ENGINEERING MATHEMATICS

MATRIX

- If $A^2 = A$ then it is called Indempotent Matrix
- Transpose of Matrix A' $R \leftrightarrow C$
  - → For symmetric Matrix $A' = A$
  - → For skew-symmetric Matrix $A' = -A$
- Square Matrix in form of Sum of symm. & skew-symm. $A = \underbrace{\frac{1}{2}(A + A')}_{\text{Symm.}} + \underbrace{\frac{1}{2}(A - A')}_{\text{Skew-symm.}}$
- Adjoint of Matrix = Transpose of Cofactor Matrix
- Inverse of Matrix $= A^{-1} = \dfrac{Adj.A}{|A|}$
  - → $(AB)^{-1} = B^{-1}A^{-1}$
- Orthogonal Matrix A if → $AA' = A'A = I$
  - → $A' = A^{-1}$ → $|A| = \pm 1$

SOLUTION OF LINER SYSTEM OF EQUATION

1. Method of determinants (CRAMER's RULE)

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{ then,}$$

$$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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2. Matrix Inversion Method

if $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ $AX = D$

$X = A^{-1}D$

NOTE: method fails if $|A| = 0$

Consistency of LINEAR SYSTEM of Eq^n

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n &= k_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n &= k_n \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{mn} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

→ Coefficient Matrix

$$K = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$$

→ Augmented Matrix.

ROUCHE's theorem:

The system of Eq^n is consistent if & only if the Coefficient Matrix A and the Augmented Matrix K are of the same Rank otherwise Inconsistent

if Rank of A → r
Rank of K → r' so,

(i) $r \neq r'$ → INCONSISTENT (NO SOL^n)

(ii) $r = r' = n$ → CONSISTENT (UNIQUE SOL^n)

(iii) $r = r' < n$ → CONSISTENT (INFINITE SOL^n)

EIGEN Values

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

→ Characteristic Eqⁿ of A

- Roots of this Eqⁿ are called Eigen Values

EIGEN Vectors:

$$[A-\lambda I][X] = 0$$

$[X]$ → Eigen Vector

CALEY-HAMILTON Theorem:

if $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ characteristic Eqⁿ of A is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$ by Caley Hamilton it is

$\Rightarrow A^2 - 4A - 5I = 0$

Similarity of Matrix

$\hat{A} = P^{-1}AP$

$\hat{A}$ is matrix of order n
$\hat{A}$ is similar to A then the above eqⁿ is satisfied and both have same Eigen Values.

→ P: (n×n) Non-Singular Matrix.

COMPLEX MATRIX

1 Conjugate of a Matrix

if $A = \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \Rightarrow$ conjugate of A ie $\bar{A}$

$$\bar{A} = \begin{bmatrix} 1-i & -1-i \\ 1-i & 1+i \end{bmatrix}$$

Transpose of $\bar{A} = A^{\theta} = \begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

2 Hermitian Matrix:

if $A' = \bar{A}$ then A is Hermition Matrix

ie, Transpose = conjugate

3 Skew-Hermition Matrix

$A' = -\bar{A}$, then A is called Skew Hermition

- if A is Hermition then, (iA) is Skew Hermition
- Eigen Values of Hermition Matrix is Real
- Eigen Values of Skew Hermition Matrix is Imaginary.

4 Unitary Matrix:

$\bar{U}' = U^{-1}$ then U is unitary Matrix

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LAPLACE TRANSFORM

- $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Transforms of Elementary function:

- $L(1) = 1/s$
- $L(t^n) = \dfrac{n!}{s^{n+1}}$ $\quad n = 0, 1, 2, \ldots\ldots$
- $L(e^{at}) = \dfrac{1}{s-a}$
- $L(\sin at) = \dfrac{a}{s^2+a^2}$
- $L(\cos at) = \dfrac{s}{s^2+a^2}$
- $L(\sinh at) = \dfrac{a}{s^2-a^2}$
- $L(\cosh at) = \dfrac{s}{s^2-a^2}$

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Properties of Laplace Transforms:

1. Linearity property

$L[af(t) + bg(t) - ch(t)] =$

$aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$

2 First shifting property

If $Lf(t) = \bar{f}(s)$ then

$L\{e^{at} f(t)\} = \bar{f}(s-a)$

eg, $L\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

3 Change of Scale property

$L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

eg, $L\left\{\dfrac{\sin at}{t}\right\} = ?$ given $L\left(\dfrac{\sin t}{t}\right) = \tan^{-1}\dfrac{1}{s}$

$L\left\{\dfrac{\sin at}{t}\right\} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{1}{s/a}\right)$

Transforms of Periodic function

if $f(t)$ is a periodic func^n with a period T ie $f(t+T) = f(t)$ then,

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Transforms of Derivatives

if $f'(t)$ be continuous & $Lf(t) = \bar{f}(s)$

then, $\quad L[f'(t)] = s\bar{f}(s) - f(0)$

if $f'(t)$ & its first $(n-1)$ derivatives be continuous, then

$L[f^n(t)] = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0)\ldots$

Transforms of Integrals:

if $L[f(t)] = \bar{f}(s)$, then

$L\left\{\int_0^t f(u)\,du\right\} = \dfrac{1}{s}\bar{f}(s)$

Multiplication by t^n:

if $L\{f(t)\} = \bar{f}(s)$ then

$L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}[\bar{f}(s)]$

Division by t

if $L\{f(t)\} = \bar{f}(s)$ then

$L\left\{\dfrac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$

INVERSE TRANSFORMS

- $L^{-1}(1/s) = 1$
- $L^{-1}(1/(s-a)) = e^{at}$
- $L^{-1}(1/s^n) = \dfrac{t^{n-1}}{(n-1)!}$
- $L^{-1}(1/(s-a)^n) = e^{at}\dfrac{t^{n-1}}{(n-1)!}$

- $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$
- $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
- $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$
- $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
- $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$
- $L^{-1}\left[\frac{(s-a)}{(s-a)^2+b^2}\right] = e^{at}\cos bt$
- $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at$
- $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

Properties of Inverse transform

- $L^{-1}[\bar{f}'(s-a)] = e^{at}f(t) = e^{at}L^{-1}[\bar{f}(s)]$
- $L^{-1}[s^n\bar{f}(s)] = \frac{d^n}{dt^n}[f(t)]$

$\rightarrow$ if $L^{-1}\bar{f}(s) = f(t)$ & $f(0)=0$

- $L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(t)\,dt$

$\rightarrow$ Also, $L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t\left[\int_0^t f(t)\,dt\right]dt$

- $t\,f(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\}$
- $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$

CONVOLUTION THEOREM

if $L^{-1}\{\bar{f}(s)\} = f(t)$ & $L^{-1}\{\bar{g}(s)\} = g(t)$

then, $L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_0^t f(u)\,g(t-u)\,du$

$= F * G$ CONVOLUTION/FALTING

Unit Step function

- $u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$ $\rightarrow$ +ve

Transform of unit func^n:

$L[u(t-a)] = \frac{e^{-as}}{s}$

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Shifting property:

if $L\,f(t) = \bar{f}(s)$, then

$L\{f(t-a).u(t-a)\} = e^{-as}.\bar{f}(s)$

EXTRA MATERIAL

- INVOLUTORY MATRIX: $A^2 = I$
- NILPOTENT MATRIX: $A^x = 0$ $(A^{x-1} \neq 0)$
- MATRIX MULTIPLY: $A_{3\times 2} \times B_{2\times 4} = C_{3\times 4}$
- MINIMUM NO. OF MULTIPLICATION REQD. to MULTIPLY $(A)_{m\times n} \times (B)_{n\times p}$ $= mnp$
- TRACE (AB) = TRACE (BA)
- $\overline{AB} = \bar{A}\bar{B}$
- $\bar{A} = A$ (REAL MATRIX)
- $\bar{A} = -A$ (PURELY IMAGINARY)
- $(AB)^\theta = B^\theta A^\theta$ Transposed conjugate

For any Matrix A

- AA^T is alway Symm. Matrix
- $\frac{A+A^T}{2}$ is alway Symm. Matrix

$A+B$ & $A-B \Rightarrow$ Symm (A & B are Symm)

- $\frac{A-A^T}{2}$ is alway Unsymm. Matrix

DIFFERENTIAL EQUATIONS

ORDER : order of the highest derivative

DEGREE : degree of the highest derivative appearing in it, after the eq^n has been expressed in a form free from radicals & $frac^n$

eg: $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = c^2\left(\frac{d^2y}{dx^2}\right)^2$

order = 2 ; Degree = 2

Solution of a differential Eq^n :

1 General / complete Sol^n : No. of arbitrary constants = order of differential Eq^n

2 Linearly Independent Sol^n : 2 sol^n $y_1(x)$ and $y_2(x)$ of the differential Eq^n

$$\frac{d^2y}{dx^2} + \alpha_1(x)\frac{dy}{dx} + \alpha_2(x)y = 0$$ are said to be linearly independent if $C_1Y_1 + C_2Y_2 = 0$ ie $C_1 = 0$ & $C_2 = 0$

if C_1 & C_2 are not both zero, then the 2 sol^n Y_1 & Y_2 are said to be linearly depend.

Eq^n of 1st Order & 1st degree :

1 Eq^n where variable are separable :

$$\int f(y)\cdot dy = \int \phi(x)\cdot dx + c$$

2 Homogeneous Eq^n :

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$$

{ $f(x,y)$ & $\phi(x,y)$ are homogeneous $func^n$ of same deg. in x & y }

Put $y = tx$,

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

3 Eq^n reducible to homogeneous form -

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$

case 1: $\frac{a}{a'} \neq \frac{b}{b'}$ put $y = tx$

$x = x+h$, $y = y+k$

$dx = dx$, $dy = dy$

$ah+bk+c = 0$, $a'h+b'k+c' = 0$

case 2 : $\frac{a}{a'} = \frac{b}{b'}$

Put $t = ax+by$

LINEAR Eq^n : A DE is said to be linear if the dependent variable & its differential coefficients occur only in the 1st degree & not multiplied together

→ the std. form of a linear Eq^n of the 1st order commonly known as LEIBNITZ Linear Eq^n.

$$\frac{dy}{dx} + Py = Q$$

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Sol^n : $y(IF) = \int Q(IF)\,dx + C$

$$IF = e^{\int P\cdot dx}$$

BERNOULLI'S Eq^n :

The Eq^n $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's eq^n and they are convertible to Leibnitz form.

divide by y^n →

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$ put $y^{1-n} = z$

ie, $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$ we have,

$$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

EXACT DIFFERENTIAL EQ^n :

A differential Eq^n of the form $M\,dx + N\,dy = 0$ is said to be exact if its left hand member is exact differential of some $func^n$ $u(x,y)$

ie, $du = M\,dx + N\,dy = 0$ its sol^n is

$u(x, y) = C$

Necessary and sufficient condition for the DE $M\,dx + N\,dy = 0$ to be EXACT:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Sol^n is given by –

$$\int_{y=\text{const}} M\,dx + \int \begin{pmatrix}\text{terms of } N \text{ not} \\ \text{containing } x\end{pmatrix} . dy = C$$

Eq^n Reducible to Exact Eq^n:

① I.F. found by Inspection:

- $x\,dy + y\,dx = d(xy)$
- $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
- $\dfrac{x\,dy - y\,dx}{xy} = d\left[\log\left(\dfrac{y}{x}\right)\right]$
- $\dfrac{x\,dy - y\,dx}{y^2} = -d\left(\dfrac{x}{y}\right)$
- $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$
- $\dfrac{x\,dy - y\,dx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$

② I.F. of a homogeneous Eq^n:

if $M\,dx + N\,dy = 0$ be a homogeneous eq^n

then, $IF = \dfrac{1}{Mx + Ny}$

③ I.F. for an Eq^n of type $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$

if the eq^n $M\,dx + N\,dy = 0$ be of this form,

then $IF = \dfrac{1}{Mx - Ny}$

④ In the Eq^n $M\,dx + N\,dy = 0$

ⓐ if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$

$I.F. = e^{\int f(x)\,.dx}$

ⓑ if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$

$I.F. = e^{\int f(y)\,.dy}$

Eq^n of 1st Order & Higher Order

Eq^n solvable for P:

$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0$

general sol^n:

$f_1(x, y, c) \cdot f_2(x, y, c) \dots f_n(x, y, c) = 0$

CLAIRAUT'S Eq^n

Eq^n of form $y = px + f(p)$

Sol^n: $y = cx + f(c)$

→ if p is eliminated from $x + f'(p) = 0$

ie, SINGULAR SOL^n which gives the envelop of family of st. lines.

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LINER DIFFERENTIAL EQUATIONS

are those in which the dependent variable & its derivative occur only in the 1st degree & are not multiplied together.

- $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = X$

where $P_1, P_2 \dots P_n$ & X are func^n of x only.

Linear Differential Eq^n with const. coeff. –

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = X$$

THEOREMS:

① If Y_1, Y_2 are only 2 sol^n of eq^n:

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = 0 \text{ then,}$$

$c_1 Y_1 + c_2 Y_2 \ (=u)$ is also its sol^n.

② Since the general sol^n of a DE of the nth order contains n arbitrary constant if $Y_1, Y_2 \dots Y_n$ are n Independent Sol^n, then $c_1 Y_1 + c_2 Y_2 + \dots + c_n Y_n \ (=u)$ is its complete sol^n.

Rules for finding CF:

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} \dots + k_n y = 0$$

$$\underbrace{(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n)}_{\text{Auxillary Eq}^n} Y = 0$$

Case 1: If all the roots be Real & different

$(D-m_1)(D-m_2)\dots(D-m_n) Y = 0$

Complete sol^n is $Y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots c_n e^{m_n x}$

Case 2: if 2 roots are Equal ie, $(m_1 = m_2)$

Complete sol^n is –

$Y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \dots c_n e^{m_n x}$

→ for 3 roots $(m_1 = m_2 = m_3)$

$Y = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$

Case 3: If one pair of roots is Imaginary ie, $m_1 = \alpha + i\beta$; $m_2 = \alpha - i\beta$ Complete sol^n is:

$Y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$.

Case 4: if 2 points of Imaginary roots be equal ie, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$. Complete sol^n :–

$Y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

INVERSE OPERATOR:

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① $\frac{1}{D} X = \int X \, dx$

② $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx$

Rules for finding Particular Integ. (PI) :–

consider the Eq^n $\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots = X$

$$PI = \frac{1}{D^n + k_1 D^{n-1} + \dots + k_n} \cdot X$$

Case I: When $X = e^{ax}$

- $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ ie, $f(a) \neq 0$

• If $f(a)=0$; $\frac{1}{f(D)}e^{ax} = x\cdot\frac{1}{f'(a)}e^{ax}$

• If $f'(a)=0$; $\frac{1}{f(D)}e^{ax} = x^2\cdot\frac{1}{f''(a)}e^{ax}$

Case 2: $X = \sin(ax+b)$ or $\cos(ax+b)$

$$\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b)$$

• if $f(-a)=0$, then

$$\frac{1}{f(D^2)}\sin(ax+b) = x\cdot\frac{1}{f'(-a^2)}\sin(ax+b)$$

and so on.

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Case 3: $X = x^m$

$$PI = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$$

Case 4: $X = e^{ax}V$ $\quad V = f(x)$

$$\frac{1}{f(D)}(e^{ax}\cdot V) = e^{ax}\frac{1}{f(D+a)}V$$

Case 5: X = Any other func^n of X.

if $f(D) = (D-m_1)(D-m_2)\ldots(D-m_n)$

$$\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \ldots + \frac{A_n}{D-m_n}$$

$$PI = \left[\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \ldots + \frac{A_n}{D-m_n}\right]X$$

$$= A_1\cdot\frac{1}{D-m_1}X + A_2\cdot\frac{1}{D-m_2}X + \ldots + A_n\cdot\frac{1}{D-m_n}X$$

$$\Rightarrow A_1 e^{m_1x}\int X\cdot e^{-m_1x}dx + A_2\cdot e^{m_2x}\int X\cdot e^{-m_2x}\cdot dx + \ldots\ldots A_n e^{m_nx}\int X\cdot e^{-m_nx}\cdot dx$$

Other Methods of finding PI –

① Method of Variation of Param

$$y'' + py' + q = X \quad \text{—(I)}$$

p, q & X is $f(x)$

$$PI = -Y_1\int\frac{Y_2X}{W}dx + Y_2\int\frac{Y_1X}{W}dx$$

Y_1 & Y_2 are sol^n of (I)

$$W = \text{Wronskian} = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix}$$

② Method of undetermined Coeff.

to find PI of $f(D)y = X$, assume a trial sol^n containing unknown constant which are determined by substitution in the given eq^n.

When $X = 2e^{3x}$, Trial sol^n $= ae^{3x}$

• $X = 3\sin 2x$, Trial sol^n $= a_1\sin 2x + a_2\cos 2x$

• $X = 2x^3$, Trial sol^n $= a_1x^3 + a_2x^2 + a_3x + a_4$

• $X = \tan x$ or $\sec x$ method fails

∵ the No. of terms obtained after diff. X here is infinite.

Eq^n Reducible to Linear Eq^n. with constant Coeff.

① CAUCHY's Homogeneous Linear Eq^n :–

Any eq^n of the form:–

$$x^n\frac{d^ny}{dx^n} + k_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + k_{n-1}x\frac{dy}{dx} + k_ny = X$$

$X = f(x)$

• Put $x = e^t$ or $t = \log_e x$

② LEGENDRE's Linear Eqⁿ:

An eqⁿ of the form -

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots$$

$$\ldots\ldots k_n y = X$$

where, $X = f(x)$

• Put $ax+b = e^t$ ie, $t = \log_e(ax+b)$

→ Solⁿ is linearly independent.

PARTIAL DIFFERENTIAL Eqⁿ

$$\frac{\partial z}{\partial x} = p \ , \quad \frac{\partial z}{\partial y} = q \ , \quad \frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{\partial^2 z}{\partial x \partial y} = s \ , \quad \frac{\partial^2 z}{\partial y^2} = t$$

Solⁿ of Partial Diff. Eqⁿ:

1 Eqⁿ Solvable by direct Integⁿ

Case 1: If Roots are Real & distinct

$$z = f(y+m_1x) + \phi(y+m_2x)$$

Case 2: If Roots be equal.

$$z = f(y+m_1x) + x\,\phi(y+m_2x)$$

Rules for finding PI:

Consider the Eqⁿ:

$$(D^2 + k_1 DD' + k_2 D'^2)\, z = f(x,y)$$

Case 1: $F(x,y) = e^{ax+by}$

$$PI = \frac{1}{f(D,D')} e^{ax+by} \Rightarrow \frac{1}{f(a,b)} e^{ax+by}$$

Case 2: $F(x,y) = \sin(mx+ny)$ or $\cos(mx+ny)$

$$\frac{1}{f(D^2, DD', D'^2)} \sin(mx+ny) = \frac{1}{f(-m^2, -mn, -n^2)} \cdot \sin(mx+ny)$$

Case 3: $F(x,y) = x^m y^n$

• $\frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$

Case 4: $F(x,y)$ is any funcⁿ of x, y

$$PI = \frac{1}{f(D,D')} F(x,y)$$

Resolve $\frac{1}{f(D,D')}$ into partial fracⁿ treating $f(D,D')$ as alone.

$$\frac{1}{D-mD'} F(x,y) = \int F(x, c-mx)\, dx$$

where c is replaced by $y+mx$ after Integⁿ.

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COMPLEX NUMBERS

- Every complex No. $(x+iy)$ can always be expressed in the form $r(\cos\theta + i\sin\theta)$

$x = r\cos\theta$ $\quad x+iy = r(\cos\theta + i\sin\theta)$
$y = r\sin\theta$

- **Modulus** $\Rightarrow$ $r = +\sqrt{x^2+y^2}$
- **Argument** $\Rightarrow$ $\theta = \tan^{-1}\frac{y}{x}$
- If conjugate of z be $\bar{z}$

1. $R(z) = \frac{1}{2}(z+\bar{z})$; $I(z) = \frac{1}{2}(z-\bar{z})$
2. $|z| = \sqrt{R^2(z) + I^2(z)} = |\bar{z}|$
3. $z\bar{z} = |z|^2$
4. $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$
5. $\overline{z_1 z_2} = \bar{z_1}\cdot\bar{z_2}$
6. $\overline{z_1/z_2} = \bar{z_1}/\bar{z_2}$

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PROPERTIES:

- $|z_1+z_2| \leq |z_1| + |z_2|$
- $|z_1-z_2| \geq |z_1| - |z_2|$
- $|z_1 z_2| = |z_1|\cdot|z_2|$
- $\text{Amp}(z_1 z_2) = \text{amp}(z_1) + \text{amp}(z_2)$
- $\text{Amp}(z_1/z_2) = \text{amp}(z_1) - \text{amp}(z_2)$

$\rightarrow$ If z_1, z_2, z_3 are vertices of Equil. Δ then,

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Complex func^n:

$$f(z) = u(x,y) + iv(x,y)$$

DEMOIVRE's THEOREM:

- $(\cos\theta + i\sin\theta)^n =$ $\rightarrow$ $\text{cis}\theta$
$\cos n\theta + i\sin n\theta$
- $\text{cis}\theta_1 . \text{cis}\theta_2 \ldots \text{cis}\theta_n =$
$\text{cis}(\theta_1 + \theta_2 + \ldots + \theta_n)$
- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$
$= (\cos\theta + i\sin\theta)^{-n}$
- $(\text{cis}\, m\theta)^n = \text{cis}\, mn\theta = (\text{cis}\, n\theta)^m$

Exponential func^n of Complex Variable:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \frac{x^n}{n!} \ldots \infty$$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots \frac{z^n}{n!} + \ldots \infty$$

Properties:

1. $\exp z = re^{i\theta}$

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \ldots \infty$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!}\right.$$

$$= \cos y + i\sin y$$

$$e^z = e^x . e^{iy} = e^x(\cos y + i\sin y)$$

$$\therefore e^z = re^{i\theta}$$

2. e^z is a periodic func^n having imaginary period $2\pi i$

$$e^{z+2\pi i} = e^z . e^{2\pi i} = e^z$$

3. $e^z \neq 0$ for any value of 'z'

4. $e^{\bar{z}} = \overline{e^{z}}$

$$e^{\bar{z}} = e^{x-iy} = e^{x}\cdot e^{-iy} = e^{x}(\cos y - i\sin y)$$
$$= \overline{e^{x}(\cos y + i\sin y)} = \overline{e^{z}}.$$

CIRCULAR funcⁿ of a COMPLEX VARIAB.

- $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$; $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$

EULER's Theorem :

$e^{iz} = \cos z + i\sin z$ * where $z = x+iy$.

Hyperbolic funcⁿ :

$\sinh x = \dfrac{e^{x}-e^{-x}}{2}$; $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

Relation between Hyberbolic & Circular fⁿ.

$\sin ix = i\sinh x$

$\cos ix = \cosh x$

$\tan ix = i\tanh x$

$\sinh ix = i\sin x$

$\cosh ix = \cos x$

$\tanh ix = i\tan x$

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Formulae for Hyperbolic funcⁿ :-

- $\cosh^2 x - \sinh^2 x = 1$
- $\operatorname{sech}^2 x + \tanh^2 x = 1$
- $\coth^2 x - \operatorname{cosech}^2 x = 1$
- $\sinh(x\pm y) = \sinh x\cosh y \pm \cosh x\cdot\sinh y$
- $\cosh(x\pm y) = \cosh x\cdot\cosh y \pm \sinh x\cdot\sinh y$
- $\tanh(x\pm y) = \dfrac{\tanh x \pm \tanh y}{1\pm \tanh x\cdot\tanh y}$

- $\sinh 2x = 2\sinh x\cdot\cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
  $= 2\cosh^2 x - 1$
  $= 2\sinh^2 x + 1$
- $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$
- $\sinh 3x = 3\sinh x + 4\sinh^3 x$
- $\cosh 3x = 4\cosh^3 x - 3\cosh x$
- $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$
- $\sinh x + \sinh y = 2\sinh\dfrac{x+y}{2}\cdot\cosh\dfrac{x-y}{2}$
- $\sinh x - \sinh y = 2\sinh\dfrac{x-y}{2}\cdot\cosh\dfrac{x+y}{2}$
- $\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cdot\cosh\dfrac{x-y}{2}$

INVERSE HYPERBOLIC

- $\sinh^{-1} z = \log\left[z+\sqrt{z^2-1}\right]$
- $\cosh^{-1} z = \log\left[z+\sqrt{z^2-1}\right]$
- $\tanh^{-1} z = \dfrac{1}{2}\log\dfrac{1+z}{1-z}$

LOGARITHMIC FUNCⁿ

if $z(=x+iy)$ & $w(=u+iv)$ be so

So related that $e^{w} = z$, then w is said to be logarithmic of z.

$$e^{w+2in\pi} = e^{w} \cdot e^{2in\pi} = z$$

$$\boxed{\underbrace{\log z}_{\text{Multivalued func}^n} = w + 2in\pi}$$

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General Values of log z :

$$\operatorname{Log} z = 2in\pi + \log z$$

$$\underbrace{\operatorname{Log}(x+iy)}_{\text{Real Part after solving the Eq}^n \Rightarrow} = 2in\pi + \log(x+iy)$$

$Re = \log\sqrt{x^2+y^2}$; $Im = 2n\pi + \tan^{-1}\frac{y}{x}$

Real and Imag. part of $(\alpha+i\beta)^{x+iy}$

$Re = x\log r - y(2n\pi + \theta)$, $\theta \rightarrow \tan^{-1}(\beta/\alpha)$

$Im = y\log r + x(2n\pi+\theta)$, $r \rightarrow \sqrt{\alpha^2+\beta^2}$

Calculus of complex func^n :

Derivative of $f(z)$

- Let $w = f(z)$

$$\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

- Necessary and sufficient Cond^n for the derivative of the func^n $w = u(x,y) + iv(x,y) = f(z)$ to exist for all values of z.

1. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are contin. func^n of x in R.

2. CAUCHY - RIEMANN Eq^n:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial y}$$

ANALYTIC FUNCTION

- A func^n $f(z)$ which is single valued & posses a unique derivative wrt z at all points of a region R is called an ANALYTIC FUNC^n.
- A func^n $f(z)$ analytic at each and every point of complex plane is called as ENTIRE FUNC^n.

$f(z) = u + iv$ be ANALYTIC FUNCTION if it follows CAUCHY-RIEMANN EQ^n

Polar form of C-R Eq^n :

- $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$
- $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Harmonic Functions

If $f(z) = u + iv$ be an Analytic funcⁿ in Some region of z-plane, then if u & v are said to Harmonic funcⁿ if they also satisfy LAPLACE EQⁿ.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

diff. wrt x –

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^2 v}{\partial x \partial y}$$

diff. wrt y –

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

+

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 v}{\partial y^2} = 0$$

LAPLACE EQⁿ

u, v are Harmonic funcⁿ. and their theory is called Potential theory.

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COMPLEX INTEGRATION

$$f(z) = u(x,y) + i\,v(x,y)$$

$$dz = dx + i\,dy$$

$$\int_c f(z)\,dz = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

→ The value of Integral is Independent of the path of Integration when the Integrand is Analytic

CAUCHY's theorem:

if $f(z)$ is Analytic and $f'(z)$ is continuous at each point within & on a closed curve C, then:

$$\oint_c f(z)\,dz = 0$$

CAUCHY's INTEGRAL FORMULA

if $f(z)$ is Analytic within & on a closed curve & if 'a' is any point within c, then

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$$

$$f^n(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)\,dz}{(z-a)^{n+1}}$$

TAYLOR's SERIES

if $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots\ldots$$

LAURENT's SERIES

if $f(z)$ is analytic in the ring shaped region R bounded by 2 concentric circles C_1 & C_2 of radii r & r_1 $(r > r_1)$ & within centres at a, then for all z in R.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

- $$\boxed{a_n = \frac{1}{2\pi i}\oint \frac{f(t)\,dt}{(t-a)^{n+1}}}$$

ZEROS OF ANALYTIC FUNCTION

- A zero of an analytic func^n $f(z)$ is that value of z for which $f(z)=0$

→ If $f(z)$ is Analytic in the Neighbourhood of a point $\underline{z=a}$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n$$

where, $a_n = \dfrac{f^n(a)}{n!}$

SINGULARITIES OF AN ANALYTIC FUNC^n

- Singular point of a func^n is the point at which func^n ceases to be Analytic.

① ISOLATED SINGULARITY :

If $z=a$ is a singularity of $f(z)$ ie, $f(z)$ is analytic at each point in its neighbourhood (ie, there exists a circle with centre 'a' which has no other singularity) then $z=a$ is called an Isolated singularity.

② Removable SINGULARITY :

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

If all the -ve powers of $(z-a)$ are zero, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$

Hence singularity can be removed by defining $f(z)$ at $z=a$ in such a way that it becomes analytic at $z=a$.

Thus, if $\lim f(z)$ exists finitely then, $z \to a$ is a removable singularity.

③ POLES :

If all the -ve powers of $(z-a)$ after the nth are missing then the singularity at $z=a$ is called a pole of order n.

- a pole of 1st order is called simple pole.

④ Essential Singularity :

If the No. of "-ve" powers of $(z-a)$ is Infinite then $z=a$ is called an essential singularity.

$\lim_{z \to a} f(z)$ does not exist.

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RESIDUES :

The coeff. of $(z-a)^{-1}$ in the exp^n of $f(z)$ around an isolated singularity is called the Residue of $f(z)$ at that point.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$
$$\dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

Res. of $f(z)$ at $z=a$ is a_{-1}

$$\oint_c f(z)\,dz = 2\pi i \text{ Res } f(a)$$

Residue theorem :

$$\oint f(z)\,dz = 2\pi i \times (\text{sum of Residues at singular pts. within } C)$$

- if $f(z)$ has a simple pole at $z=a$ then,

$$\text{Res } f(a) = \lim_{z \to a} [(z-a) f(z)]$$

- if $f(z)$ has a pole of order n at $z=a$, then

$$\text{Res } f(a) = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)] \right\}_{z=a}$$

- Let $f(z) = \dfrac{\phi(z)}{\psi(z)} \Rightarrow \text{Res } f(a) = \dfrac{\phi(a)}{\psi'(a)}$

$(z-a)F(z)$

nk DIFFERENTIAL CALCULUS

Fundamental Theorems

① ROLLE's Theorem:

If:-

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) $f'(x)$ exists for every value of x in the open interval (a, b) &

(iii) $f(a) = f(b)$

then there is atleast one value 'c' of x in (a, b) for which $f'(c) = 0$

2. LAGRANGE's Mean Value Theorem (LMVT):- If

(i) $f(x)$ is continuous in the closed interval $[a, b]$ and

(ii) $f'(x)$ exists in the open interval (a, b)

then there is atleast one value 'c' of x in (a, b) for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

3 CAUCHY's Mean Value theorem: If

(i) $f(x)$ & $g(x)$ be continuous in $[a, b]$

(ii) $f'(x)$ & $g'(x)$ exist in (a, b) &

(iii) $g'(x) \neq 0$ for any value of x in (a, b)

then there is atleast one value 'c' of x in (a, b) ie,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

4. TAYLOR's Theorem: (generalised MVT)

If

(i) $f(x)$ & its first $(n-1)$ derivatives be continuous in $[a, a+h]$ &

(ii) $f^n(x)$ exists for every value of 'x' in $(a, a+h)$, then there is atleast one no. 'θ' $(0 < \theta < 1)$ ie, $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots \frac{h^n}{n!} f^n(a+\theta h)$

cor. 1 : For $n = 1$, Taylor's theorem reduces to Lagrange's MVT

cor. 2 : Put $a = 0$, $h = x$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots\dots + \frac{x^n}{n!} f^n(\theta x)$$

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$\Downarrow$

Maclaurin's Theorem.

Expansion of FUNCTIONS:

1) Maclaurin's Series:

If $f(x)$ can be expanded as an Infinite series, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots\dots$$

- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots\dots$
- $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \dots\dots$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots\dots$
- $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \dots\dots$
- $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15} x^5 + \dots\dots$
- $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} \dots\dots$
- $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dots$
- $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots$
- $\log(1-x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \dots\right)$
- $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!} x^2 + \dfrac{n(n-1)(n-2)}{3!} x^3 + \dots\dots$

2) Taylor's Series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots\ldots$$

• If $f(x)$ posses derivatives of all orders of $R_n \to 0$ as $n \to \infty$ then Taylor's theor. becomes Taylor's Series.

Cor. :- Replacing $x \to a$ & $h \to x-a$ we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots$$

For $a=0$, Maclaurin's series.

INDETERMINATE FORMS:

• $$\lim_{x\to a} \frac{f(x)}{\phi(x)} = \lim_{x\to a} f(x) / \lim_{x\to a} \phi(x)$$

① if Form $\frac{0}{0}$:-

$$\lim_{x\to a} \frac{f(x)}{\phi(x)} = \frac{f^n(a)}{\phi^n(a)} = \lim_{x\to a} \frac{f^n(x)}{\phi^n(x)}$$

② if Form $\frac{\infty}{\infty}$:-

Use L' Hospital's Rule:

• $$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$$

• $$\lim_{x\to 0}(1+x)^{1/x} = e$$

③ Forms reducible to $\frac{0}{0}$ forms :-

I) Form $0 \times \infty$

If $\lim_{x\to 0} f(x) = 0$ & $\lim_{x\to\infty} \phi(x) = \infty$

$\therefore \lim_{x\to a} f(x)\cdot g(x)$ assumes form $0\times\infty$

$$\therefore f(x)g(x) = \frac{f(x)}{1/\phi(x)} \quad (0/0)$$

$$= \frac{\phi(x)}{1/f(x)} \quad (\infty/\infty)$$

II Forms $\infty - \infty$

if $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} \phi(x)$

then, $\lim_{x\to a}[f(x) - \phi(x)]$ assumes the form $\infty - \infty$

$$f(x) - g(x) = \left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\cdot\phi(x)}$$

III Forms $0^0, 1^\infty, \infty^0$

If $y = \lim_{x\to a}[f(x)]^{\phi(x)}$ assumes one of these forms, then

$$\log y = \lim_{x\to a} \phi(x)\log f(x)$$

$$\log y = l \quad \therefore \quad y = e^l$$

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TANGENTS & NORMALS

1) Eq^n of the Tangent

$$Y - y = \frac{dY}{dx}(X - x)$$

Intercept which the tangent cuts off from x- axis $= x - Y\Big/\frac{dY}{dx}$

Intercept which the tangent cuts off from y-axis $= Y - x\Big/\frac{dY}{dx}$

2) Eq^n of the Normal

$$Y - y = -\frac{dx}{dy}(X - x)$$

$$\underbrace{m}_{\text{Slope of tangent}} \cdot \underbrace{m'}_{\text{Slope of Normal}} = -1$$

3) Angle of Intersection of 2 Curves:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

where m_1 & m_2 are slopes of tangent at point of intersec^n.

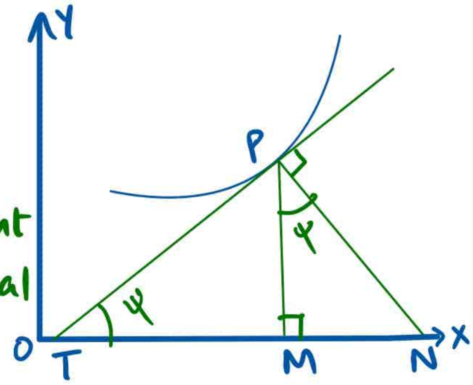
4) Length of Tangent, Normal, Subtangent & SubNormal:

PT = Length of Tangent

PN = Length of Normal

TM = Length of Sub-Tangent

MN = Length of Sub-Normal



• Length of Tangent $= y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

• Length of Normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

• Length of Sub-Tangent $= y\,dx/dy$

• Length of Sub-Normal $= y\,dy/dx$

Derivative of Arc

1) For the curve $y=f(x)$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

2) For the curve $x=f(y)$

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

3) For the parametric form, $x=f(t)$, $y=\phi(t)$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

4) For the curve, $r=f(\theta)$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

5) For the curve, $\theta=f(r)$

$$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

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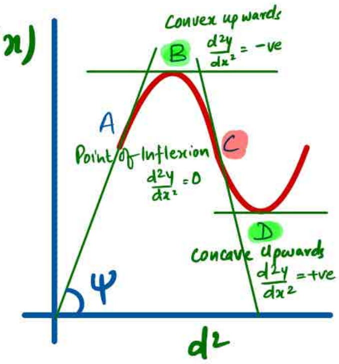
INCREASING & DECREASING FUN

In the func^n $y=f(x)$

if $y\uparrow$ as $x\uparrow$

(as at A) it is

called Increasing



In the func^n $y=f(x)$

if $y\downarrow$ as $x\uparrow$

(as at A) it is

called decreasing function

$$\frac{dy}{dx} = \psi$$

• ψ is Acute, ie $\frac{dy}{dx}$ is '+ve'

it is Increasing

• ψ is obtuse, ie $\frac{dy}{dx}$ is '−ve'

it is decreasing

• ψ is zero, ie $\frac{dy}{dx} = 0$ then it is neither $\uparrow$ing nor $\downarrow$ing it is called stationary function.

Point of Inflexion: $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$

MAXIMA & MINIMA

• if a func^n is max/min at $x=a$, then

$(dy/dx)_{x=a} = 0$

• Around a Max point, the curve is increasing in a small interval $(a-h, a)$ & decreasing in $(a, a+h)$ where h is +ve & small.

$(a-h, a)$, $\frac{dy}{dx} \geq 0$

at $x=a$, $\frac{dy}{dx} = 0$

$(a, a+h)$, $\frac{dy}{dx} \leq 0$ hence, $f(x)$ is Max^m at $x=a$

if $f'(a)=0$ & $f''(a) = -ve$

Similarly

→ $f(x)$ is minimum at $x=a$ if $f'(a)=0$ & $f''(a) = +ve$

PARTIAL DIFFERENTIATION

Partial Derivatives :

Let $z = f(x,y)$

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

- $\frac{\partial^2 z}{\partial x^2} = f_{xx}$ • $\frac{\partial^2 z}{\partial y^2} = f_{yy}$
- $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}$ $\Rightarrow$ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$
- $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}$

Homogeneous functions

$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots a_n y^n$

$= x^n[a_0 + a_1 (y/x) + a_2 (y/x)^2 + \ldots a_n (y/x)^n]$

$\Rightarrow$ Every term is of n^{th} degree.

EULER's Theorem on homogeneous func^n :

If 'u' be a homogeneous func^n of degree 'n' in x & y then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Total Derivative :

If $u = f(x,y)$, where $x = \phi(t)$ & $y = \psi(t)$ then we can express 'u' as a func^n of 't' alone by substituting value of x & y in $f(x,y)$

$\frac{du}{dt}$ = Total derivative of u

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Differentiation of Implicit func^n :

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}$$

- if $f(x,y) = c$, $\frac{dy}{dx}$ represents the 1st diff coeff. of an Implicit func^n
- if $u = f(x,y)$

$$du = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy$$

Maxima and Minima of func^n of 2 Variables

- find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ & equate each to zero

Let (a,b) & (c,d) be pair of Values.

- Calculate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

for each pair of Values.

→ if $rt - s^2 > 0$ and $r < 0$ at (a,b) f(a,b) is a max. value.

→ if $rt - s^2 > 0$ and $r > 0$ at (a,b) f(a,b) is a min. value.

→ if $rt - s^2 < 0$ at (a,b) f(a,b) is not an extreme value ie, (a,b) is a SADDLE POINT.

→ if $rt - s^2 = 0$ at (a,b) case is doubtful & needs further Investigation.

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INTEGRAL CALCULUS

Definite Integrals Properties :

I : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ ie, the value of a definite integral depends on the limits and not on the variable of integration.

II : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

III : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx$

IV : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

V : $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx \text{ ; } f(x) \text{ is an EVEN func}^n \\ 0 \text{, if } f(x) \text{ is an ODD func}^n \end{cases}$

- If $f(x)$ is EVEN func^n $\therefore f(-x) = f(x)$
- If $f(x)$ is ODD func^n $\therefore f(-x) = -f(x)$

VI : $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

if $f(2a-x) = f(x)$

$\int_0^{2a} f(x)\,dx = 0$

if $f(2a-x) = -f(x)$

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Corollary : If n is even

$\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cdot \cos^n x\,dx$

if n is odd

$\int_0^{\pi} \sin^m x \cos^n x\,dx = 0$

if m is even

$\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi} \sin^m x \cdot \cos^n x\,dx$

if m is odd

$\int_0^{\pi} \sin^m x \cos^n x\,dx = 0$

* $\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\cos x)\,dx = -\frac{\pi}{2}\log 2$

$\int_0^{\pi} \log(\sin x)\,dx = -\pi \log 2$

* $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

$= \frac{(m-1)(m-3)\ldots\ldots(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)\ldots\ldots} \times \frac{\pi}{2}$ (m & n are even)

Walle's formula

$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)\ldots\ldots 2}{n(n-2)(n-4)\ldots 3}$

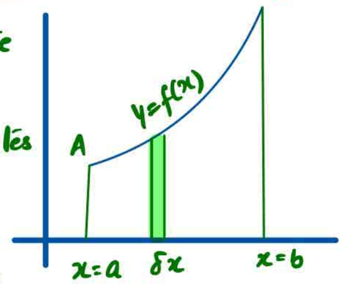
(n = odd)

$= \frac{(n-1)(n-3)\ldots\ldots}{n(n-2)\ldots\ldots} \cdot \frac{\pi}{2}$ (n = even)

AREA of Cartesian Curves :

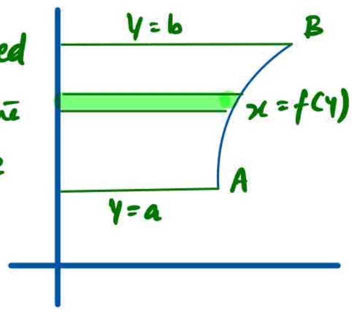
① Area bounded by the curve $y = f(x)$, the x-axis & the ordinates $x=a$, $x=b$ is

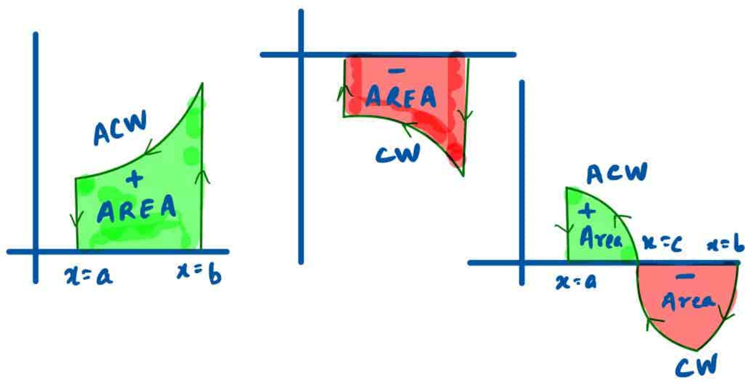
$\int_a^b y\,dx = \int_a^b f(x)\,dx$



② Similarly, Area bounded by the curve $x = f(y)$, the y-axis & the abscissae $y=a$ & $y=b$ is

$\int_a^b x\,dy = \int_a^b f(y)\,dy$





- Area of polar curves bounded by $r = f(\theta)$ & the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

Length of CURVES

1.) The length of the arc of the curve $y = f(x)$ b/w the points $x=a$ & $x=b$ is given by –

$$l_{arc} = \int_a^b \sqrt{\left(1+\left(dy/dx\right)^2\right)}\,.dx$$

2.) $x = f(y)$ from $y=a$ to $y=b$

$$l_{arc} = \int_a^b \sqrt{\left(1+\left(dx/dy\right)^2\right)}\,.dy$$

3) $x = f(t)$, $y = \phi(t)$ from $t=a$ to $t=b$ is

$$l_{arc} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

4) $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$

$$l_{arc} = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,.d\theta$$

5) $\theta = f(r)$ from $r = a$ to $r = b$.

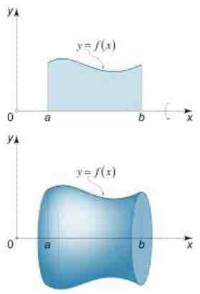
$$l_{arc} = \int_a^b \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\,dr$$

Volumes of Revolution

1) Revolution about x-axis:

Volume of solid generated by the revol^n about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis & $x=a$ & $x=b$ is

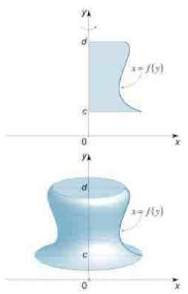
$$Vol_{x\text{-}axis} = \int_a^b \pi y^2\,dx$$



2) Revolution about y-axis:

Volume of solid generated by the revol^n about the y-axis, of the area bounded by the curve $x = f(y)$, the y-axis & $y=a$ & $y=b$ is

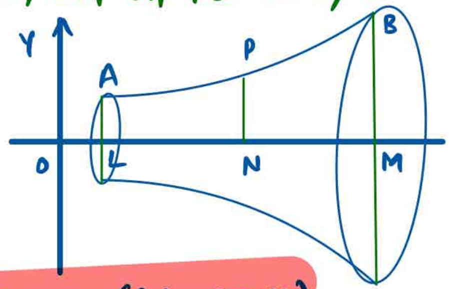
$$Vol_{y\text{-}axis} = \int_a^b \pi x^2\,dy$$



3) Revolution about any axis

The Volume of the solid generated by the revolu^n about any axis LM of the area bounded by the curve AB, the axis LM & the perpendiculars AL, BM on the axis, is

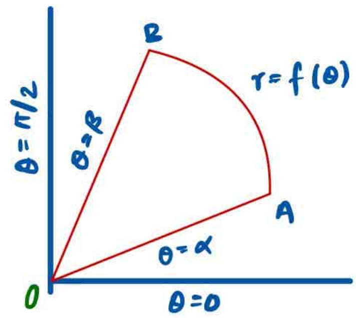
$$\int_{OL}^{OM} \pi (PN)^2\,d(ON)$$



4) Volumes of Revolution (Polar curves)

* About OX ($\theta = 0$)

$$= \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\,d\theta$$



* About OY ($\theta = \pi/2$)

$$= \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta\,d\theta$$

SURFACE AREA OF REVOLUTION

① Revolution about x-axis:

Curve $y = f(x)$ from $x = a$ to $x = b$ is

$$S = \int_a^b 2\pi y\,ds$$

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* For curve $y = f(x)$

$$S = \int 2\pi y \frac{ds}{dx}\,.dx, \quad \frac{ds}{dx} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

* For curve $x = f(t)$ & $y = \phi(t)$

$$S = \int 2\pi y \frac{ds}{dt}\,.dt, \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

* For curve $r = f(\theta)$

$$S = \int 2\pi y \frac{ds}{d\theta}\,d\theta, \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$y = r\sin\theta$$

② Revolution about y-axis:

$x = f(y)$ from $y = a$ to $y = b$.

$$S = \int_a^b 2\pi x\,ds$$

MULTIPLE INTEGRALS

Area enclosed by plane curves:

① Cartesian Coordinates

$$\text{Area} = \int_{y_1}^{y_2}\int_{f_1(x)}^{f_2(x)} dx\,dy$$

② Polar Coordinates

$$\text{Area} = \iint r\,d\theta\,dr$$

Volume of Solids

① Volume as double Integral

$$\text{Vol} = \iint z\,dx\,dy$$

② Volume as triple Integral

$$\text{Vol} = \iiint dx\,dy\,dz$$

③ Volume of Solid of Revolution

$$= \iint_A 2\pi y\,dx\,dy \quad (\text{about } x\text{-axis})$$

$$= \iint_A 2\pi x\,dx\,dy \quad (\text{about } y\text{-axis})$$

$$(\text{Vol.})_{polar} = \iint_A 2\pi r^2 \sin\theta\,d\theta\,dr$$

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VECTOR CALCULUS

Differentiation of Vectors.

ϕ – Scalar funcn   F, G, H – Vector funcn

1) $\frac{d}{dt}(F+G-H) = \frac{dF}{dt} + \frac{dG}{dt} - \frac{dH}{dt}$

2) $\frac{d}{dt}(F\phi) = F\frac{d\phi}{dt} + \frac{dF}{dt}\phi$

3) $\frac{d}{dt}(F\cdot G) = F\cdot\frac{dG}{dt} + \frac{dF}{dt}\cdot G$

* 4) $\frac{d}{dt}(F\times G) = F\times\frac{dG}{dt} + \frac{dF}{dt}\times G.$

5) $\frac{d}{dt}(FGH) = \left[\frac{dF}{dt}GH\right] + \left[F\frac{dG}{dt}H\right] + \left[FG\frac{dH}{dt}\right]$

6) $\frac{d}{dt}[(F\times G)\times H] = \left(\frac{dF}{dt}\times G\right)\times H + \left(F\times\frac{dG}{dt}\right)\times H$

$+ (F\times G)\times\frac{dH}{dt}$

* If $F(t)$ has a constant Magnitude,

then $F\cdot\frac{dF}{dt} = 0$

$F(t)\cdot F(t) = [F(t)]^2 = \text{const}$

* If $F(t)$ has constant direcn (FIXED),

then, $F\times\frac{dF}{dt} = 0$

SCALAR and Vector Point Funcn :

$$dF = \frac{\partial F}{\partial x}\cdot dx + \frac{\partial F}{\partial y}\cdot dy + \frac{\partial F}{\partial z}\cdot dz$$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$dF = (\nabla\cdot dR)F$$

GRADIENT :– (∇ applied to Scalar Point funcn)

$$\text{grad } F = \nabla F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$$

grad F is a vector ⊥ to the surface $f = \text{const}$ & has a mag. equal to the rate of change of f along this Normal.

DIRECTIONAL DERIVATIVE

- directional derivative of 'f' in the direcn of unit vector (N')

$$= \nabla f \cdot \frac{N'}{|N|} \quad *$$

∇f gives the Max. rate of change of 'f'

$$\nabla f \cdot N' = |\nabla f| \cos\alpha \leq |\nabla f|$$

DEL Applied to Vector Point FUNCTION

(i) DIVERGENCE

- $\text{div } F = \nabla \cdot F = i \cdot \frac{\partial F}{\partial x} + j \cdot \frac{\partial F}{\partial y} + \hat{k} \cdot \frac{\partial F}{\partial z}$

if $F = f\hat{i} + \phi\hat{j} + \psi\hat{k}$

- $\nabla \cdot F = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$

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Physical Interpretation

Divergence V (velocity of fluid) gives the rate at which fluid is originating at a pt. per unit volume. For incompressible fluids div V = 0, such a point funcn is also called Solenoid Vector Function ($\nabla \cdot V = 0$)

2) CURL

$$\text{curl } F = \nabla \times F = \hat{i} \times \frac{\partial F}{\partial x} + \hat{j} \times \frac{\partial F}{\partial y} + \hat{k} \times \frac{\partial F}{\partial z}$$

- if $F = f\hat{i} + \phi\hat{j} + \psi\hat{k}$.

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & \phi & \psi \end{vmatrix}$$

Physical Interpretation :

if Ω is the angular velocity then velocity of any particle of the body

$$V = \Omega \times R$$

$$\Omega = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k} \ ; R = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$\Omega = \frac{1}{2} \underbrace{(\nabla \times V)}_{\text{CURL}}$$

if $\nabla \times V =$ IRROTATIONAL

SOME BASIC FORMULAE

- $\text{div grad } F = \nabla^2 F$
- * $\text{curl grad } F = \nabla \times \nabla F = 0$
- * $\text{div curl } F = \nabla \cdot \nabla \times F = 0$
- $\text{curl curl } F = \nabla \times (\nabla \times F)$
$= \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F$
$= \nabla(\nabla \cdot F) - \nabla^2 F$
- $\text{grad div } F = \nabla(\nabla \cdot F)$
$= \nabla \times (\nabla \times F) + \nabla^2 F$
- $\nabla \cdot \nabla F = \nabla^2 F$
- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla \cdot (fG) = \nabla f \cdot G + f \nabla \cdot G$
- $\nabla \times (fG) = \nabla f \times G + f \nabla \times G$

INTEGRATION OF VECTORS

if $\frac{dG(t)}{dt} = F(t)$ [$F(t)$ & $G(t)$ are Vector func^n]

then $\int F(t)dt = G(t) + C$

C → Arbitrary Const

LINE INTEGRAL

$$\int_c F(R).dR \text{ or } \int_c F(R).\frac{dR}{dt}dt$$

If $F(R) = \hat{i}f(x,y,z) + \hat{j}\phi(x,y,z) + \hat{k}\psi(x,y,z)$

& $dR = \hat{i}dx + \hat{j}dy + \hat{k}dz$ then,

$$\int_c F(R).dR = \int_c f dx + \phi dy + \psi dz$$

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CIRCULATION

If F represents the Velocity of a Fluid particle then the Integral $\int_c F.dR$ is called the Circulation of F around the Curve.

→ When the circulation of F around closed curve in a region E vanishes, F is said to be IRROTATIONAL in E

$$\text{Work done} = \int_A^B F.dR.$$

SURFACE INTEGRAL

* Normal Surface Integral of F(R) over S

$\int_S F.ds$ or $\int_S F.Nds$ N: Unit outward Normal

FLUX across a Surface

$$= \int_S F.ds$$

→ When the flux of F across every closed surface 'S' in a region 'E' vanishes, F is a said to be a Solenoidal Vector point func^n in E.

GREEN's Theorem in the Plane

→ This theorem converts a line Integral around a closed curve into a double Integral & is a special case of STOKES' THEOREM

$$\int_c (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right)dxdy$$

STOKE's THEOREM :

(Relation b/w line and Surface Integral)

If S be an open Surface bound by a closed curve C & $F = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any contin. differentiable Vector point func^n

then,

$$\int_C F.dR = \int_S \text{curl } F.Nds$$

$N = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is a unit Normal at any point of S.

$$ds = \frac{dxdy}{\cos\gamma}$$

* For any closed Surface (S),

$$\int_S \text{curl } F.ds = 0$$

VOLUME INTEGRAL

• The Volume Integral of F(R) over E is $\int_E F dV$

$$\int_E F dV = \hat{i}\iiint f dxdydz + \hat{j}\iiint \phi dxdydz + \hat{k}\iiint \psi dxdydz$$

GAUSS DIVERGENCE THEOREM

(Relation b/w Surface & Volume Integral)

If 'F' is a continously differentiable vector funcⁿ in the region E bounded by the closed surface S, then

$$\int_S F\cdot N\,ds = \int_E \operatorname{div} F\,dV$$

$$\iint_S (f\,dy\,dz + \phi\,dz\,dx + \psi\,dx\,dy) =$$

$$\iiint_E \left(\frac{\partial F}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}\right) dx\,dy\,dz$$

IRROTATIONAL FIELDS

An Irrotational field F is characterised

① $\nabla \times F = 0$

② Circulation $\int F\cdot dR$ along every closed surface is 0.

③ $F = \nabla \phi$, if the domain is simply connected.

Such a scalar funcⁿ ϕ is called

POTENTIAL

SOLENOIDAL FIELDS

A solenoidal field F is characterised by

① $\nabla \cdot F = 0$

② flux $\int F\cdot N\,ds$ across every closed surface is 0.

③ $F = \nabla \times N$

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EXTRA

BASIC INTEGRATION

- $\int \tan x = \log(\sec x)$
- $\int \cot x = \log(\sin x)$
- $\int \sec x = \log(\sec x + \tan x)$
- $\int \operatorname{cosec} x = \log(\operatorname{cosec} x - \cot x)$
- $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} = \log\left[x+\sqrt{x^2+a^2}\right]$
- * $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} = \log\left[x+\sqrt{x^2-a^2}\right]$
- $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left[x+\sqrt{a^2+x^2}\right]$
- $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$
- $\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos ax)$
- $\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

LIMIT

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- Limit of a func^n can be any real No., ∞ or $-\infty$.

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$$

Expansion:

- $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3\ldots$
- $(1-x)^{-1} = 1+x+x^2+x^3+\ldots\ldots$
- $a^x = 1+x\log a+\frac{x^2}{2!}(\log a)^2+\frac{x^3}{3!}(\log a)^3\ldots$
- $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots$
- * $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\ldots$
- $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\ldots$
- $\tan x = x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\ldots$
- $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$
- $\log(1-x) = -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$
- $\sin^{-1}x = x+\frac{x^3}{6}+\frac{3}{40}x^5$
- $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-$
- $\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$
- $\cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$

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- $\lim_{x\to 0}(1+x)^{1/x} = e$
- $\lim_{x\to 0}(1+nx)^{1/x} = e^n$
- $\lim_{x\to \infty}(1+nx)^{1/x} = e$
- $\lim_{x\to \infty}\left(1+\frac{a}{x}\right)^x = e^a$

CONTINUITY

$f(x)$ is said to be continuous at $x=a$ if:

1) If $f(a)$ is a definite No.
2) $\lim_{x\to a} f(x)$ exist & is equal to $f(a)$

ie, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

DIFFERENTIABILITY

$f(x)$ is said to be differentiable at $x=a$ if:

$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$ or $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

exist finitely and denoted by $f'(a)$

LHD $\lim_{h\to 0}\frac{f(a-h)-f(a)}{(a-h)-a}$, $h>0$

RHD $\lim_{h\to 0}\frac{f(a-h)-f(a)}{(a+h)-a}$, $h<0$

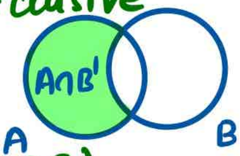
$$RHD = LHD$$

* If a func^n is differentiable at any point, then it is necess. continuous at that pt., but the converse is not true.

PROBABILITY & DISTRIBUTION

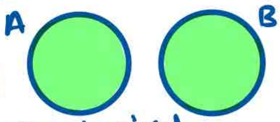
Notation :

- A or B $\Rightarrow$ $P(A+B)$ or $P(A \cup B)$
- A and B $\Rightarrow$ $P(AB)$ or $P(A \cap B)$
- A and B are Mutually Exclusive
  $A \cap B = \phi$ (NULL)



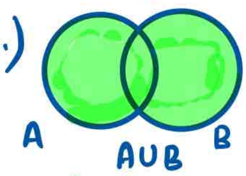
- $P(A \cap B') = P(A) - P(A \cap B)$

ADDITION LAW OF PROBABILITY / THEOREM OF TOTAL PROBABILITY.

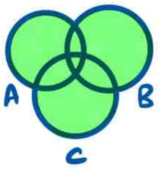


① $P(A \cup B) = P(A) + P(B)$
(if A & B are Mutually Exclusive)

② $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(if A & B not Mutually excl.)



③ $P(A+B+C) = P(A) + P(B) + P(C)$
$- P(AB) - P(BC) - P(CA) + P(ABC)$



or

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

INDEPENDENT EVENTS

→ 2 events are said to be Independent if happening or failure of one doesnot affect the hapenning or failure of other

① CONDITIONAL PROBABILITY

$P(B/A)$: the probability of occurence of B, when A has already occured.

② MULTIPLICATION LAW OF PROBAB.

$P(AB)$ or $P(A \cap B) = P(A) \cdot P(B/A)$

③ if A and B are Independ. events ie, happening of B doesnot depend on whether A has happened or not, then

$P(B/A) = P(B)$ & $P(A/B) = P(A)$

$P(AB) = P(A) \cdot P(B)$

COROLLARY : if P_1 & P_2 are probabities of happening of 2 Independent Events, then :

1 Probability that the 1st event happens and the 2nd fails is $P_1(1-P_2)$

2 Probability that both the events fail to happen is
$(1-P_1)(1-P_2)$

3 Probability that atleast one of the events happens is :
$1-(1-P_1)(1-P_2)$

Discrete Probability Distribution

If the probability that x takes the values x_i as P_i, then

$P(X = x_i) = P(x_i)$

(i) $P(x_i) \geq 0$ & (ii) $\Sigma P(x_i) = 1$

Eg:-

$X=x_i$	2	3	4	5	6	7	8	9
$P(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$

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DISTRIBUTION FUNCTION

The distribution func^n^ $F(x)$ of the discrete variate X is defined by:

$$F(x) = P(X \le x) = \sum_{i=1}^{x} P(x_i)$$

The graph of F will be stair step form

CONTINUOUS PROBABILITY DISTRIB.

When a variate X takes every value in an Interval, it give rise to continuous distrib^n^ of X.

$$P\left(x - \tfrac{1}{2}dx \le x \le x + \tfrac{1}{2}dx\right) = f(x) \cdot dx$$

$f(x)$ is called Probability density func^n^

we get a continuous curve.

EXPECTATION:

The mean value (μ) of the probability distribution of a variate X is commonly known as Expectation.

$$E(x) = \sum_i x_i f(x_i) \quad \text{— DISCRETE}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx \quad \text{— CONTINUOUS}$$

Expectation of any func^n^ $\phi(x)$

$$E[\phi(x)] = \sum_i \phi(x_i) f(x_i)$$

$$E(x) = \int_{-\infty}^{\infty} \phi(x) f(x)\,dx.$$

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VARIANCE:

$$\sigma^2 = \sum_i (x_i - \mu)^2 f(x_i)$$

σ : Std. dev.

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx.$$

Mean deviation from Mean.

- $\sum |x_i - \mu| f(x_i)$
- $\int_{-\infty}^{\infty} |x - \mu| f(x)\,dx$

REPEATED TRIALS

*Prob. of r success $= {}^nC_r\, p^r q^{n-r}$

Prob. of atleast r success in n trials:

= Sum of prob. of $r, r+1, \ldots n$ successes.

$$= {}^nC_r\, p^r q^{n-r} + {}^nC_{r+1}\, p^{r+1} q^{n-r-1} + \ldots \ldots\ {}^nC_n\, p^n$$

BINOMIAL DISTRIBUTION

it is used where only the occurence or Non-occurence, Success or failure, acceptance or Rejection, Yes or No of a particular event is of Interest.

Probabilities of $0, 1, 2, \ldots r, \ldots n$ successes are given by

$$q^n,\ {}^nC_1\, p q^{n-1},\ {}^nC_2\, p^2 q^{n-2}, \ldots\ {}^nC_r\, p^r q^{n-r} \ldots, p^n$$

Sum of Probabilities $= (p+q)^n = 1$

Mean $= np$

$S.D. (\sigma) = \sqrt{npq}$

POISSON DISTRIBUTION

→ It is used where the occurence is extremely Rare but have a large No. of Independent opportunities for occurence.

* Limiting case of Binomial Distrib. ie, $n \to \infty$ & $P \to 0$, keeping $(np = m)$ fixed.

$$P(r) = \frac{m^r}{r!} e^{-m} \qquad (m = n \cdot p)$$

$$\sigma = \sqrt{m}$$

NORMAL DISTRIBUTION

Let us define a variate $z = \dfrac{x - np}{\sqrt{npq}}$

x is a binomial variate with Mean $= np$ and $\sigma = \sqrt{npq}$ so that z is a variate with mean zero & variance unity

* The limiting form of Binomial with a large values of n when neither p nor q is very small, is called Normal Distrib.

Normal curve is of the form

$$Y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Properties of Normal Distrib.

- Bell shaped
- Max^{m} ordinate $= \dfrac{1}{\sigma\sqrt{2\pi}}$
- It is symm. so Mean, Median & Mode are same.
- Point of Inflexion $x = \mu \pm \sigma$.
- Mean deviation from Mean μ

$$= \int_{-\infty}^{\infty} |x-\mu| \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{4}{5}\sigma$$

- Probability of x lying b/w x_1 and x_2: $P(x_1 \le x \le x_2) =$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Also, $$P(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz$$

- Normal frequency distrib.

$$Y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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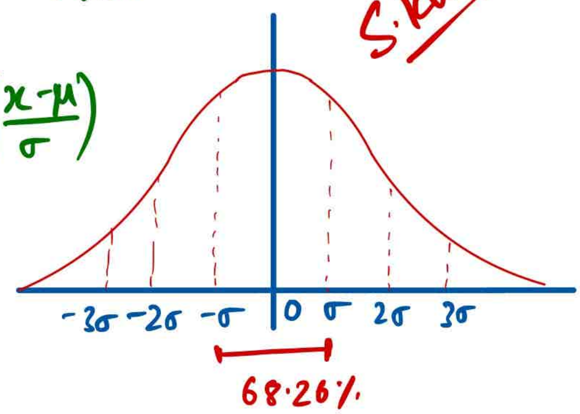
$$z = \left(\frac{x-\mu}{\sigma}\right)$$



$$\sigma^2 = \text{Variance} = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

* Normal Distribution Ques can be easily solves by Area under the curve, Area represents the probability.

NUMERICAL METHODS

Numerical Integration

1 Newton-cotes Quadrature formula

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \dots\right].$$

2. Trapezoidal Rule

$n=1$ in Quadrature formula.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots y_{n-1})\right]$$

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3 SIMPSON's 1/3 RULE

$n=2$ in quadrature formula

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\Big[(y_0+y_n) + 4(y_1+y_3+\dots y_{n-1}) + 2(y_2+y_4+\dots y_{n-2})\Big]$$

$$= \frac{h}{3}\Big[(y_0+y_n) + 4\underbrace{O}_{\Sigma \text{ODD}} + 2\underbrace{E}_{\Sigma \text{Even.}}\Big]$$

4. SIMPSON 3/8 Rule

$n=3$ in quadrature formula

$$\int_{x_0}^{x_0+nh} = \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+\dots y_{n-1}) + 2(y_3+y_6+\dots y_{n-3})\Big]$$

Simpson 1/3 Rule for 3 points or 2 Intervals

$$I = \frac{h}{3}(y_0+4y_1+y_2)$$

Error in Quadrat. formula

$$E = \int_a^b y\,dx - \int_a^b p(x)\,dx$$

$p(x)$ is the polynomial represent. func^n $y=f(x)$ in $[a,b]$

Error in Trapezoidal

Total Error $E =$

$$-\frac{h^3}{12}\left[y_0'' + y_1'' + \dots y_{n-1}''\right]$$

expanding it we get terms of h^2

Error in Trapezoidal Rule is of ORDER(h^2)

Error in Simpson $\frac{1}{3}$ Rule is of the ORDER (h^4)

Methods used for Ist Order DE :-

1. Euler's Method
2. Mod. Euler's Method
3. Runge kutta method

EULER's METHOD

- $Y_1 = Y_0 + hf(x_0, Y_0)$
- $Y_2 = Y_1 + hf(x_0+h, Y_1)$
- $Y_n = Y_{n-1} + hf(x_0 + \overline{n-1}\,h, Y_{n-1})$

where $f(x,y) = \frac{dy}{dx}$; $y(x_0) = Y_0$.

MODIFIED EULER's METHOD :

$Y_1 = Y_0 + hf(x_0, Y_0)$

for better approx. $Y_1^{(1)}$ of $y(x_0+h)$ by taking the slope of curve as mean of the slope of the tangent.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, Y_0) + f(x_0+h, Y_1)\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1^{(1)})\right]$$

$$y_2 = y_1 + hf(x_0+h, y_1)$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_0+h, y_1) + f(x_0+2h, y_2)\right]$$

RUNGE's METHOD

$K_1 = hf(x_0, y_0)$ $\qquad Y = Y_0 + K$

$K_2 = hf\left(x_0 + \frac{h}{2}, Y_0 + \frac{K_1}{2}\right)$

$K' = hf(x_0+h, Y_0+K_1)$

$K_3 = hf(x_0+h, Y_0+K')$

$$K = \frac{1}{6}(K_1 + 4K_2 + K_3)$$

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Also, referred as RUNGE-KUTTA METHOD

RUNGE-KUTTA METHOD

1 1st order R-K Method

- $Y_1 = Y_0 + hf(x_0, Y_0)$
  $= Y_0 + hY_0'$

>> Euler's Method is the R-K method of 1st order

2 2nd order R-K Method

- $K_1 = hf(x_0, Y_0)$
  $K_2 = hf(x_0+h, Y_0+K_1)$

$$\boxed{Y_1 = Y_0 + \frac{1}{2}(K_1 + K_2)}$$

>> Modified Euler's is R-K Method of 2nd order.

3. 3rd order R-K Method

- $K_1 = hf(x_0, Y_0)$
- $K_2 = hf\left(x_0 + \frac{h}{2}, Y_0 + \frac{K_1}{2}\right)$
- $K' = hf(x_0+h, Y_0+K_1)$
- $K_3 = hf(x_0+h, Y_0+K')$

$$\boxed{Y_1 = Y_0 + 1/6\,(K_1 + 4K_2 + K_3)}$$

4) 4th order R-K Method

- $K_1 = hf(x_0, y_0)$
- $K_2 = hf\left(x_0 + \frac{h}{2}, Y_0 + \frac{K_1}{2}\right)$
- $K_3 = hf\left(x_0 + \frac{h}{2}, Y_0 + \frac{K_2}{2}\right)$
- $K_4 = hf(x_0+h, Y_0+K_3)$

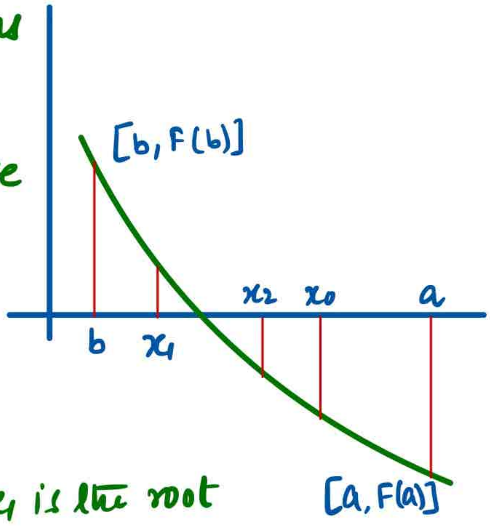
$$\boxed{Y_1 = Y_0 + 1/6\,(K_1 + 2K_2 + 2K_3 + K_4)}$$

NUMERICAL METHODS FOR ROOT FINDING

(Iterative)

① BISECTION METHOD:

- $f(x)$ be a continuous funcⁿ b/w a, b
  $f(a) = -ve$, $f(b) = +ve$



1st approx. to root,

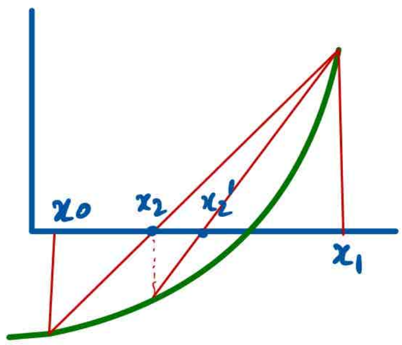
$$x_1 = \frac{1}{2}(a+b)$$

- If $f(x_1) = 0$, then x_1 is the root otherwise root lies b/w a & x_1 or x_1 & b accⁿ as $f(x_1)$ is +ve or −ve.
- At the end of nth step, the New Interval will be of length $\left(\frac{b-a}{2^n}\right)$

$$\frac{b-a}{2^n} \le \epsilon$$ → Accuracy within error Margin

② Method of FALSE POSITION / REGULA-FALSI METHOD

- 2 points 'x_0' & 'x_1' are chosen ie $f(x_0)$ & $f(x_1)$ are of opp. sign ie, a root lies b/w x_0 & x_1
  $f(x_0) \cdot f(x_1) < 0$



- Approx. to the Root $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$

* Rate of Convergence is faster than the BISECTION METHOD

③ SECANT METHOD:

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n)$$

Approx. to the root is

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_1)$$

* If the Secant method once converges its rate of is 1.6 which is faster than Method of false posⁿ

④ NEWTON-RAPHSON'S METHOD

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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