conjugate beam,

$+\circlearrowright \Sigma M_B = 0$ [Right part]

$$\Rightarrow \frac{1}{2} \times 4 \times \frac{4}{EI} \times \frac{4}{3} - R_c' \times 4 = 0 \qquad \Rightarrow R_c' = \frac{8}{3EI}$$

$$(+\uparrow)\ \Sigma F_y = 0 \Rightarrow R_A' - \frac{1}{2} \times 8 \times \frac{4}{EI} + \frac{8}{3EI} = 0$$

$$\Rightarrow R_A' = \frac{40}{3EI}$$

$+\circlearrowright \Sigma M_B = 0$ [left]

$$-M_A + \frac{40}{3EI} \times 4 - \frac{1}{2} \times 4 \times \frac{4}{EI} \times \frac{1}{3} \times 4 = 0$$

$$M_A = \frac{128}{3EI}$$

$M_x = Y_{XA}$ = Deflection at X
= Moment at 'x' of conjugate beam
$= \frac{40 \times x}{3EI} - \frac{128}{3EI} - \frac{x \times x \times x}{2 \times EI \times 3}$ From $x=0$ from'A

Again,

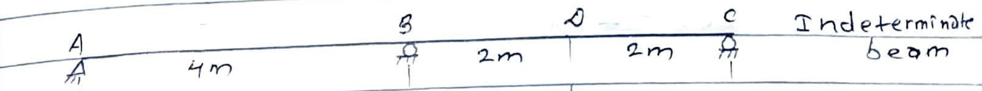
$M_x = Y_{XA}$ = deflection at X
= Moment at 'x' of conjugate beam   From $x = 0$ to 4m from
$= \frac{8 \times x}{3EI} - \frac{x \times x \times x}{2EI \times 3}$ Here,

$$Y_{AA} = M_A = \frac{128}{3EI}$$

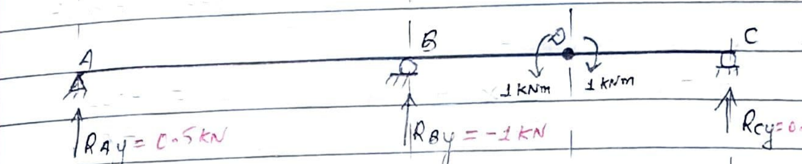
section from A	Y_{XA}	Y_{XA}/Y_{AA}
0	−42.67/EI	−1
1	−29.5/EI	−0.691
2	−17.33/EI	−0.406
3	−7.167/EI	−0.168
4	0	0

section from C	Y_{XA}	Y_{XA}/Y_{AA}
0	0	0
1	2.5/EI	0.058
2	4/EI	0.093
3	3.5/EI	0.082
4	0	0

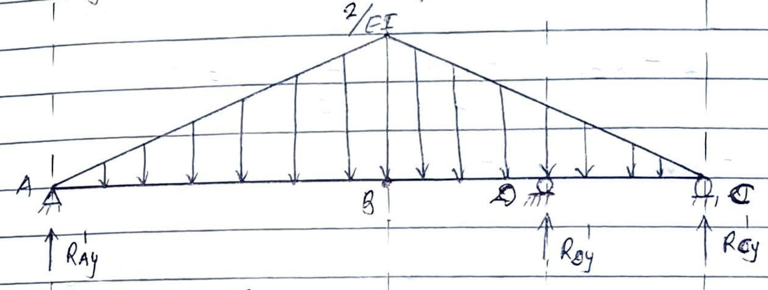
✲ Draw ILD for moment at D at an interval of 1m.



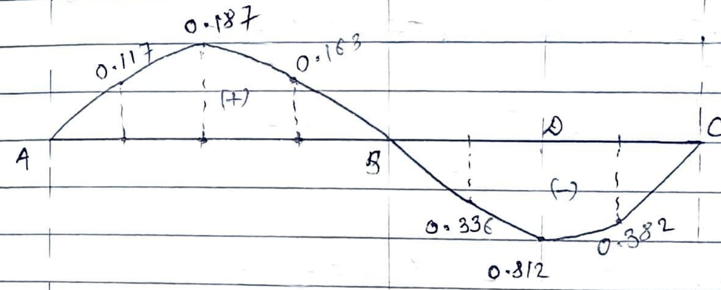
Indeterminate beam



Releasing unit moment at D



Conjugate beam with M/EI loading



ILD for M_D

Releasing unit moment at D,

$+\circlearrowright \Sigma M_D = 0$ [Right]

$-R_{cy} \times 2 + 1 = 0 \Rightarrow R_{cy} = 0.5$KN

$+\circlearrowright \Sigma M_D = 0$ [left]

$-1 + 6R_{Ay} + 2R_{By} = 0 \Rightarrow 6R_{Ay} + 2R_{By} = 1$ --- ①

$(+\uparrow)\Sigma F_y = 0; \Rightarrow R_{Ay} + R_{By} = -0.5$ --- ⑪

On solving ① & ⑪

$R_{Ay} = 0.5\ kN$

$R_{By} = -1\ kN$

Calculation of reactions in conjugate beam:-

$+\circlearrowright \Sigma M_B = 0$ [Left]

$R'_{Ay} \times 4 - \frac{1}{2} \times 4 \times \frac{2}{EI} \times \frac{4}{3} = 0 \Rightarrow R'_{Ay} = \frac{1.33}{EI}$

$+\circlearrowright \Sigma M_B = 0$ [Right]

$-R'_{Dy} \times 2 - R'_{Cy} \times 4 + \frac{1}{2} \times 4 \times \frac{2}{EI} \times \frac{4}{3} = 0$

$4R'_{Cy} + 2R'_{Dy} = \frac{16}{3EI}$ --- ⑪⑪

$(+\uparrow) \Sigma F_y = 0$

$\frac{1.33}{EI} + R'_{Cy} + R'_{Dy} - \frac{1}{2} \times 8 \times \frac{2}{EI} = 0$

$R'_{Cy} + R'_{Dy} = \frac{6.67}{EI}$ --- (iv)

On solving ⑪⑪ & (iv),

$R'_{Cy} = \frac{-4}{EI}$ & $R'_{Dy} = \frac{10.67}{EI}$

Now,

θ_{DD} = slope of D

= SF at D of conjugate beam.

$= \frac{10.67}{EI}$

Y_{xD} = Deflection at 'x'

= Moment at point 'x' of conjugate beam with M/EI diagram.

For moment,

From A to B (i.e for n = 0−4)

$M_x = \frac{1.33x}{EI} - \frac{1}{2} \times x \times \frac{x}{2EI} \times \frac{x}{3} = \frac{1.33x}{EI} - \frac{x^3}{12EI}$

From C to D

$M_x = \frac{-4x}{EI} - \frac{1}{2} \times x \times \frac{x}{2EI} \times \frac{x}{3} = \frac{-4x}{EI} - \frac{x^3}{12EI}$

From D to B,

$M_x = \frac{-4x}{EI} + \frac{10.67}{EI}(x-2) - \frac{1}{2} \times x \times \frac{x}{2EI} \times \frac{x}{3}$

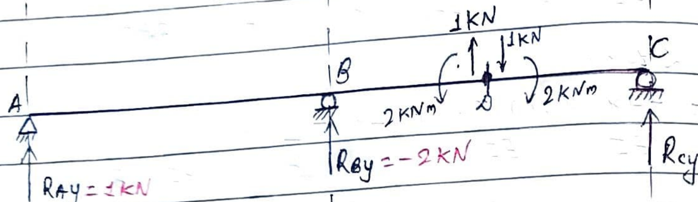
$= \frac{-4x}{EI} + \frac{10.67(x-2)}{EI} - \frac{x^3}{12EI}$

From A to B,

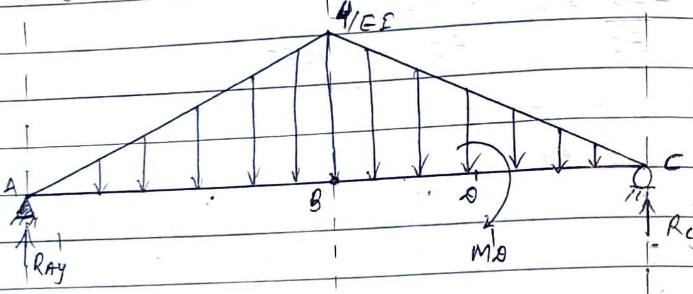
Section	y_{xD}	y_{xD}/θ_{DD}
0	0	0
1	1.25/EI	0.117
2	2/EI	0.187
3	1.74/EI	0.163
4	0	0
5	-3.58/EI	-0.336
6	-8.67/EI	-0.812
7	-4.08/EI	-0.382
8	0	0

(Mistake)

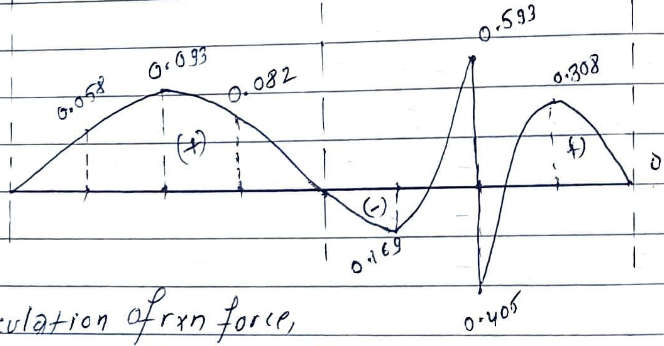
* Draw the ILD for shear force at D of the following indeterminate beam.



Released beam with unit load.



for Conjugate beam.



ILD for f

Calculation of rxn force,

$(+\uparrow)\ \Sigma F_y = 0$ [Right of D]

$R_{cy} = 1\ KN$ & $M_D = 2\ KNm$

$+\circlearrowright\ \Sigma M_A = 0$ [Left of D]

$-R_{By} \times 4 - 1 \times 6 - 2 = 0 \quad \Rightarrow R_{By} = -8/4 = -2KN$

$R_{Ay} = 1\ KN$

Calculation of rxn in conjugate beam

$+\circlearrowright\ \Sigma M_B = 0 \ ; \quad R'_{Ay} \times 4 - \frac{1}{2} \times 4 \times 4/EI \times 4/3 = 0$

$R'_{Ay} = 8/3EI$

$(+\uparrow)\ \Sigma F_y = 0 \ ; \quad \frac{8}{3EI} + R'_{cy} - \frac{1}{2} \times 8 \times \frac{4}{EI} = 0$

$R'_{cy} = \frac{13.33}{EI}$

$+\circlearrowright\ \Sigma M_B = 0$ [Right part only]

$M'_D + \frac{1}{2} \times 4 \times \frac{4}{EI} \times \frac{4}{3} - \frac{13.33}{EI} \times 4 = 0 \quad \Rightarrow M'_D = \frac{42.65}{EI}$

M_x for AB

$M_x = \frac{8x}{3EI} - \frac{1}{2} \times x \times \frac{x}{EI} \times \frac{x}{3} = \frac{8x}{3EI} - \frac{x^3}{6EI}$

For CD

$M_x = \frac{13.33x}{EI} - \frac{1}{2} \times x \times \frac{x}{EI} \times \frac{x}{3} = \frac{13.33x}{EI} - \frac{x^3}{6EI}$

For DB

$M_x = \frac{13.33x}{EI} - \frac{42.65}{EI} - \frac{x^3}{6EI}$

Section	0	1	2	3	4	5	6	7	8
y_{xD}	0	$\frac{2.5}{EI}$	$\frac{4}{EI}$	$\frac{3.5}{EI}$	0	$\frac{-7.25}{EI}$	L = 25.32/EI $\frac{-17.28}{EI}$ -R	$\frac{13.16}{EI}$	0
y_{xD}/y_{DD}	0	0.058	0.093	0.082	0	-0.169	L = 0.593 R = -0.405	0.308	0

Ch-6 Two Hinged Arch

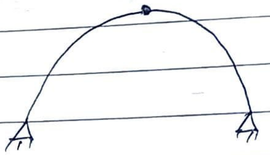
→ As we know, arch is a curved beam having differ cross-sections rectangular, circular, parabolic etc.

Types of arch:-

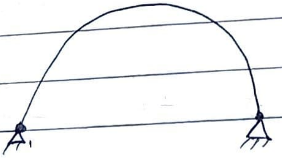
a) On the basis of material Used
- → Steel arch
- → Reinforced Concrete arch

b) On the basis of shape:-
- → parabolic
- → Circular
- → Elliptical

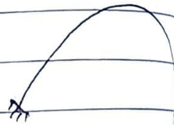
c) On the basis of structural behaviour
- → Three Hinged arch
- → Two hinged arch
- → Fixed/Hingeless arch



Three hinged Arch



Two hinged Arch

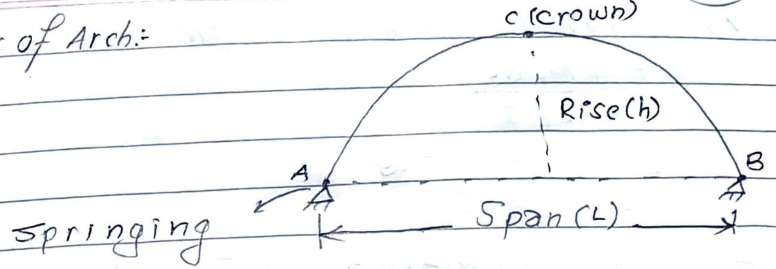


Fixed Arch

→ Arch transfer the load to the abutment by axial compression & bending.

→ Three hinge arch is statically determinate whereas two hinge arch & fixed arch are indeterminate stru

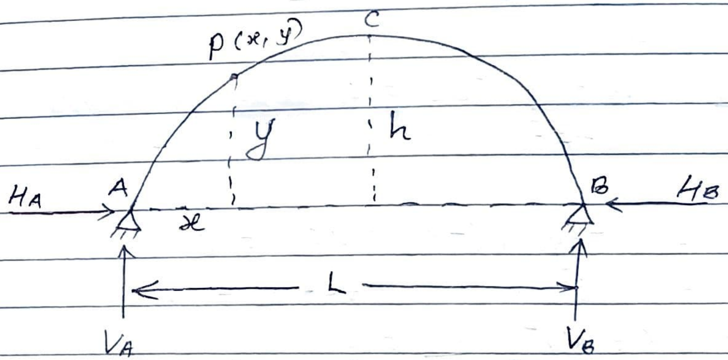
* parts of Arch:-



* Method of analysis of two hinged arch:-

i) Unit load method [Consistent Deformation method]

ii) First theorem of Castiglanio [Strain Energy principle]



Consider two hinged arch of span 'L'. let 'h' be the central rise.

According to first theorem of castiglanio's, we have,

$$M_x = M' - H \cdot y$$

Here,

M_x = Moment at 'x' distance

M' = Beam moment

H = Horizontal thrust

y = vertical distance

From strain energy principle,

$$U = \int \frac{M_x^2 \, ds}{2EI}$$

$$U = \int \frac{(M' - Hy)^2 \, ds}{2EI}$$

Also, From castiglanio's first theorem,

$$\frac{du}{dH} = \Delta$$

Since, supports are unyielding. so, $\Delta = 0$

Then,

$$\frac{du}{dH} = 0$$

$$\Rightarrow \frac{d}{dH} \int \frac{(M' - Hy)^2 ds}{2EI} = 0$$

$$\Rightarrow \int 2(M' - Hy) \times (-y) \times \frac{ds}{2EI} = 0$$

$$\Rightarrow (-M'y + Hy^2) \frac{ds}{EI} = 0$$

$$\Rightarrow H \int y^2 \frac{ds}{EI} = \int M'y \left(\frac{ds}{EI}\right)$$

$$\Rightarrow H = \frac{\int M'y \times \frac{ds}{EI}}{\int y^2 \times \frac{ds}{EI}} \quad \text{---} ①$$

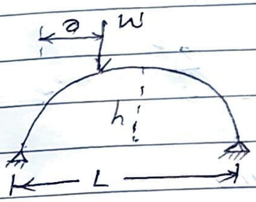
From both methods, we can obtained same result for 'H' as given by eqn ①.

* Some important formulas..

a)

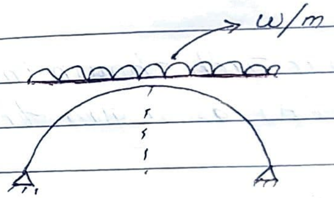
$$H = \frac{25}{128} \times \frac{WL}{h}$$

b)



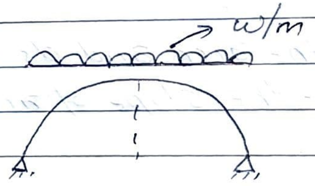
$$H = \frac{5}{8}\left(\frac{W}{hL^3}\right) a(L-a)(L^2 + La - a^2)$$

c)



$$H = \frac{WL^2}{8h}$$

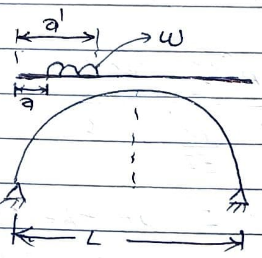
d)



$$H = \frac{WL^2}{16h}$$

e) $\int y^2 dx = \frac{8}{15} h^2 L$

f)



$\alpha_1 = a/L$ & $\alpha_2 = a'/L$

Then,

$$H = \frac{5WL^2}{8h} \int_{\alpha_1}^{\alpha_2} (\alpha - 2\alpha^3 + \alpha^4) \, d\alpha$$

* Analysis of two hinged parabolic Arch

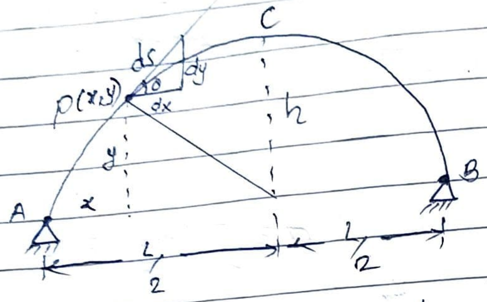


Fig: A parabolic arch.

Consider a typical parabolic arch shown in figure. Taking left springing point (A) as origin, the eqn of parabola is given by,

$$y = \frac{4hx(L-x)}{L^2}$$

Let p(x,y) be a point on the arch and ds be the elemental length. Let, 'θ' be the slope of arch to the horizontal at 'p'. Then,

$$\frac{dx}{ds} = \cos\theta$$

$$ds = dx \times \sec\theta$$

The horizontal thrust 'H' is given by,

$$H = \frac{\int M'y\left(\frac{ds}{EI}\right)}{\int y^2\left(\frac{ds}{EI}\right)} = \frac{\int M'y\left(\frac{dx}{EI}\right)\sec\theta}{\int y^2\left(\frac{dx}{EI}\right)\sec\theta}$$

For parabolic arch, the value of 'I' at the springing is greater than at the crown & it is assumed tha

cross section varies in such a sway that,

$$I = I_o \sec\theta$$

where, I_o = Moment of inertia at section of crown.

Now,

$$H = \frac{\int My\left(\frac{dx}{EI_o\sec\theta}\right)\sec\theta}{\int y^2 \times \frac{dx}{EI_o\sec\theta} \times \sec\theta}$$

$$H = \frac{\int My\left(\frac{dx}{EI_o}\right)}{\int y^2\left(\frac{dx}{EI_o}\right)}$$

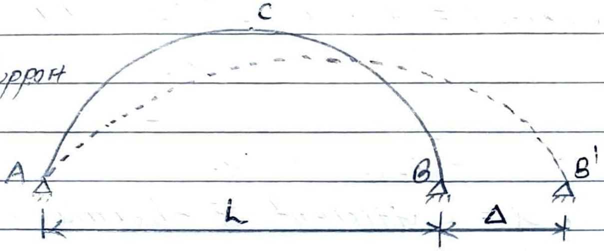
If EI_o is constant, $H = \frac{\int My\,dx}{\int y^2\,dx}$

* Effect of yielding of Supports:-

Let,

Δ = yielding of Support

$$H = \frac{\int M'y\left(\frac{ds}{EI}\right) - \Delta}{\int y^2 \frac{ds}{EI}}$$



If the support is elastic & it yield by 'k' due to unit horizontal force at support level

$$\Delta = KH$$

Then,

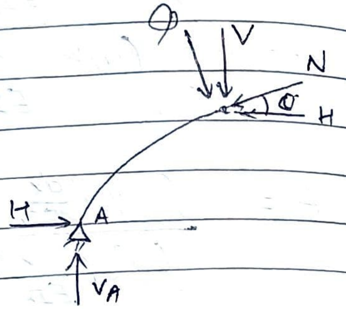
$$H = \frac{\int M'y\frac{ds}{EI}}{\int y^2\left(\frac{ds}{EI}\right) + K}$$

* Effect of shortening of RIB:-

The cross-section of arch is subjected to normal thrust also. The arch being made up of elastic material shortening of rib takes place. This shortening reduces the horizontal thrust developed.

$$H = \frac{\int M'y\left(\frac{ds}{EI}\right)}{\int y^2\left(\frac{ds}{EI}\right) + \frac{L}{EA_m}}$$

where, $A_m = \frac{A}{\cos\theta}$



If both yielding of support & rib shortening are to be considered.

$$H = \frac{\int M'y\left(\frac{ds}{EI}\right)}{\int y^2\left(\frac{ds}{EI}\right) + \frac{L}{EA_m} + k}$$

where, k = yielding of support per unit horizontal force (mm/kN)

* Effect of tempr changes.

Let, α = coefficient of thermal expansion

$\Delta = L\alpha t$

Then, due to change in tempr,

$$H = \frac{L\alpha t}{\int y^2 \frac{ds}{EI}}$$

If tempr changes, loading & settlement of supports, the total horizontal thrust is,

$$H = \frac{\int M'y\left(\frac{ds}{EI}\right) + L\alpha t - \Delta}{\int y^2\left(\frac{ds}{EI}\right)}$$

If effect of rib shortening is also to be accounted, then,

$$H = \frac{\int M'y\left(\frac{ds}{EI}\right) + L\alpha t - \overset{\text{settlement}}{\Delta}}{\int y^2\left(\frac{ds}{EI}\right) + \frac{L}{EA_m}}$$

If it is an elastic support with yielding of k per unit horizontal thrust, then,

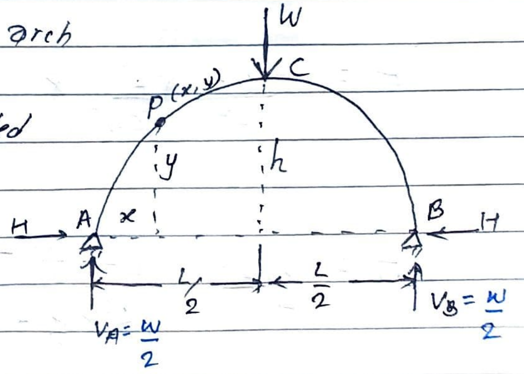
$$H = \frac{\int M'y\left(\frac{ds}{EI}\right) + \overset{\text{tempr}}{L\alpha t}}{\int y^2\left(\frac{ds}{EI}\right) + \underset{\text{rib shortening}}{\frac{L}{EA_m}} + \underset{\text{yielding}}{k}}$$

Q. A two hinged parabolic arch of span L and rise 'h' carries a concentrated load (W) at the crown. Determine the expression for horizontal thrust developed at springing.

→ Consider a parabolic arch having central rise 'h' carries a concentrated load 'W' at the crown.

The eqn of parabola is,

$$y = \frac{4hx(L-x)}{L^2}$$



The general expression for horizontal thrust is,

$$H = \frac{\int M' y \frac{dx}{EI}}{\int y^2 \frac{dx}{EI}}$$

Since, EI is constant throughout section of arch, so,

$$H = \frac{\int M' y dx}{\int y^2 dx}$$

Here, $M' = \frac{w}{2} \times x$ [Beam moment]

$$H = \frac{2\int_0^{L/2} \frac{w}{2} \times x \times \frac{4hx}{L^2}(L-x)\,dx}{\int_0^L \left[\frac{4hx}{L^2}(L-x)\right]^2 dx}$$

$$H = \frac{\frac{4hw}{L^2}\int_0^{L/2}(Lx^2 - x^3)\,dx}{\frac{16h^2}{L^4}\int_0^L x^2(L^2 - 2Lx + x^2)\,dx}$$

$$H = \frac{4hw}{L^2} \times \frac{L^4}{16h^2} \times \frac{\int_0^{L/2}(Lx^2 - x^3)\,dx}{\int_0^L [x^2L^2 - 2Lx^3 + x^4]\,dx}$$

$$H = \frac{wL^2}{4h} \frac{\left[L \times \frac{x^3}{3} - \frac{x^4}{4}\right]_0^{L/2}}{\left[L^2 \times \frac{x^3}{3} - 2L \times \frac{x^4}{4} + \frac{x^5}{5}\right]_0^L}$$

$$H = \frac{wL^2}{4h} \times \frac{\left[\frac{L}{3} \times \frac{L^3}{8} - \frac{L^4}{64}\right]}{\left[\frac{L^2}{3} * \cancel{L^3} - \frac{2L}{4} \times L^4 + \frac{L^5}{5}\right]}$$

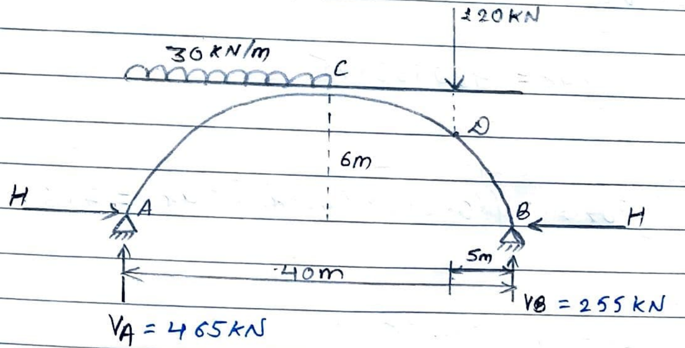
$$H = \frac{wL^2}{4h} \times \frac{5L^4}{192} \times \frac{30}{L^5}$$

$$H = \frac{25}{128} \times \left(\frac{wL}{h}\right)$$

which is required expression for horizontal thrust produced in an arch having concentrated load 'w' at crown 'c'.

Q A two hinged parabolic arch is loaded as shown in figure. Determine the horizontal thrust, maximum positive & negative moments, shear force & normal thrust at 10m from right support.
Assume $I = I_0 \sec\theta$, where I_0 is moment of inertia at the crown & θ is the slope at the section under consideration.

→



$+\circlearrowleft \Sigma M_A = 0$; $30 \times 20 \times 10 + 120 \times 35 - V_B \times 40 = 0$; $V_B = 255$ KN

$(+\uparrow) \Sigma F_y = 0$; $V_A - 30 \times 20 - 120 + 255 = 0$; $V_A = 465$ KN

The horizontal thrust 'H' is,

$$H = \frac{\int M' y \, dx}{\int y^2 dx}$$

Section	AC	CD	DB
Origin	A	B	B
Limit	0-20	5-20	0-5
M'	$465x - 15x^2$	$255x - 120x(x-5)$	$255x$

$$\int M' y \, dx = \int_0^{20} (465x - 15x^2) y \, dx + \int_5^{20} \{255x - 120(x-5)\} y \, dx$$

$$+ \int_0^{5} 255x \times y \, dx \quad \text{---} ①$$

But,

$$y = \frac{4hx}{l^2}(l-x) = \frac{4 \times 6 \times x}{40^2}(40-x)$$

$$y = 0.015x(40-x)$$

put value of 'y' in eqn ①,

$$\int M' y \, dx = 430593.75$$

Also,

$$\int y^2 dx = \int_0^{40} \{0.015x(40-x)\}^2 dx = 768.$$

Then,

$$H = \frac{430593.75}{768}$$

$$H = 560.669 \text{ kN}$$

For Maxm positive & negative moment,

(a) For portion AC,

$$M_x = 465 * x - 15x^2 - 560.669 \times y$$

$$= 465x - 15x^2 - 560.669 \times 0.015x(40-x)$$

$$= 128.59x - 6.589x^2$$

For Mmax,

$$\frac{dM_x}{dx} = 0$$

$$\Rightarrow 128.59 - 13.18x = 0$$

$$\Rightarrow x = 9.75 \text{ m}$$

Then,

$$M_{max} = 627.388 \text{ kNm}$$

(b) For portion CD,

Considering 'x' from right support,

$$M_x = 255x - 120(x-5) - 560.669 \times y$$

$$= 600 - 201.4x + 8.41x^2$$

Then,

$$\frac{dM_x}{dx} = 0 \quad \Rightarrow x = 11.974 \text{ m}$$

$$M_x = -605.766 \text{ kNm}$$

c) For portion BD,

$$M_x = 255x - 560.669y$$

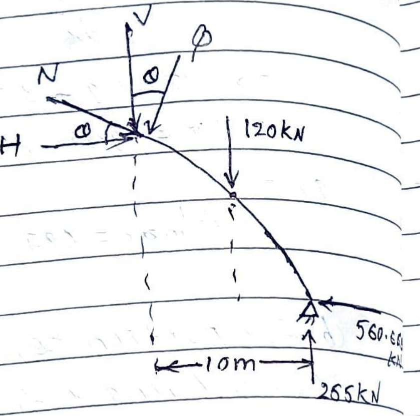
$$= -81.4x + 8.41x^2$$

$$\frac{dM_x}{dx} = 0; \quad x = 4.889 \text{ m}.$$

Then, $M = -196.967 \text{ kNm}$

Therefore, maxm positive moment is 627.388kN at 9.757m from left support and maximum negative moment = −605.766 kNm at 11.974m from right support.

For shear force, N & φ at a distance of 10m from right supp



Here,

$y = 0.015x(40-x)$

$\tan\theta = \frac{dy}{dx}$

$\tan\theta = 0.015(40-2x)$

$\theta = \tan^{-1}\left[0.015(40-2\times10)\right]$

$\theta = 16.7°$

Then,

$V = 255 - 120 = 135$ kN(↓)

$N = V\sin\theta + H\cos\theta$

$= 135\sin16.7° + 560.669\cos(16.7°)$

$= 575.815$ kN.

$\phi = V\cos\theta - H\sin\theta$

$= 135\cos16.7° - 560.669\sin(16.7°)$

$= -31.808$ kN.

Q. A two hinged parabolic arch of span 50m & rise 5m is subjected to a central concentrated load of 60kN. It has an elastic support which yields by 0.0001 mm/kN. Take, $E = 200\,kN/mm^2$, $I = 5\times10^9\,mm^4$,

Average Area, $A_m = 10000\,mm^2$

$\alpha = 10\times10^{-6}/°C$

& assuming secant variation, calculate the horizontal thrust developed when tempr rises by 20°c.

(i) neglecting rib shortening

(ii) Considering rib shortening

→ Here,

$y = \frac{4\times5\times x}{50^2}(50-x)$

$= 0.008x(50-x)$

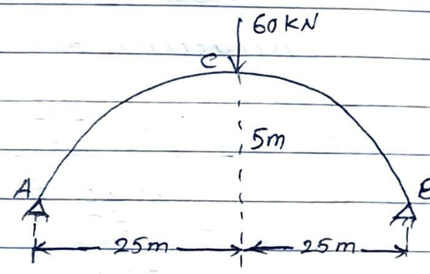


fig: Two hinged parab arch.

$\int_0^{50} y^2\frac{dx}{EI} = \frac{666.67}{EI}$

$M' = 30x,$

Then,

$\int M'y\frac{dx}{EI} = 2\int_0^{25} 30x\times0.008x(50-x)\frac{dx}{EI} = \frac{78125}{EI}$

But,

$EI = 200\times10^6\times5\times10^9\times10^{-12} = 10^6\,kNm^2$

$EA_m = 200\times10^6\times10\,000\times10^{-6} = 2\times10^6$

Now,

$\int_0^{50} y^2\frac{dx}{EI} = \frac{666.67}{10^6} = 6.66\times10^{-4}$ ~~m~~

$\int M'y\frac{dx}{EI} = \frac{78125}{10^6} = 0.0781'$

$L\alpha t = 50 \times 12 \times 10^{-6} \times 20 = 0.012$ [+ve due to rise in tempr]

$k = 0.0001.$

(i) Neglecting rib shortening.

$$H = \frac{\int M'y\left(\frac{dx}{EI}\right) + L\alpha t}{\int y^2 \frac{dx}{EI} + k}$$

$$H = \frac{0.0781 + 0.012}{6.66 \times 10^{-4} + 0.0001} = 117.554\,KN.$$

ii) Considering Rib shortening,

$$H = \frac{\int M'y\left(\frac{dx}{EI}\right) + L\alpha t}{\int y^2 \frac{dx}{EI} + \frac{L}{EAm} + k}$$

$$H = \frac{0.0781 + 0.012}{6.66 \times 10^{-4} + \frac{50}{2 \times 10^6} + 0.0001} = 113.78\,kN$$

iii) without considering all effect,

$$H = \frac{\int M'y \frac{dx}{EI}}{\int y^2 dx/EI} = \frac{0.0781}{6.66 \times 10^{-4}} = 117.267\,kN$$

iii) Without considering yielding & Rib shortening,

$$H = \frac{\int M'y\left(\frac{dx}{EI}\right) + L\alpha t}{\int y^2 \frac{dx}{EI}}$$

$$H = \frac{0.0781 + 0.012}{6.66 \times 10^{-4}} = 135.285\,KN$$

(iv) Considering yielding of support only,

$$H = \frac{\int M'y\left(\frac{dx}{EI}\right)}{\int y^2 \frac{dx}{EI} + K} = \frac{0.0781}{6.66 \times 10^{-4} + 0.0001} = 101.958\,KN$$

v) Considering rib shortening only,

$$H = \frac{\int M'y\left(\frac{dx}{EI}\right)}{\int y^2 \frac{dx}{EI} + \frac{L}{EAm}}$$

$$H = \frac{0.0781}{6.66 \times 10^{-4} + \frac{50}{2 \times 10^6}} = 113.024\,KN$$

ch-7 Introduction to plastic Analysis

→ In elastic theory, the designer has a concept that, the structure would fail if the design load applied is equal to factor of safety times working load. But this is not the correct concept.

→ The elastic theory under-estimates the load carrying capacity of structure.

→ For indeterminate structure, only the term factor of safety is not giving the correct idea about load carrying capacity of structures. Hence, a new theory called plastic theory has been developed.

→ "plastic theory" gives the correct idea about the load carrying capacity of the structures.
It is based on the concept that a structure will carry load till the plastic hinges are formed at the sufficient points to cause collapse of the structure.

→ To make the theory simple, strain hardening is neglected.

✻ plastic Hinge:-

It is the section at which all the fibers are yielded & hence for further load rotation takes place at the section without resisting any additional moment.

* Plastic Moment Capacity:-

plastic Moment capacity of a section may be defined as the moment which makes all the fibers that section to yield & thereby form a plastic hinge.

* Shape factor:-

The term shape factor may be defined as the ratio of plastic Moment capacity to yield moment capacity.

$$S = \frac{M_p}{M_y}$$

$M_p = f_y \times Z_p$

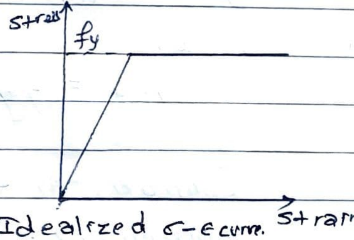
where, M_p = plastic Moment
f_y = yield stress
Z_p = section modulus / plastic modulus.

$M_y = f_y \times Z$

where, M_y = yield Moment
f_y = yield stress
Z = section modulus = I/y_{max}

✻ Assumptions of plastic theory:-

a) stress strain relationship is idealized to two straight lines. This means, strain hardening effect is neglected.



Idealized σ-ε curve.

b) Plane section before bending, remains plane even after bending.

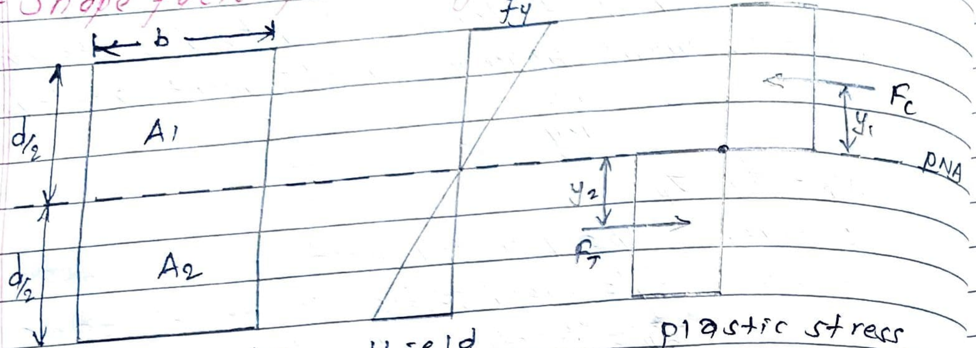
c) The relationship betn compressive stress & strain is same as between tensile stress and strain.

d) Whenever a fully plastic moment is attained at any cross-section, a plastic hinge is formed at that section.

e) Effect of axial load & shear on fully plastic moment capacity of section is neglected.

f) Eq^b eq^n can be applied due to small deflection.

* Shape factor for rectangular Section:-



Rectangular section.   yield stress   plastic stress diagram

Consider the rectangular section of width 'b' & depth 'd' as shown in figure. The stress diagram corresponding to yield moment & plastic moment is also shown in figure.

Now,

$$M_y = f_y \times Z = f_y \times \frac{bd^2}{6}$$

Consider, A_1 & A_2 be area of section of compression & tension zone.

Considering horizontal equilibrium condition,

$$F_c = F_t$$

$$\Rightarrow f_y \times A_1 = f_y \times A_2$$

$$\Rightarrow A_1 = A_2 = A/2$$

Thus, plastic neutral axis divides the area into two equal halves.

Now,

$$M_p = F_c \times y_1 + F_T \times y_2$$

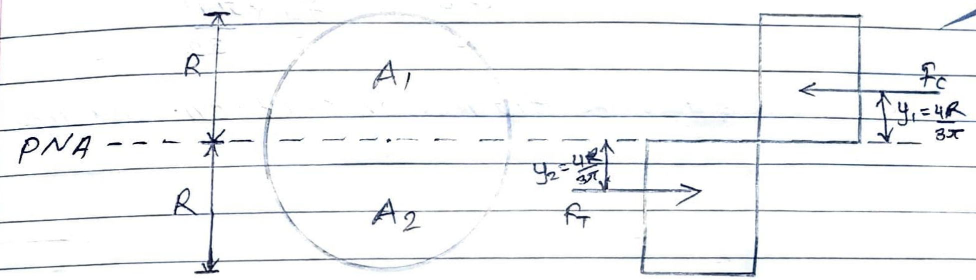
$$= f_y A_1 y_1 + f_y A_2 y_2$$

$$M_p = f_y \times \frac{A}{2} (y_1 + y_2)$$

$$M_p = f_y \times \frac{bd}{2} \times \left(\frac{d}{4} + \frac{d}{4}\right) = f_y \times \frac{bd^2}{4}$$

$$\text{Shape factor (S)} = \frac{M_p}{M_y} = 1.5$$

* Shape factor for circular section:-



Consider a circular section of radius R. As we know, plastic N.A divides the area into two equal halves.

$$A_1 = A_2 = A/2$$

$$Z = \frac{I}{y_{max}} = \frac{\pi d^4}{64 \times d/2} = \frac{\pi d^3}{32}$$

Then,

$$M_y = f_y \times \frac{\pi d^3}{32}$$

The C.G of compressive area & tensile area lies at a distance of 4R/3π from PNA.

Now,

$$M_p = F_c \times \frac{4R}{3\pi} + F_T \times \frac{4R}{3\pi}$$

$$= f_y \times A_1 \times 4R/3\pi + f_y \times A_2 \times 4R/3\pi$$

$$M_p = f_y \times \frac{A}{\cancel{2}} \times \frac{4R}{3\pi} \times \cancel{2}$$

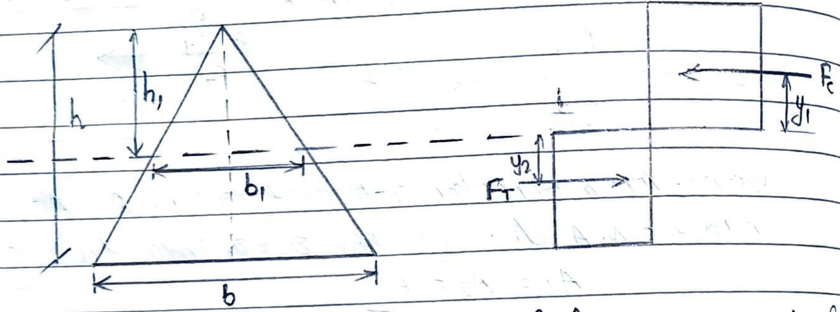
$$M_p = f_y \times \frac{\pi d^2}{4} \times \frac{4}{3\pi} \times \frac{d}{2}$$

$$M_p = f_y \times \frac{\cancel{\pi} d^3}{6\cancel{\pi}} = f_y \times \frac{d^3}{6}$$

Then,

$$\text{Shape factor (S)} = \frac{M_p}{M_y} = \frac{f_y \times d^3/6}{f_y \times \frac{\pi d^3}{32}} = 1.698$$

* Shape factor for traingular section.



Consider a traingular section of base width 'b' & depth 'h' as shown in figure.

Then,

$$Z = \frac{I}{y_{max}} = \frac{bh^3}{36 \times (2h/3)} = \frac{bh^2}{24}$$

$$M_y = f_y \times \frac{bh^2}{24}$$

Let, the plastic neutral axis be at a depth h_1 from apex. At this level,

$$b_1 = b \times \left(\frac{h_1}{h}\right)$$

Then, $\frac{1}{2} \times b_1 \times h_1 = \frac{1}{2}\left(\frac{1}{2} \times b \times h\right)$

$$b_1 \times h_1 = \frac{bh}{2}$$

$$\cancel{b} \times h_1^2 = \frac{\cancel{b}h^2}{2} \Rightarrow h_1 = \frac{h}{\sqrt{2}}$$

Then, $b_1 = b/\sqrt{2}$

Distance of centroid of compression area from PNA is,

$$y_1 = \frac{h_1}{3} = \frac{h}{3\sqrt{2}}$$

Distance of centroid of tensile area from plastic NA,

$$y_2 = \left(\frac{h-h_1}{3}\right) \times \left(\frac{b_1+2b}{b_1+b}\right)$$

$$y_2 = \left(\frac{h-h/\sqrt{2}}{3}\right) \times \left(\frac{b/\sqrt{2}+2b}{b/\sqrt{2}+b}\right)$$

$$y_2 = 0.1548h$$

Now,

$$M_p = f_y \times A_1 \times y_1 + f_y \times A_2 \times y_2$$

$$= f_y \times \frac{A}{2}\left[\frac{h}{3\sqrt{2}} + 0.1548h\right]$$

$$= 0.09763\, bh^2 f_y$$

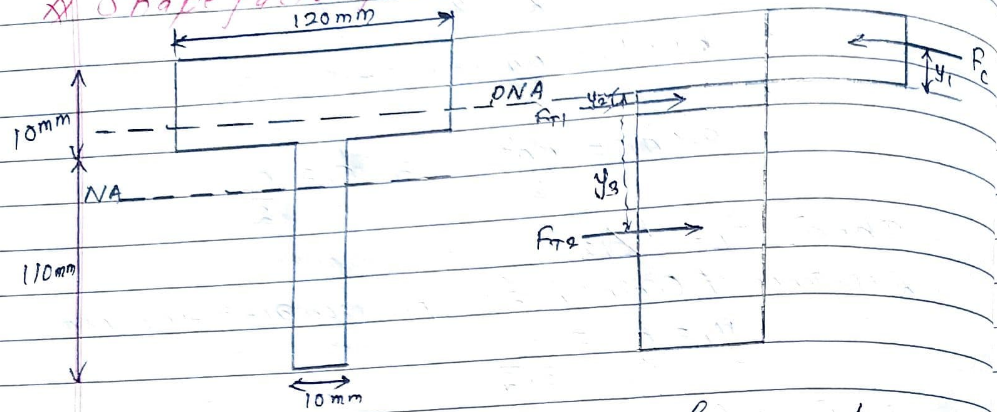
Then,

$$\text{Shape factor (S)} = \frac{M_p}{M_y}$$

$$= \frac{0.09763\, bh^2 \times f_y}{f_y \times bh^2/24}$$

$$= 2.343$$

* Shape factor for 'T' section.



To find 'Z' we reqd I & ymax. So, first we have to find $\bar{y}$ for neutral axis.

$$\bar{y}\text{ (from top)} = \frac{(120\times10)\times5 + 110\times10\times(10+55)}{120\times10+110\times10} = 33.695\text{mm}$$

$$I_{xx} = \frac{120\times10^3}{12} + 120\times10\times(5-33.695)^2 + \frac{10\times110^3}{12} + 110\times10\times(65-33.$$

$$= 3.18\times10^6\text{mm}^4$$

ymax = 120 − 33.70 = 86.3 mm.

Then,

$$M_y = f_y\times Z = f_y\times\frac{3.18\times10^6}{86.3} = 36848.203f_y$$

Consider plastic N.A lies on flange at a distance y_p from top. We know,

$$y_p\times120 = \frac{A}{2} = \frac{1}{2}[120\times10+110\times10]$$

$$y_p = 9.58\text{mm} < 10\text{ mm ok}$$

Now,

$$M_p = F_c\times y_1 + F_{T1}\times y_2 + F_{T2}\times y_3$$

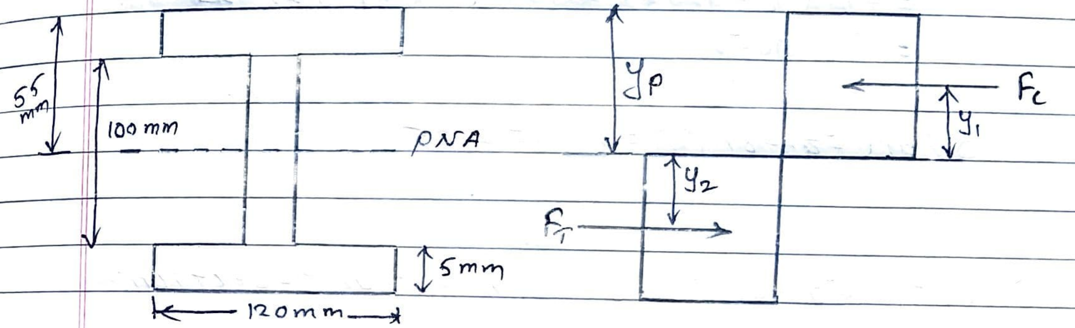
$$= f_y\times(9.58\times120)\times\frac{9.58}{2} + f_y\times(0.42\times120)\times\frac{0.42}{2}$$

$$+ f_y\times(110\times10)\times\left(0.42+\frac{110}{2}\right)$$

$$= 66479.16f_y$$

$$S = \frac{M_p}{M_y} = 1.804$$

* Shape factor for Symmetrical 'I'-section:-



For calculation of My, We have to calculate 'Z'!

$$I_{xx} = 2\left[\frac{120\times5^3}{12} + 120\times5\times(2.5-55)^2\right] + \frac{20\times100^3}{12} + 100\times20\times(55-55)^2$$

$$= 4.976\times10^6\text{ mm}^4$$

ymax = 55 mm.

Then,

$$M_y = f_y\times Z$$

$$= f_y\times\frac{4.976\times10^6}{55}$$

$$= 90484.84f_y.$$

As we know that, P.N.A divides area of section into two equal halves.

Let, p.NA lies in web at a distance 'yp' from top then.

$$120\times5+(y_p-5)\times20=\frac{1}{2}[2\times120\times5+100\times20]$$

$$y_p=55\,mm.$$

for y_1 & y_2,

$$y_1=y_2=\frac{(120\times5)\times(50+2.5)+(50\times20)\times25}{120\times5+50\times20}$$

$$=35.3125\,mm$$

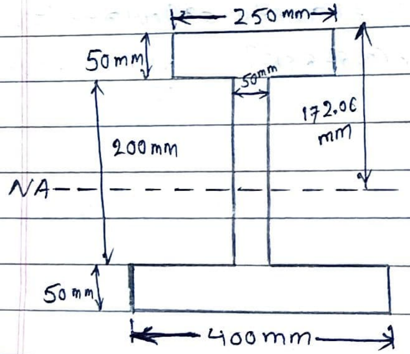
$$M_p=f_y\times A_1\times y_1+f_y\times A_2\times y_2$$

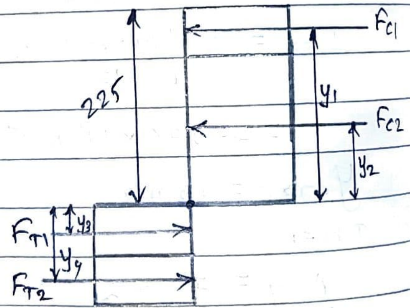
$$=f_y\times[\{120\times5+50\times20\}\times35.3125\times2]$$

$$=113000f_y$$

$$\text{Shape factor }(s)=\frac{M_p}{M_y}=1.25$$

※ Shape factor for unsymmetrical I-section:-





plastic stress diagram.

For centroid from top,

$$\bar{y}=\frac{250\times50\times25+50\times200\times(50+100)+400\times50\times(250+25)}{250\times50+50\times200+400\times50}$$

$$=172.06\,mm$$

$$I_{xx}=\frac{250\times50^3}{12}+(250\times50)\times(25-172.06)^2+\frac{200^3\times50}{12}+(50\times200)\times(150-172.06)^2+\frac{400\times50^3}{12}+400\times50\times(172.06-275)^2$$

$$=5.272\times10^8\,mm^4$$

$$y_{max}=172.06\,mm$$

Then,

$$M_y=f_y\times Z=3064259.7f_y$$

For plastic Moment (M_p):-

Let, PNA be at a distance y_p from top fibre. Assuming it to fall in web,

$$250\times50+50\times(y_p-50)=\frac{(250+200+400)\times50}{2}$$

$$y_p=225\,mm.$$

Our assumption is correct.

Now,

$$M_p=F_{C1}\times y_1+F_{C2}\times y_2+F_{T1}\times y_3+F_{T2}\times y_4$$

$$=f_y\times(A_1y_1+A_2y_2+A_3y_3+A_4y_4)$$

$$=f_y\times[250\times50\times(225-25)+(175\times50)\times87.5+25\times50\times12.5+50\times400\times50]$$

$$=4281250f_y$$

$$\text{Shape factor }(S)=\frac{M_p}{M_y}$$

$$=1.397$$

* Collapse load:-

A structure is said to have collapsed if the entire structure or part of the structure starts undergoing unlimited deformation. This happens when the number of independent static equilibrium equations available are more than the number of reaction components. The state at which this condition develops is said to be collapse mechanism & load carried at this state is called collapse load.

Collapse load = Load factor × Working load.

* Basic theorems for finding collapse loads.

a) static theorem: $W \le W_c$

W = working load

W_c = Collapse load.

→ Lower bound theorem

b) kinematic theorem: $W \ge W_c$

→ Upper bound theorem.

c) Uniqueness theorem,

$W > W_c$

** Methods of plastic Analysis

Based on uniqueness theorem, there are two methods of plastic analysis.

a) statical method

b) kimematic method.

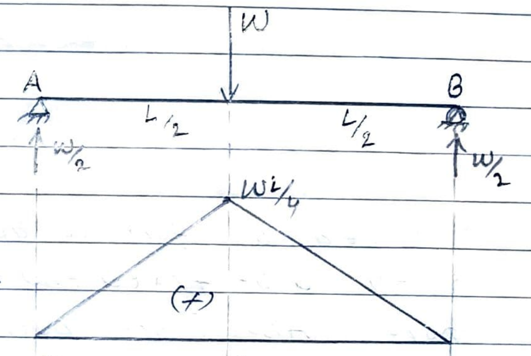
Statical Method:-

→ This method is suitable for the analysis of structure for which the shape of bending moment diagram is easily known.

→ In this method, the maximum bending moment will be equal to plastic moment.

Then, we can calculate collapse load (W_c).

Q1 Determine the collapse load W_c in Simply supported beam shown in figure.



→ The bending moment diagram for this beam is shown in figure. Since, simply supported beam is determinate structure, formation of one hinge in the beam creates collapse mechanism. Since, the moment is maximum under the load, the hinge will form at that place.

By statistical method, Max^m B.M will be equal to plastic moment.

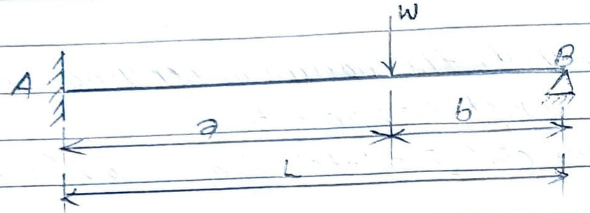
$$\frac{WL}{4} = M_p$$

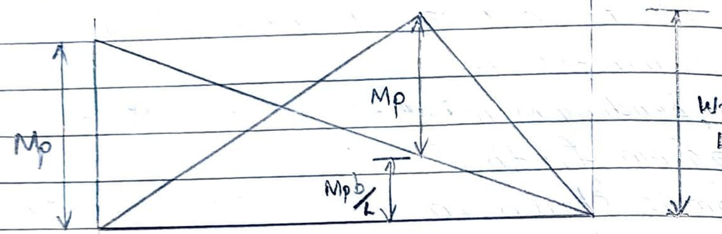
$$\Rightarrow W = \frac{4M_p}{L}$$

$$\Rightarrow W_c = \frac{4M_p}{L}$$

Q2 Determine the collapse load in case of propped cantilever loaded beam shown in figure.

→





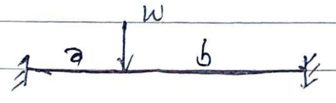
This beam will form collapse mechanism, if a hinge is formed at fixed end and another hinge in the portion AB. Since, positive bending moment is maximum under the load, naturally inner hinge will develop at this point. At collapse condition, bending moment at A & under the load will be equal to Mp.

$$\frac{W_c ab}{L} = Mp + Mp \times \frac{b}{L} = Mp\left(\frac{L+b}{L}\right)$$

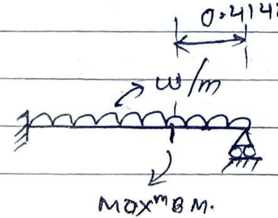
Then,

$$W_c = \left(\frac{L+b}{ab}\right) Mp.$$

Note:-



$W_c = 2M_p\left(\frac{L}{ab}\right)$



$W_c = 11.655 \times \frac{M_p}{L^2}$

* Kinematic Method:-

→ This method starts with an assumed collapse mechanism. After collapse mechanism is formed, there can be no change of curvature at any cross-section except where plastic hinges are formed. Hence, if a virtual displacement is given to the structure just after collapse mechanism is formed, the internal work is done only at plastic hinges, where plastic moment Mp is acting. Hence, by equating internal workdone by plastic moments at plastic hinges to the external workdone by loads, we can get collapse load.

Q Determine the collapse load in the simply supported beam shown in figure.

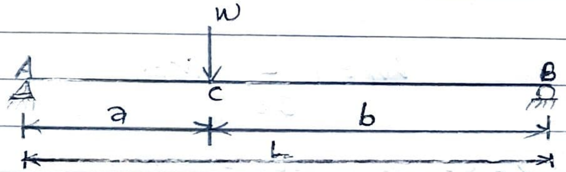


fig a: Simply Supported beam.

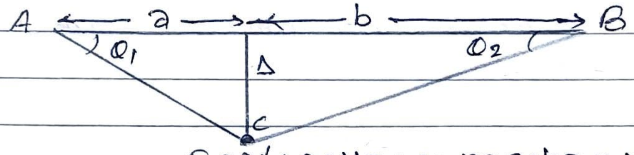


Fig b: collapse mechanism.

Since, bending moment is maximum under the load, interior hinge will be under the load as shown in fig(b).

Let, θ_1 & θ_2 be rotations in AC & CB. Then, from geometry,

$$a\theta_1 = \Delta = b\theta_2$$

$$\theta_2 = \left(\frac{a}{b}\right)\theta_1$$

$$\text{Internal workdone} = M_p\theta_1 + M_p\theta_2$$
$$= M_p\left[\theta_1 + \frac{a}{b}\theta_1\right]$$

$$= M_p\left(\frac{L}{b}\right)\theta_1 \text{ --- ①}$$

$$\text{External workdone} = W_c \times \Delta$$
$$= W_c \times a\theta_1 \text{ --- (II)}$$

Equating ① & (II),

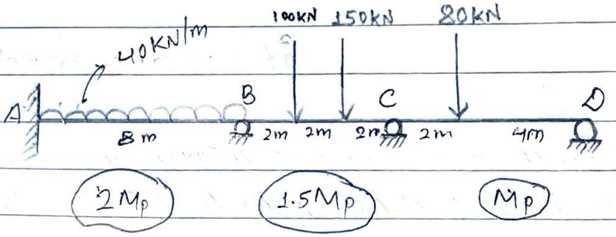
External workdone = Internal workdone.

$$M_p \times \left(\frac{L}{b}\right)\theta_1 = W_c\, a\theta_1$$

$$W_c = \left(\frac{L}{ab}\right) M_p.$$

If $a = b = L/2$, $W_c = \frac{4M_p}{L}$

Q. Calculate the plastic moment for the given beam as shown in figure.



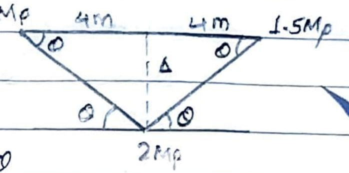
Beam mechanism

↳ Sway mechanism

↳ Combined mechanism.

Solution:-

Beam mechanism for AB,



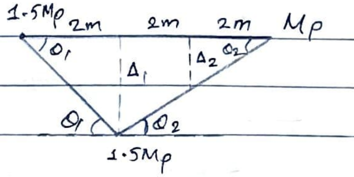
$\Delta = 4\theta$.

E.W = I.W

$$\Rightarrow 40 \times 8 \times \frac{\Delta}{2} = 2M_p \times \theta + 2M_p \times (\theta + \theta) + 1.5M_p \times \theta$$

$$\Rightarrow M_p = 85.33 \text{ KNm}.$$

Beam mechanism for BC.



(a) Considering 100KN load,

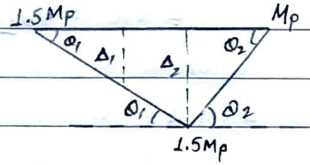
$$1.5M_p \times \theta_1 + 1.5M_p(\theta_1 + \theta_2) + M_p \times \theta_2$$
$$= 100 \times \Delta_1 + 150 \times \Delta_2$$

Also,

$$\Delta_1 = 2\theta_1 = 4\theta_2, \quad \Delta_2 = \frac{\Delta_1}{2}, \quad \Delta_2 = 2\theta_2$$

Then,

$$M_p = 82.35 \text{ KN}$$



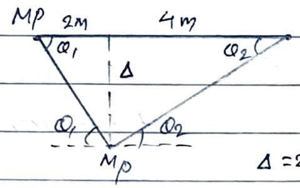
(b) Considering 150KN load:-

$$1.5M_p \times \theta_1 + 1.5M_p(\theta_1 + \theta_2) + M_p \times \theta_2$$
$$= 100 \times \Delta_1 + 150 \times \Delta_2$$

Here, $\Delta_1 = 2\theta_1$, $\Delta_2 = 4\theta_1 = 2\theta_2$ $\Delta_2 = 2\Delta_1$

Then,

$$M_p = 100 \text{ KNm}.$$



Beam Mechanism for CD.

$$M_p \times \theta_1 + M_p \times (\theta_1 + \theta_2) = 80 \times \Delta$$

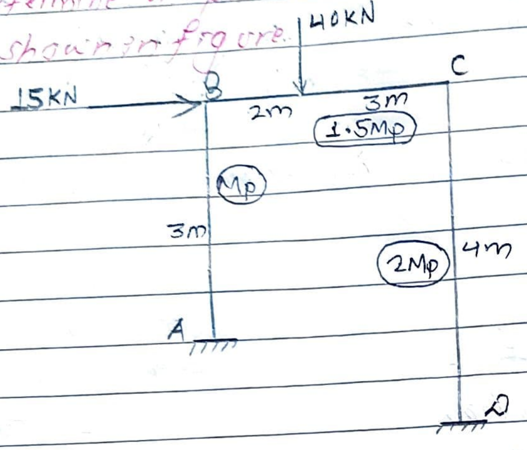
$\Delta = 2\theta_1 = 4\theta_2$

Substituting conditions,

$$M_p = 64 \text{ KNm}$$

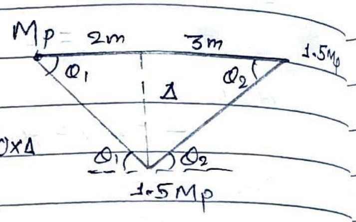
∴ plastic moment for AB = 85.33KNm, BC = 100KNm, CD = 64KNm.

Q Determine the plastic moment for the given frame as shown in figure.



→

(a) Beam mechanism for BC:-



Here, $\Delta = 2\theta_1 = 3\theta_2$

$M_p \times \theta_1 + 1.5M_p(\theta_1 + \theta_2) + 1.5M_p \times \theta_2 = 40 \times \Delta$

$M_p = 17.78$ KNm

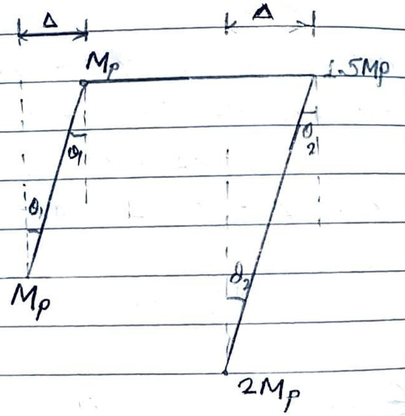
(b) Sway Mechanism:-

$\Delta = 3\theta_1 = 4\theta_2$

Now,

$15 \times \Delta = M_p \times \theta_1 + M_p \times \theta_1$
$+ 1.5M_p \times \theta_2 + 2M_p \times \theta_2$

$M_p = 9.73$ KNm



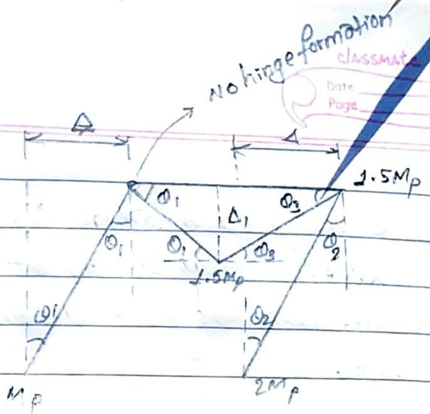
(c) Combined Mechanism:-

$M_p \times \theta_1 + 1.5M_p(\theta_1 + \theta_3) + 1.5M_p(\theta_2 + \theta_3) + 2M_p \times \theta_2$
$= 15 \times \Delta + 40 \times \Delta_1$

put $\Delta = 3\theta_1$, $\Delta_1 = 2\theta_1 = 3\theta_3$ & $\Delta = 4\theta_2$

$M_p = 17.519$ KNm

Therefore, the real plastic moment
$= 17.78$ KNm



Ch-8 Stiffness Matrix Method:-

* Stiffness:-

It is defined as the force required to produce the given deflection at that point whereas flexibility is defined as the displacement due to applied force.

* Stiffness matrix:-

The square matrix of order 'n' which gives the force corresponding to the displacement & represented as,

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & --- \\ K_{21} & K_{22} & K_{23} & --- \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

- The element K_{ij} of stiffness matrix is the force at coordinate 'i' due to unit displacement at j
- stiffness matrix is symmetrical matrix $[K_{ij} = K_{ji}]$
- Flexibility matrix & stiffness matrix are inverse of each other. [Follow Ch-2 note for their relation]

$$[K][S] = [I]$$

⇒ Let $\Delta_1, \Delta_2, \Delta_3 ---- \Delta_n$ be the displacements in coordinate 1, 2, 3 ---- n respectively. Let the forces developed due to applied loads in the restrained structure in the coordinate directions be $P_1, P_2, ... P_n$. Let $[K]$ be the stiffness matrix & $[\Delta]$ be the displacement vector. Let, the final forces be $[P]$. Then, the forces developed in coordinate dir^n are,

$$P_1 = P_{1L} + K_{11}\Delta_1 + K_{12}\Delta_2 + ---- + K_{1n}\Delta_n$$

$$P_2 = P_{22} + K_{21}\Delta_1 + K_{22}\Delta_2 + ---- + K_{2n}\Delta_n$$

- - - - - - - -

$$P_n = P_{nL} + K_{n1}\Delta_1 + K_{n2}\Delta_2 + ---- + K_{nn}\Delta_n$$

In matrix form,

$$[P] = [P_L] + [K][\Delta]$$

$$[K][\Delta] = [P - P_L]$$

$$[\Delta] = [K]^{-1}[P - P_L]$$

* Steps for stiffness matrix method:-

1. Determine degree of kinematic indeterminancy 'n'.
2. Assign the coordinate numbers to the unknown displacements.
3. Impose restraints in all coordinate directions to get a fully restrained structure
4. Determine the forces developed in each of the coordinate directions of a fully restrained structure $[P_L]$.
5. Determine the stiffness matrix $[K]$ by giving unit displacement to the restrained structure in each of the coordinate directions and find the forces developed in all the coordinate directions. For this, only ~~the~~ structure approach is explained in this text.
6. Observing the final forces in various coordinate directions, note the final forces $[P]$.
7. Solve the stiffness equation $[\Delta] = [K]^{-1}[P - P_L]$ to get displacements in coordinate directions.
8. Calculate the member forces using these joint displacements.

$$K_{11} = \frac{4EI}{L} \times \theta_A$$

$$K_{21} = \frac{2EI}{L}\theta_A$$

* Transverse displacement without rotation of one end of a prism

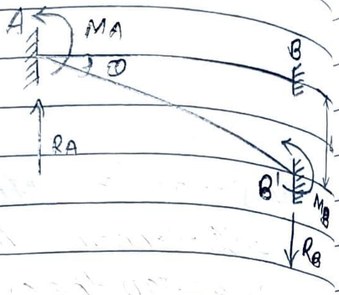
(a) When right end settle by Δ.

$$M_A = M_B = \frac{6EI\Delta}{L^2}$$

$$R_A = R_B = \frac{12EI\Delta}{L^3}$$

For Δ = 1, $M_A = M_B = \frac{6EI}{L^2}$ (↻ ↻)

$$R_A = R_B = \frac{12EI}{L^3}$$



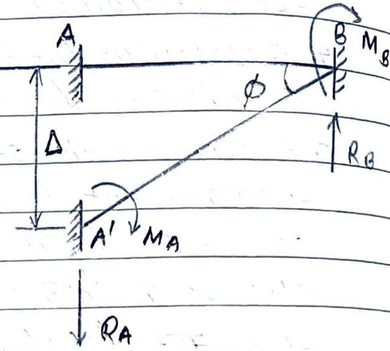
(b) When left end settle by Δ.

$$M_A = M_B = \frac{6EI\Delta}{L^2}$$

$$R_A = R_B = \frac{12EI\Delta}{L^3}$$

For Δ = 1, $M_A = M_B = \frac{6EI}{L^2}$

$$R_A = R_B = \frac{12EI}{L^2}$$



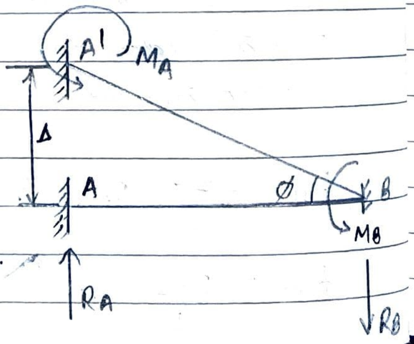
(c) When left support move upward by Δ

$$M_A = M_B = \frac{6EI\Delta}{L^2}$$

$$R_A = R_B = \frac{12EI\Delta}{L^3}$$

For Δ = 1, $M_A = M_B = \frac{6EI}{L^2}$ (↺)

$R_A = R_B = \frac{12EI}{L^3}$



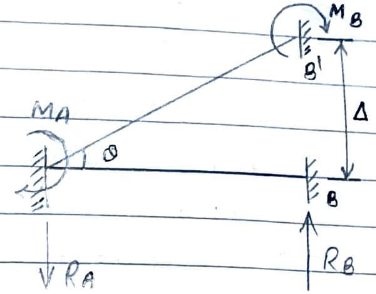
(d) When right support moves upward by Δ.

$$M_A = M_B = \frac{6EI\Delta}{L^2}$$

$$R_A = R_B = \frac{12EI\Delta}{L^3}$$

For Δ = 1, $M_A = M_B = \frac{6EI}{L^2}$ (↺)

$R_A = \frac{12EI}{L^3}$ (↓) & $R_B = \frac{12EI}{L^3}$ (↑)



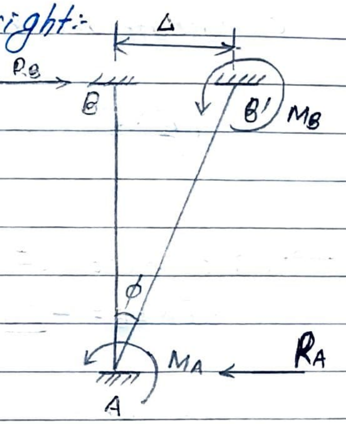
(e) When column sways towards right:-

For Δ = 1,

$$M_A = M_B = \frac{6EI}{L^2} \ (↻)$$

$$R_A = \frac{12EI}{L^3} \ (←)$$

$$R_B = \frac{12EI}{L^3} \ (→)$$



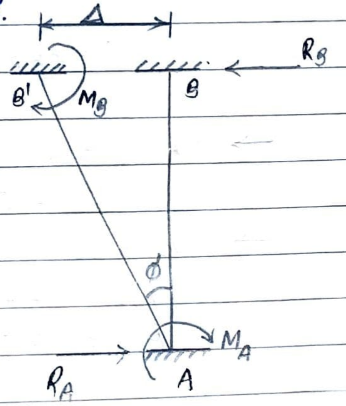
(f) When column sways towards left.

For Δ = 1,

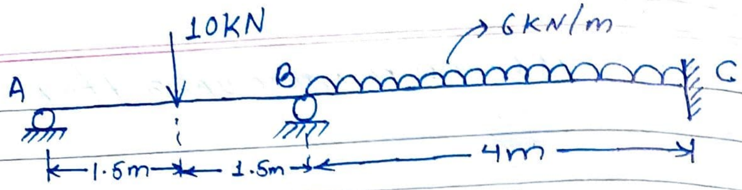
$$M_A = M_B = \frac{6EI}{L^2} \ (↻)$$

$$R_A = \frac{12EI}{L^3} \ (→)$$

$$R_B = \frac{12EI}{L^3} \ (←)$$



Q



Using stiffness matrix method, find support rxns & draw bending moment diagram for the given loaded continuous beam.

→ The independent displacement components are rotation at A and B. Hence, degree of freedom is two. The unknown displacement are denoted as Δ_1 & Δ_2 & coordinate are assigned as shown in figure.

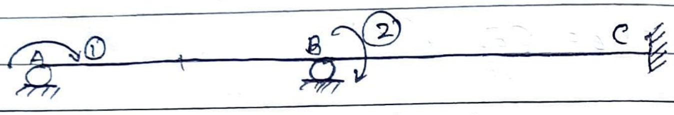


Fig: coordinate assign.

→ Calculation of FEM,

$FEM_{AB} = -\frac{10 \times 3}{8} = -3.75 \text{ kNm}$

$FEM_{BA} = 3.75 \text{ kNm}$

$FEM_{BC} = -\frac{6 \times 4^2}{12} = -8 \text{ kNm}$

$FEM_{CB} = \frac{6 \times 4^2}{12} = 8 \text{ kNm}$

→ There are no external load at coordinate 1 & 2. So,

$P_1 = 0$ & $P_2 = 0$

→ P_{1L} & P_{2L} be forces due to external loading at ① & ② Then,

$P_{1L} = (FEM)_{AB} = -3.75 \text{ kN}$

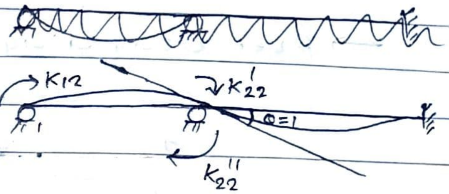
$P_{2L} = (FEM)_{BA} + (FEM)_{BC} = 3.75 - 8 = -4.25 \text{ kNm}.$

→ Calculation of stiffness matrix.

→ Applying unit rotation at 1,

$K_{11} = \frac{4EI}{3}$ & $K_{21} = \frac{2EI}{3}$

→ Applying unit rotation at 2,



$K_{12} = \frac{2EI}{3}$

$K_{22} = K_{22}' + K_{22}''$

$= \frac{4EI}{4} + \frac{4EI}{3} = \frac{7EI}{3}$

Then,

Stiffness matrix is, $K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$

$K = \begin{bmatrix} 4EI/3 & 2EI/3 \\ 2EI/3 & 7EI/3 \end{bmatrix}$

Then,

$\Delta = [K]^{-1} [P - P_L]$

$\Rightarrow \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 4EI/3 & 2EI/3 \\ 2EI/3 & 7EI/3 \end{bmatrix}^{-1} \begin{bmatrix} 0 - (-3.75) \\ 0 - (-4.25) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 2.22 \\ 1.19 \end{bmatrix}$

Hence,

$\theta_A = \Delta_1 = 2.22/EI$ & $\theta_B = \Delta_2 = 1.19/EI$

Now,

Applying slope deflection eqns,

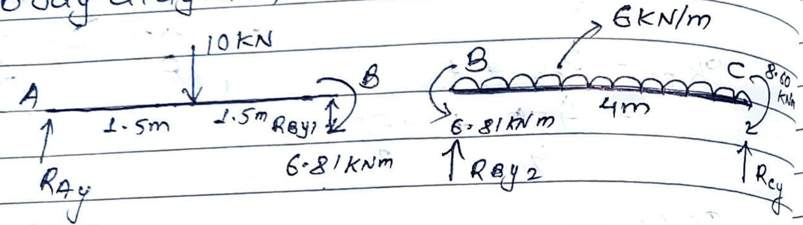
$M_{AB} = \frac{2EI}{3} \left[2 \times \frac{2.22}{EI} + \frac{1.19}{EI} - 0 \right] - 3.75 = 0$

$M_{BA} = \frac{2EI}{3} \left[2 \times \frac{1.19}{EI} + \frac{2.22}{EI} - 0 \right] + 3.75 = 6.81 \text{ kNm}$

$$M_{BC} = \frac{2EI}{4}\left[2\times\frac{1.19}{EI} + 0 - 0\right] - 8 = -6.81\ KNm$$

$$M_{CB} = \frac{2EI}{4}\left[\frac{1.19}{EI} + 0 - 0\right] + 8 = 8.60\ KNm$$

→ Free body diagram,



→ Rxn calculation,

$+\circlearrowright \Sigma M_A = 0$

$\Rightarrow 10\times1.5 + 6.81 - R_{By1}\times3 = 0 \quad \Rightarrow R_{By1} = 7.27\,KN$

$\Rightarrow R_{Ay} = 2.73\,KN$

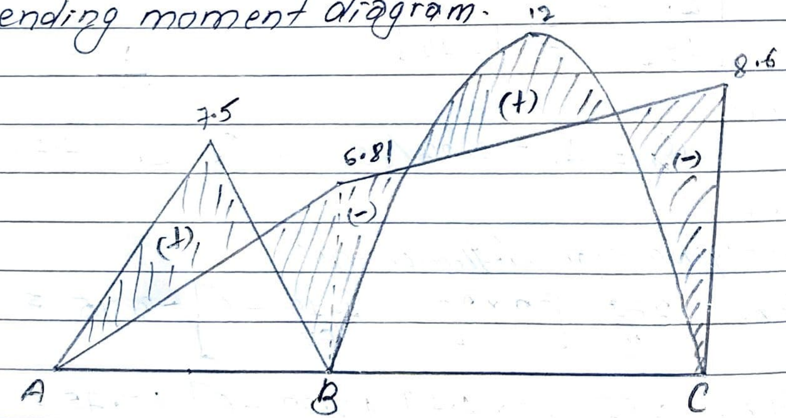
$+\circlearrowright \Sigma M_B = 0$

$\Rightarrow -6.81 + 6\times4\times2 + 8.60 - R_{cy}\times4 = 0 \quad \Rightarrow R_{cy} = 12.44\,KN$

$\Rightarrow R_{By2} = 11.56\,KN$

$\therefore R_{Ay} = 2.73\,KN,\ R_{By} = 18.83\,KN,\ R_{cy} = 12.44\,KN$ #

→ Bending moment diagram.



BMD (KNm)

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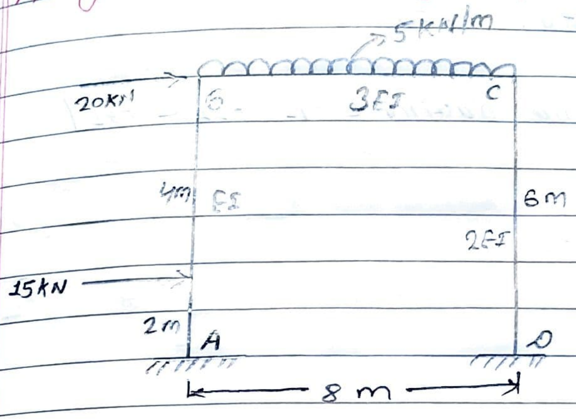
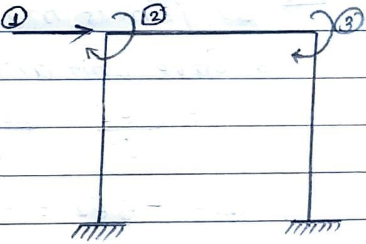
* Analyse the given frame by using stiffness matrix method.



→ Here, degree of freedom of structure is 3. These are sway, rotation of B & B are the unknown displacements.

→ It may be noted that the horizontal displacement of joint 'C' is same as that of joint 'B'. The displacements are denoted as Δ_1, Δ_2 & Δ_3 at coordinates 1, 2 & 3 as shown in figure (i).



→ Calculation of FEM,

$$(FEM)_{AB} = -\frac{15\times2\times4^2}{6^2} = -\frac{40}{3}\ KNm$$

$$(FEM)_{BA} = \frac{15\times2^2\times4}{6^2} = \frac{20}{3}\ KNm$$

$$(FEM)_{BC} = -\frac{wl^2}{12} = -\frac{5\times8^2}{12} = -\frac{80}{3}\ KNm$$

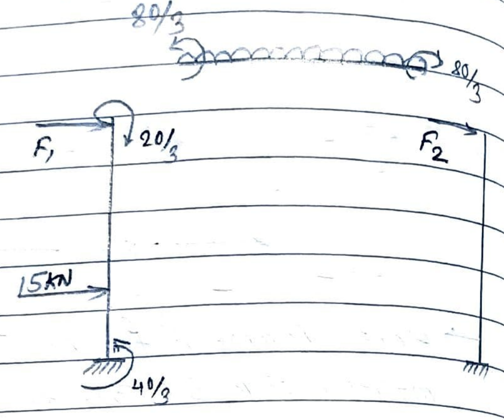
$$(FEM)_{CB} = \frac{80}{3}\ KNm$$

$$(FEM)_{CD} = (FEM)_{DC} = 0.$$

→ The external load acting at coordinates are,

$P_1 = 20kN, \quad P_2 = 0, \quad P_3 = 0$

→ Force due to external loading $[P_{1L}, P_{2L} \& P_{3L}]$



$+\circlearrowright \Sigma M_A = 0$

$F_1 \times 6 + \frac{20}{3} - \frac{40}{3} + 15 \times 2 = 0$

$F_1 = -35/9 \text{ kN}$

$F_2 = 0$

Then,

$P_{1L} = F_1 + F_2 = -35/9 \text{ kN}$

$P_{2L} = \frac{20}{3} - \frac{80}{3} = -20 \text{ kN}$

$P_{3L} = 80/3 + 0 = 80/3 \text{ kN}$

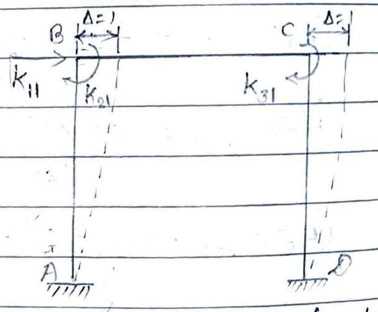
→ Stiffness matrix calculation.

(a) Give unit displacement at coordinate 1,

$K_{11} = \frac{12EI}{6^3} + \frac{12E(2I)}{6^3} = \frac{EI}{6}$

$K_{21} = \frac{-6EI}{6^2} = -\frac{EI}{6}$

$K_{31} = \frac{-6E(2I)}{6^2} = -\frac{EI}{3}$



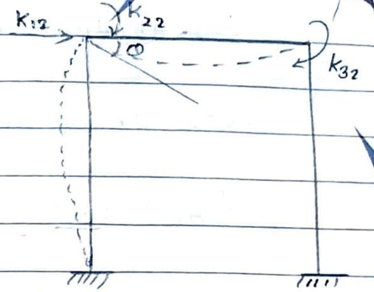
Unit displacement at coordinate 1

(b) Apply unit rotation at coordinate 2

$K_{12} = -\frac{6EI}{6^2} = -\frac{EI}{6}$

$K_{22} = \frac{4EI}{6} + \frac{4E(3I)}{8} = \frac{13EI}{6}$

$K_{32} = \frac{2E(3I)}{8} = \frac{3EI}{4}$



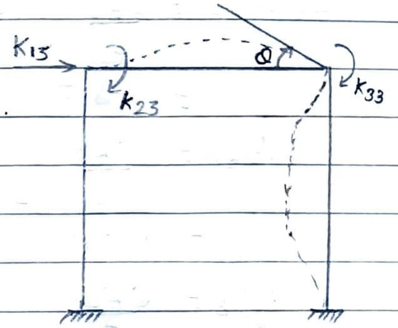
Unit rotation at coordinate 2.

(c) Apply unit rotation at coordinate-3,

$K_{13} = -\frac{6E \times (2I)}{6^2} = -\frac{EI}{3}$

$K_{23} = \frac{2E \times (3I)}{8} = \frac{3EI}{4}$

$K_{33} = \frac{4E \times (3I)}{8} + \frac{4E(2I)}{6} = \frac{17EI}{6}$



Unit rotation at coordinate-3

Hence, stiffness matrix is,

$$K = EI \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 13/6 & 3/4 \\ -1/3 & 3/4 & 17/6 \end{bmatrix}$$

→ The displacement can be calculated as,

$$\Delta = [K]^{-1} [P - P_L]$$

$$\Rightarrow \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}^{-1} \begin{bmatrix} P_1 - P_{1L} \\ P_2 - P_{2L} \\ P_3 - P_{3L} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 13/6 & 3/4 \\ -1/3 & 3/4 & 17/6 \end{bmatrix}^{-1} \begin{bmatrix} 20 + 35/9 \\ 0 + 20 \\ 0 - 80/3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 175.58 \\ 20.745 \\ 5.754 \end{bmatrix}$$

$\therefore \Delta_{AB}/\Delta_{BA} = \frac{175.58}{EI}$ $\theta_B = \frac{20.745}{EI}$ $\theta_C = \frac{5.754}{EI}$

Now applying slope deflection ~~angle~~, method,

$$M_{AB} = \frac{2EI}{6}\left[2\times 0 + \frac{20.745}{EI} - \frac{175.58}{EI}\times\frac{3}{6}\right] - \frac{40}{3}$$

$= -35.681$ kNm

$$M_{BA} = \frac{2EI}{6}\left[2\times\frac{20.745}{EI} - \frac{175.58}{EI}\times\frac{3}{6}\right] + \frac{20}{3}$$

$= -8.76$ kNm

$$M_{BC} = \frac{6EI}{8}\left[2\times\frac{20.745}{EI} + \frac{5.754}{EI} - 0\right] - \frac{80}{3}$$

$= 8.76$ kNm

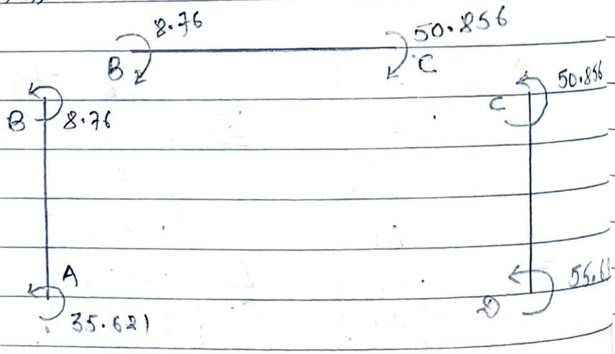
$$M_{CB} = \frac{6EI}{8}\left[2\times\frac{5.754}{EI} + \frac{20.745}{EI}\right] + \frac{80}{3}$$

$= 50.856$ kNm

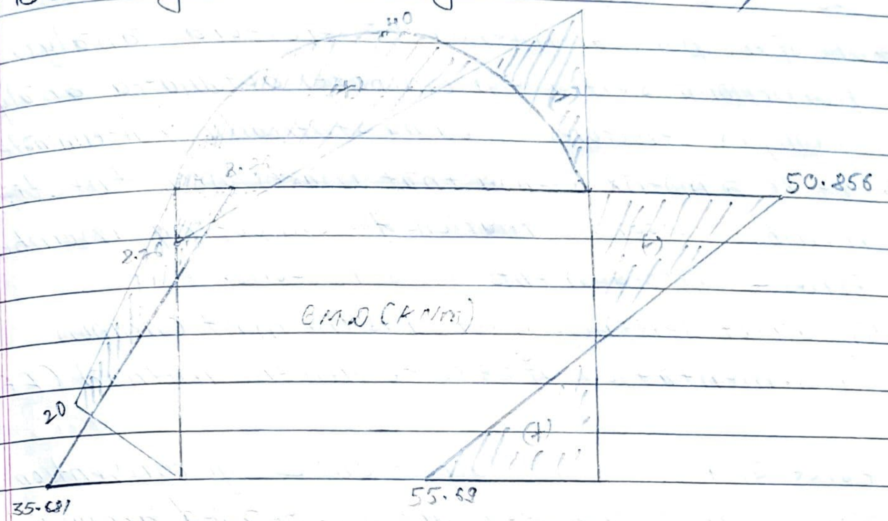
$$M_{CD} = \frac{4EI}{6}\left[2\times\frac{5.754}{EI}\cancel{+0} - \frac{3}{6}\times\frac{175.58}{EI}\right] = -50.85 \text{ kN}$$

$$M_{DC} = \frac{4EI}{6}\left[\frac{5.754}{EI}\cancel{+0} - \frac{3}{6}\times\frac{175.58}{EI}\right] = -55.69$$

→ Free Body Diagram,



→ Bending moment diagram.



ch-9

* Direct stiffness method:-

→ One of the popular method of structural analysis, particularly suited for computer automated analysis of complex structures including statically indeterminate type.

→ It is a matrix method that makes use of the stiffness members stiffness relations for computing member forces & displacements in structures.

→ The direct stiffness method is the most common implementation of finite element method (FEM).

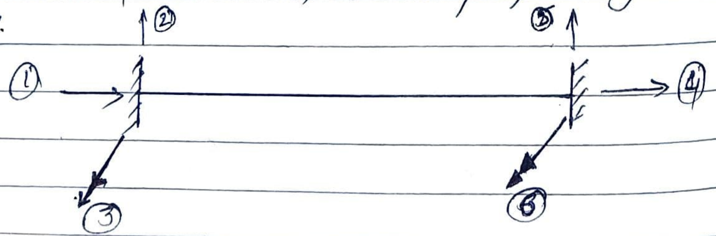
* Force and moment directions & sign convention.

Force / Rxn is indicated by single headed arrow which represent the translation. For beam, vertical forces are positive.

Moment is indicated by double headed arrow which represent the rotation. Anticlockwise moments are positive.

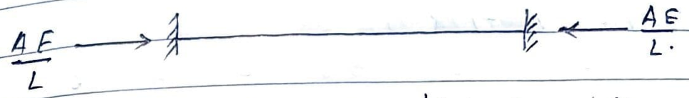
* Coordinate System:-

While assigning the number to the ~~coordinate~~ unknown displacement & rotations, we can prefer anyone method.



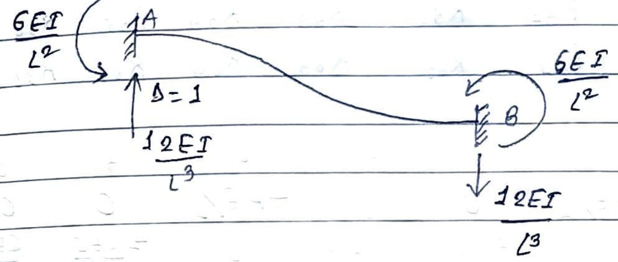
first - horizontal force, second - vertical, Third - Moment

* Displacement due to force 1,



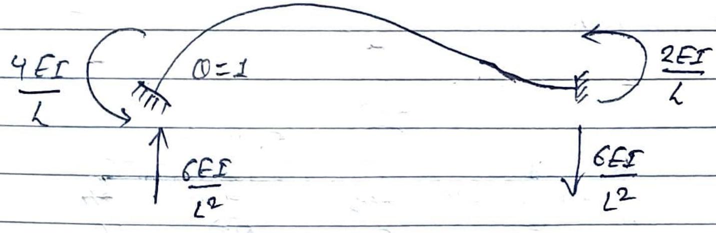
$K_{11} = \frac{AE}{L}$, $K_{21} = 0$, $K_{31} = 0$, $K_{41} = -\frac{AE}{L}$, $K_{51} = 0$, $K_{61} = 0$

* Displacement due to force 2,



$K_{12} = 0$, $K_{22} = \frac{12EI}{L^3}$, $K_{32} = \frac{6EI}{L^2}$, $K_{42} = 0$, $K_{52} = -\frac{12EI}{L^3}$, $K_{62} = \frac{6EI}{L^2}$

* Displacement due to force 3,



$K_{13} = 0$, $K_{23} = \frac{6EI}{L^2}$, $K_{33} = \frac{4EI}{L}$, $K_{43} = 0$, $K_{53} = -\frac{6EI}{L^2}$,

$K_{63} = \frac{2EI}{L}$

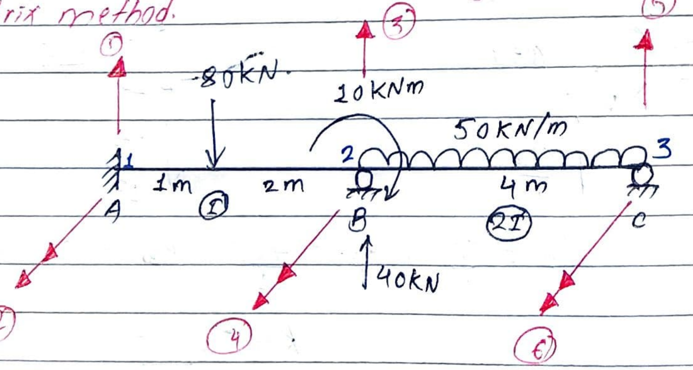
In similar way, we can calculate remaining stiffness (k) due to force 4, 5 & 6.

Then, stiffness matrix is,

$$[k] = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & 1 \\ & K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 2 \\ & K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & 3 \\ & K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & 4 \\ & K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & 5 \\ & K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & 6 \end{array}$$

$$[k] = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & AE/L & 0 & 0 & -AE/L & 0 & 0 & 1 \\ & 0 & 12EI/L^3 & 6EI/L^2 & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 2 \\ & 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & \frac{2EI}{L} & 3 \\ & -AE/L & 0 & 0 & AE/L & 0 & 0 & 4 \\ & \cancel{AE/L}\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -\frac{6EI}{L^2} & 5 \\ & 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & \frac{4EI}{L} & 6 \end{array}$$

Q Analyse the given beam using direct stiffness matrix method.



Total stiffness matrix = stiffness matrix of element 1-2 & stiffness matrix of element 2-3

stiffness matrix for element 1-2,

$$[K_{12}] = \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \hline 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 & 1 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L & 2 \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 & 3 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L & 4 \end{array}$$

Substituting value of I & L,

$$[K_{12}] = EI \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \hline 12/27 & 6/9 & -12/27 & 6/9 & 1 \\ 6/9 & 4/3 & -6/9 & 2/3 & 2 \\ -12/27 & -6/9 & 12/27 & -6/9 & 3 \\ 6/9 & 2/3 & -6/9 & 4/3 & 4 \end{array}$$

Stiffness matrix for element 2-3

$$[K_{23}] = \begin{array}{cccc|c} 3 & 4 & 5 & 6 & \\ \hline 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 & 3 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L & 4 \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 & 5 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L & 6 \end{array}$$

substituting value of I & L,

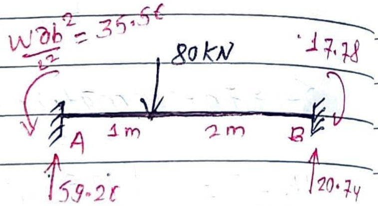
$$[K_{23}] = EI \begin{array}{cccc|c} 3 & 4 & 5 & 6 & \\ \hline 24/64 & 12/16 & -24/64 & 12/16 & 3 \\ 12/16 & 8/4 & -12/16 & 4/4 & 4 \\ -24/64 & -12/16 & 24/64 & -12/16 & 5 \\ 12/16 & 4/4 & -12/16 & 8/4 & 6 \end{array}$$

Then, totalstiffness matrix will be,

$$K = EI \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ 12/27 & 6/9 & -12/27 & 6/9 & 0 & 0 & 1 \\ 6/9 & 4/3 & -6/9 & 2/3 & 0 & 0 & 2 \\ -12/27 & -6/9 & 59/72 & 1/12 & -24/64 & 12/16 & 3 \\ 6/9 & 2/3 & 1/12 & 10/3 & -12/16 & 4/4 & 4 \\ 0 & 0 & -24/64 & -12/16 & 24/64 & -12/16 & 5 \\ 0 & 0 & 12/16 & 4/4 & -12/16 & 8/4 & 6 \end{array}$$

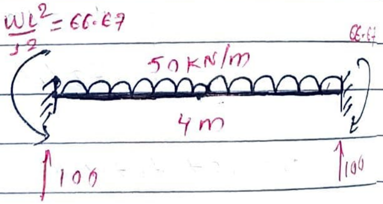
→ Joint load matrix,

$$A_j = \begin{array}{|c|c} 0 & 1 \\ 0 & 2 \\ 40 & 3 \\ -10 & 4 \\ 0 & 5 \\ 0 & 6 \end{array}$$



→ Member reation matrix,

$$A_r = \begin{array}{|c|c} 59.26 & 1 \\ 35.56 & 2 \\ 120.74 & 3 \\ 48.89 & 4 \\ 100 & 5 \\ -66.67 & 6 \end{array}$$



→ Combined load matrix [Ac]

$$[A_c] = [A_j] - [A_r] = \begin{bmatrix} 0 \\ 0 \\ 40 \\ -10 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 59.26 \\ 35.56 \\ 120.74 \\ 48.89 \\ 100 \\ -66.67 \end{bmatrix} = \begin{bmatrix} -59.26 \\ -35.56 \\ -80.74 \\ -58.89 \\ -100 \\ 66.67 \end{bmatrix}$$

→ Assemble stiffness equation $[K][\delta] = [A_c]$

$$EI \begin{bmatrix} 12/27 & 6/9 & -12/27 & 6/9 & 0 & 0 \\ 6/9 & 4/3 & -6/9 & 2/3 & 0 & 0 \\ -12/27 & -6/9 & 59/72 & 1/12 & -24/64 & 12/16 \\ 6/9 & 2/3 & 1/12 & 10/3 & -12/16 & 4/4 \\ 0 & 0 & -24/64 & -12/16 & 24/64 & -12/16 \\ 0 & 0 & 12/16 & 4/4 & -12/16 & 8/4 \end{bmatrix} \times \begin{bmatrix} \delta_A \\ \theta_A \\ \delta_B \\ \theta_B \\ \delta_C \\ \theta_C \end{bmatrix} = \begin{bmatrix} -59.26 \\ -35.56 \\ -80.74 \\ -58.89 \\ -100 \\ 66.67 \end{bmatrix}$$

Since, $\delta_A = 0$, $\theta_A = 0$ [No deflection & rotation at fix support]
$\delta_B = 0$ [No vertical displacement of roller support]
$\delta_C = 0$ [No vertical displacement of roller support]

Then,

$$\begin{bmatrix} 10EI/3 & 4EI/4 \\ 4EI/4 & 8EI/4 \end{bmatrix} \times \begin{bmatrix} \theta_A \\ \theta_C \end{bmatrix} = \begin{bmatrix} -58.89 \\ 66.67 \end{bmatrix}$$

On solving,

$\theta_B = -\frac{32.55}{EI}$ rad & $\theta_C = \frac{49.61}{EI}$ rad.

→ Finding reactions.

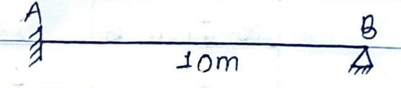
$$[R] = [k][\delta] - [A_c]$$

$$\Rightarrow EI\begin{bmatrix} 12/27 & 6/9 & -12/27 & 6/9 & 0 & 0 \\ 6/9 & 4/3 & -6/9 & 2/3 & 0 & 0 \\ -12/27 & -6/9 & 59/72 & 1/12 & \frac{-24}{64} & 12/16 \\ 6/9 & 2/3 & 1/12 & 10/3 & -12/16 & 4/4 \\ 0 & 0 & -24/64 & -\frac{12}{16} & 24/64 & -12/16 \\ 0 & 0 & 12/16 & 4/4 & -12/16 & 8/4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -32.55/EI \\ 0 \\ 49.61/EI \end{bmatrix} - \begin{bmatrix} +59.26 \\ -35.56 \\ -80.74 \\ -58.89 \\ -100 \\ 66.67 \end{bmatrix} = \begin{bmatrix} 37.56 \\ 13.86 \\ 115.235 \\ 0 \\ 87.205 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} V_A \\ M_A \\ V_B \\ M_B \\ V_C \\ M_C \end{bmatrix} = \begin{bmatrix} 37.56 \\ 13.86 \\ 115.235 \\ 0 \\ 87.205 \\ 0 \end{bmatrix} \begin{matrix} kN \\ kNm \\ kN \\ kNm \\ kN \\ kNm \end{matrix}$$

ILD Rev

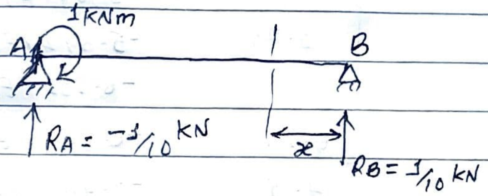
① Using Muller Breslau principle, draw ILD for moment at A for beam shown in figure at 1.25m interval.



→ According to Muller Breslau principle,

$$M_A = \frac{Y_{xA}}{\theta_{AA}}$$

Apply unit moment at A, Using slope deflection eqn



$$EI \times \frac{d^2y}{dx^2} = M_x = \frac{x}{10}$$

$$\Rightarrow EI\frac{d^2y}{dx^2} = \frac{x}{10}$$

On integration,

$$EI \times \frac{dy}{dx} = \frac{x^2}{20} + C_1$$

Again, on integration,

$$EI \times y = \frac{x^3}{60} + C_1x + C_2$$

When $x=0$, $y=0$, $C_2 = 0$

When $x = 10$, $y = 0$, $C_1 = -1.67$

Then,

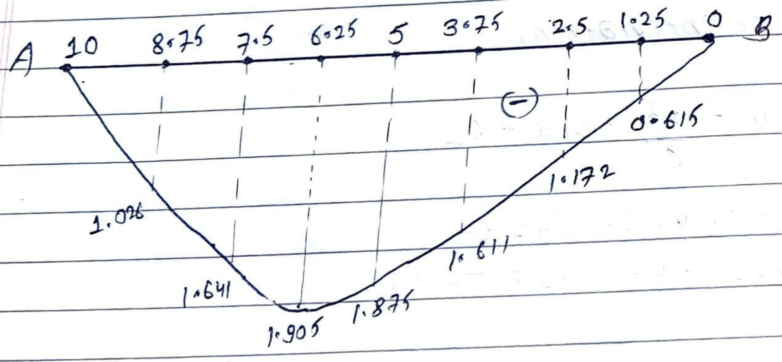
$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{x^2}{20} - 1.67\right] \quad \text{----} (I)$$

&

$$y = \frac{1}{EI}\left[\frac{x^3}{60} - 1.67x\right] \quad \text{----} (II)$$

At $x = 10m$, $\frac{dy}{dx} = \frac{3.33}{EI}$ $\therefore \theta_{AA} = \frac{3.33}{EI}$

x from B	y_{xA}	y_{xA}/θ_{AA}
0	0	0
1.25	-2.05 /EI	-0.615
2.5	-3.907 /EI	-1.172
3.75	-5.372 /EI	-1.611
5	-6.251 /EI	-1.875
6.25	-6.349 /EI	-1.905
7.5	-5.471 /EI	-1.641
8.75	-3.421 /EI	-1.026
10	0	0

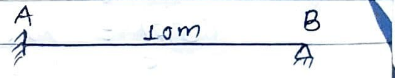


ILD for M_A.

Q Draw ILD for reaction at B by using Muller's Breslau principle at an interval of 1.25m.

→ According to Muller Breslau principle,

$R_B = \frac{y_{xB}}{y_{BB}}$



propped cantilevel beam

Applying unit load at B,

$(+\uparrow)\Sigma F_y = 0;$ $R_{Ay} = 1kN$

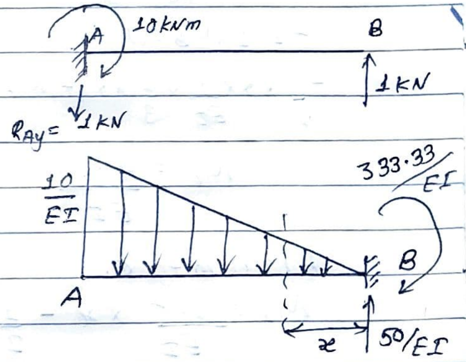
$M_A = 10kNm$

conjugate beam is,

calculate rxn in conjugate beam,

$R'_{By} = 50/EI$

$M'_B = 333.33/EI$

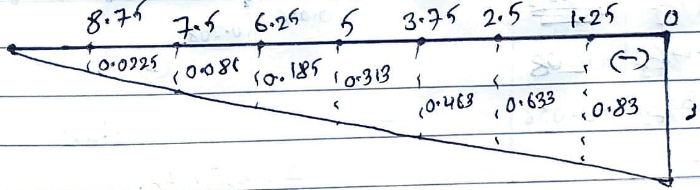


Here,

$y_{BB} = \frac{333.33}{EI}$

Now, $M_x = -\frac{333.33}{EI} + \frac{50x}{EI} - \frac{1}{2} \times x \times \frac{x}{EI} \times \frac{x}{3}$

x	0	1.25	2.5	3.75	5	6.25	7.5	8.75	0
y_{xB}	-333.33/EI	-271.15/EI	-210.93/EI	-154.62/EI	-104.163/EI	-61.25/EI	-28.64/EI	-7.48/EI	0
y_{xB}/y_{BB}	-1	-0.813	-0.633	-0.463	-0.313	-0.185	-0.086	-0.0225	0

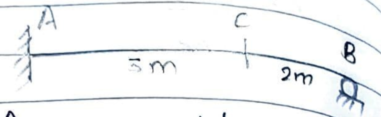


ILD for R_B.

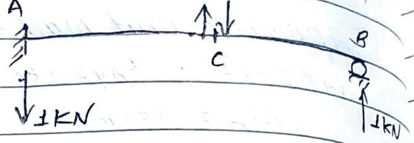
Draw ILD for SF at point 'c' of proped cantilever having length of 5m & constant flexural rigidity. Give ordinate value at 1m interval.

→ we know,

$$(SF)_c = \frac{y_{xc}}{y_{cc}}.$$



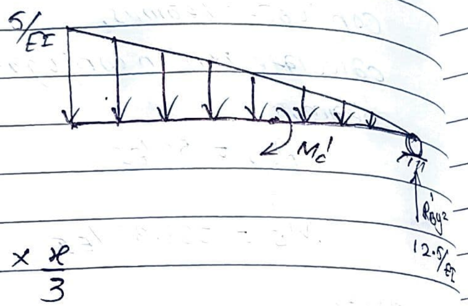
Applying unit shear at C,



$+\circlearrowleft \Sigma M_B = 0$

$$-\frac{1}{2} \times 5 \times \frac{5}{EI} \times \frac{2}{3} \times 5 + M_c' = 0$$

$$M_c' = \frac{41.67}{EI}$$



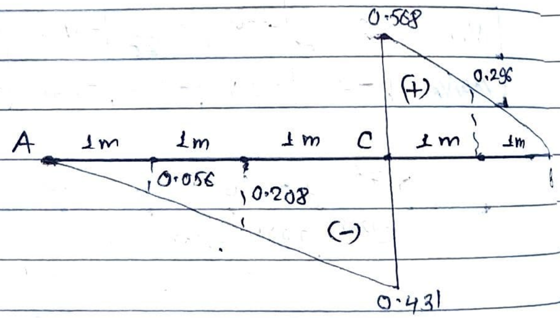
From B to C,

$$M_x = \frac{12.5}{EI} \times x - \frac{1}{2} \times x \times \frac{x}{EI} \times \frac{x}{3}$$

From C to A,

$$M_x = \frac{12.5}{EI} \times x - \frac{x^3}{6EI} - \frac{41.67}{EI}$$

x from B	y_{xc}	
0	0	0
1	12.33/EI	0·296
2 (R)	23.67/EI	0·568
2(L)	-18/EI	~~0·208~~ -0·431
3	-8.67/EI	-0·208
4	-2.33/EI	-0·056
5	0	0



ILD for Fc.