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Derivatives and its Applications

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Chapter - 1

Derivatives

1.1 Introduction:

The subject Calculus is one of the branches of Mathematics and was found by Newton and Leibniz. This is primarily concerned with two basic operations called differentiation and integration .In this section; we shall briefly formalize the idea of differentiation in terms of the notion of the limit of a function.

1.2 Derivative

A function f(x) is said to be *derivable* if

$$\lim_{x\to 0} \frac{f(x+h)-f(x)}{h} \text{ exist.}$$

We write $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ read it as *differential coefficient of f(x) with respect to x.*

A function f(x) is said to be *right hand derivative* if

$$\lim_{h\to 0+} \frac{f(x+h)-f(x)}{h} \text{ exist.}$$

We write

$$R\,f'(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}$$

Similarly, a function f(x) is said to be *left hand derivative* if

$$\lim_{h\to 0-} \frac{f(x-h)-f(x)}{-h} \text{ exist.}$$

We write

$$Lf'(x) = \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h}$$

Exercise-1

1. Find, from the first principles, the derivatives of the following functions

i. $\sqrt{x}$ ii. $\tan^{-1} x$ iii. $\log \cos x$

iv. $e^{\sqrt{x}}$ v. $e^{\cos x}$ vi. x^x

Answers

1. i. $\frac{1}{2\sqrt{x}}$ ii. $\frac{1}{1+x^2}$ iii. $-\tan x$

iv. $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ v. $-e^{\cos x} \sin x$ vi. $x^x (1 + \log x)$

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Chapter - 2

Higher Order Derivatives

2.1 Introduction:

If $y = f(x)$ is differentiable in an interval, then its derivative with respect to x is $f'(x)$, y' or $\frac{dy}{dx}$. If $f'(x)$ is also differentiable in the interval, then its derivative is given by $f''(x)$, y'' or $\frac{d^2y}{dx^2}$.

Similarly, if the f(x) is differentiable in the interval n times, then its n^{th} derivative is $f^n(x)$, y^n or $\frac{d^ny}{dx^n}$.

If $y = f(x)$, then the *higher order derivatives* are denoted by

$\frac{dy}{dx}$,	$f'(x)$,	y_1,	Dy,	$\frac{d}{dx} f(x)$
$\frac{d^2y}{dx^2}$,	$f''(x)$,	y_2,	D^2y,	$\frac{d^2}{dx^2} f(x)$
$\frac{d^3y}{dx^3}$,	$f'''(x)$,	y_3,	D^3y,	$\frac{d^3}{dx^3} f(x)$
:	:	:	:	:
:	:	:	:	:
$\frac{d^ny}{dx^n}$,	$f^n(x)$,	y_n,	D^ny,	$\frac{d^n}{dx^n} f(x)$

2.2. The n^{th} derivatives of some special functions

The n^{th} derivatives of the following functions are to be obtained by the method of induction.

1. **If $y = x^m$, then $y_n = \frac{m!}{(m-n)!} x^{m-n}$ where m is positive integer, $m > n$.**

Here $y = x^m$

Differentiating it n times successively,

$y_1 = mx^{m-1}$

or $y_2 = m(m-1)x^{m-2}$

or $y_3 = m(m-1)(m-2)x^{m-3}$

or $y_4 = m(m-1)(m-3)x^{m-4}$

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$y_n = m(m-1)(m-2)(m-3) \ldots\ldots (m-n+1)x^{m-n}$

or $y_n = m(m-1)(m-2)\ldots(m-n+1)(m-n)\ldots 3.2.1$

or $y_n = \dfrac{m(m-1)(m-2)\ldots(m-n+1)(m-n)\ldots 3.2.1}{(m-n)\ldots\ldots\ldots 3.2.1} x^{m-n}$, m>

$\therefore \quad y_n = \dfrac{m!}{(m-n)!} x^{m-n}$

Cor.

When m = n

$y_n = n!$

For example,

If $y = x^3$, then $y_3 = 3!$

For $y = x^3$

Differentiating it successively,

$y_1 = 3x^2$

or $y_2 = 3\times 2x$

or $y_3 = 3\times 2\times 1$

$\therefore \quad y_3 = 3!$

2. If $y = (ax+b)^m$, then $y_n = a^n \dfrac{m!}{(m-n)!}(ax+b)^{m-n}$ where m i positive integer, m > n.

Here $y = (ax+b)^m$

Differentiating it successively,

$y_1 = a.m(ax+b)^{m-1}$

$y_2 = a^2 m(m-1)(ax+b)^{m-2}$

$y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$

: : :

$y_n = a^n m(m-1)(m-2) \ldots\ldots\ldots (m-n+1)(ax+b)^{m-n}$

$\therefore \quad y_n = a^n \dfrac{m!}{(m-n)!}(ax+b)^{m-n}$

3. If $y = e^{ax}$, then $y_n = a^n e^{ax}$

Here $y = e^{ax}$

Differentiating it successively,

$y_1 = ae^{ax}$

or $y_2 = a^2 e^{ax}$

or $y_3 = a^3 e^{ax}$

: :

: :

$\therefore \quad y_n = a^n e^{ax}$

4. If $y = \dfrac{1}{(ax+b)}$, then $y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

Here $y = \dfrac{1}{(ax+b)}$

Differentiating it successively,

$y_1 = (-1)\dfrac{a.1}{(ax+b)^2}$

or $y_2 = (-1)^2 \dfrac{a^2.1.2}{(ax+b)^3}$

or $y_3 = (-1)^3 \dfrac{a^3.1.2.3}{(ax+b)^4}$

: :

$\therefore \quad y_n = (-1)^n \dfrac{a^n.n!}{(ax+b)^{n+1}}$

Cor. 1.

If $y = \dfrac{1}{x}$, then $y_n = \dfrac{(-1)^n n!}{x^{n+1}}$

Cor. 2.

If $y = \dfrac{1}{x+a}$, then $y_n = \dfrac{(-1)^n n!}{(x+a)^{n+1}}$

Cor. 3. If $y = \dfrac{1}{x-a}$, then $y_n = \dfrac{(-1)^n n!}{(x-a)^{n+1}}$

5. **If $y = \sin(ax + b)$, then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$.**

Here $y = \sin(ax + b)$

Differentiating it successively,

$$y_1 = a\cos(ax + b) \quad = a\sin\left(1.\frac{\pi}{2} + ax + b\right)$$

$$y_2 = -a^2 \sin(ax + b) = a^2 \sin\left(2.\frac{\pi}{2} + ax + b\right)$$

$$y_3 = -a^3 \cos(ax + b) = a^3 \sin\left(3.\frac{\pi}{2} + ax + b\right)$$

$\vdots \qquad \vdots$

$\vdots \qquad \vdots$

$$\therefore \quad y_n = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$$

Cor.

Put $b = 0$, if $y = \sin ax$, then $y_n = a^n \sin\left(\frac{n\pi}{2} + ax\right)$

6. **If $y = \cos(ax + b)$, then $y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$**

Here $y = \cos(ax + b)$

Differentiating it successively,

$$y_1 = -a\sin(ax + b) = a\cos\left(1.\frac{\pi}{2} + ax + b\right)$$

or $\quad y_2 = -a^2 \cos(ax + b) = a^2 \cos\left(2.\frac{\pi}{2} + ax + b\right)$

or $\quad y_3 = a^3 \sin(ax + b) = a^3 \cos\left(3.\frac{\pi}{2} + ax + b\right)$

$\vdots \qquad \vdots$

$\vdots \qquad \vdots$

$$\therefore \quad y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

Cor.

Put $b = 0$, if $y = \cos ax$, then $y_n = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

7. **If $y = e^{ax} \sin bx$, then $y_n = (a^2 + b^2)^{n/2} e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right)$**

Here $y = e^{ax} \sin bx$

Differentiating it with respect to x,

$$y_1 = be^{ax} \cos bx + ae^{ax} \sin bx$$

or $\quad y_1 = e^{ax}(a\sin bx + b\cos bx)$

Put $a = r\cos\theta$, $b = r\sin\theta$, so that, $a^2 + b^2 = r^2$ and $\tan\theta = \frac{b}{a}$

Thus

$$y_1 = e^{ax}(r\cos\theta \sin bx + r\sin\theta \cos bx)$$

$$= r e^{ax}(\cos\theta \sin bx + \sin\theta \cos bx)$$

$$y_1 = re^{ax} \sin(bx + \theta)$$

Differentiating it successively,

$$y_2 = r^2 e^{ax} \sin(bx + 2\theta)$$

or $\quad y_3 = r^3 e^{ax} \sin(bx + 3\theta)$

$\vdots \qquad \vdots$

$\vdots \qquad \vdots$

$$y_n = r^n e^{ax} \sin(bx + n\theta)$$

$$\therefore \quad y_n = (a^2 + b^2)^{n/2} e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right)$$

8. **If $y = e^{ax} \cos bx$, then $y_n = (a^2 + b^2)^{n/2} \cos\left(bx + n\tan^{-1}\frac{b}{a}\right)$**

We have $\quad y = e^{ax} \cos bx$

Differentiating it with respect to x, we get

$$y_1 = -b\,e^{ax} \sin bx + ae^{ax} \cos bx$$

or $\quad y_1 = e^{ax}(a\cos bx - b\sin bx)$

Put $a = r\cos\theta$, $b = r\sin\theta$ so that $a^2 + b^2 = r^2$ and $\tan\theta = \frac{b}{a}$

Thus

$$y_1 = e^{ax}(r\cos\theta\cos bx - r\sin\theta\sin bx)$$

$$= r e^{ax}(\cos\theta\cos bx - \sin\theta\sin bx)$$

$$y_1 = r e^{ax}\cos(bx + \theta)$$

Differentiating it successively,

$$y_2 = r^2 e^{ax}\cos(bx + 2\theta)$$

or $y_3 = r^3 e^{ax}\cos(bx + 3\theta)$.

: : :

: : :

or $y_n = r^n e^{ax}\cos(bx + n\theta)$

$\therefore$ $y_n = (a^2 + b^2)^{n/2} e^{ax}\cos\left(bx + n\tan^{-1}\dfrac{b}{a}\right)$

9. **If $y = \dfrac{1}{x^2 - a^2}$, then $y_n = \dfrac{(-1)^n n!}{2a}\left[\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(n+a)^{n+1}}\right]$**

Here $y = \dfrac{1}{x^2 - a^2}$

or $y = \dfrac{1}{(x-a)(x+a)}$

or $y = \dfrac{1}{2a}\left[\dfrac{1}{x-a} - \dfrac{1}{x+a}\right]$

Thus

$$y_n = \frac{1}{2a}\left[\frac{(-1)^n n!}{(x-a)^{n+1}} - \frac{(-1)^n n!}{(x+a)^{n+1}}\right]$$

$$\therefore\ y_n = \frac{(-1)^n n!}{2a}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right]$$

10. **If $y = \dfrac{1}{x^2 + a^2}$, then $y_n = \dfrac{(-1)n!}{a^{n+2}}\sin^{n+1}\left(\tan^{-1}\dfrac{a}{x}\right)\sin\left\{(n+1)\tan^{-1}\dfrac{a}{x}\right\}$**

Here $y = \dfrac{1}{x^2 + a^2}$

or $y = \dfrac{1}{x^2 - i^2a^2} = \dfrac{1}{(x+ia)(x-ia)}$

$$= \frac{1}{2ai}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$y_n = \frac{1}{2ai}\left[\frac{(-1)^n n!}{(x-ia)^{n+1}} - \frac{(-1)^n n!}{(x+ia)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2ai}\left[(x-ia)^{-(n+1)} - ((x+ia))^{-(n+1)}\right]$$

Put $x = r\cos\theta$, $a = r\sin\theta$, so that, $x^2 + a^2 = r^2$, $\tan\theta = \dfrac{a}{x}$

Thus

$$y_n = \frac{(-1)^n n!}{2ai}\left[(r\cos\theta - i\,r\sin\theta)^{-(n+1)} - (r\cos\theta + ir\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^n n!}{2ai} r^{-(n+1)}\left[\cos(n+1)\theta + i\sin(n+1)\theta - \cos(n+1)\theta + i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n n!}{2ai\,r^{n+1}} 2i\sin(n+1)\theta = \frac{(-1)^n n!}{a\,r^{n+1}}\sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$$

$$\therefore\ y_n = \frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\left(\tan^{-1}\frac{a}{x}\right)\sin\left\{(n+1)\tan^{-1}\frac{a}{x}\right\}$$

Leibnitz's Theorem

If u and v are two function of x, then the n^{th} derivative of their product is

$$(uv)_n = u_n v + c(n,1)\,u_{n-1}v_1 + c(n,2)\,u_{n-2}v_2 + \ldots\ldots\ldots + c(n,r)\,u_{n-r}v_r + \ldots\ldots\ldots + uv_n$$

where the suffices of u and v denote the order of differentiation of u and v with respect to x.

Proof:

Let $y = uv$

Differentiating it with respect to x, we get

$$y_1 = u_1v + uv_1$$

$$y_2 = u_2v + 2u_1v_1 + uv_2$$

$$y_3 = u_3v + 3u_2v_1 + 3u_1v_2 + uv_3$$

It shows that the theorem is true for n = 1, 2, 3.

Let us suppose that the theorem is true for n = m and need to show that it is true for n = m + 1.

Let $y_m = u_m v + C(m, 1)\, u_{m-1} v_1 + C(m, 2)\, u_{m-2} v_2 + \dots\dots$

$\dots\dots + C(m, r-1)\, u_{m-r+1} v_{r-1} + C(m, r)\, u_{m-r} v_r + \dots\dots + u v_m$

Differentiating it with respect to x, we get

$$y_{m+1} = u_{m+1} v + u_m v_1 + C(m, 1)\, u_m v_1 + C(m, 1)\, u_{m-1} v_2$$
$$+ C(m, 2)\, u_{m-1} v_2 + C(m, 2)\, u_{m-2} v_3 + \dots\dots$$
$$\dots\dots + C(m, r-1)\, u_{m-r+2} v_{r-1} + C(m, r-1)\, u_{m-r+1} v_r$$
$$+ C(m, r)\, u_{m-r+1} v_r + \dots\dots + u v_{m+1}$$
$$= u_{m+1} v + [1 + C(m, 1)]\, u_m v_1 + [C(m, 1) + C(m, 2)]\, u_{m-1} v_2$$
$$+ \dots\dots + [C(m, r-1) + C(m, r)]\, u_{m-r+1} v_r + \dots\dots + u v_{m+1}$$

Using $1 + C(m, 1) = C(m + 1, 1)$,

$C(m, 1) + C(m, 2) = C(m + 1, 2)$,

$C(m, r-1) + C(m, r) = C(m + 1, r)$,

We get

$$y_{m+1} = u_{m+1} v + C(m + 1, 1)\, u_m v_1 + C(m + 1, 2)\, u_{m-1} v_2 + \dots$$
$$\dots\dots + C(m + 1, r)\, u_{m-r+1} v_r + \dots\dots + u v_{m+1}$$

Thus, the theorem is true for $n = m + 1$ if it is true for $n = m$, and hence it is true for $n = 1, 2, 3$ and so on. It can be concluded that the theorem is true for every positive integral value of n.

Worked out Examples

Ex. 1: If $y = \sqrt{x}$ then find y_n.

Solution:

Given function is

$$y = \sqrt{x}$$

Differentiating it successively,

$$y_1 = \frac{1}{2} x^{-1/2}$$

or $y_2 = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-(1/2)-1}$

or $y_3 = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right) x^{-(1/2)-2}$

:: :: :: :: :: ::

or $y_n = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right) \dots\dots \left(-\frac{1}{2} - n + 2\right) x^{-(1/2)-n+1}$

$\therefore$ $y_n = \frac{(-1)^{n-1}}{2^n}\, [1.3.5 \dots\dots (2n-3)]\, x^{-(1/2)-n}$

Ex. 2: If $y = \frac{x^n}{x+1}$, then find y_n.

Solution:

Here $y = \frac{x^n}{x+1}$

By actual division

$$y = x^{n-1} - x^{n-2} + x^{n-3} \dots\dots + (-1)^n \frac{1}{x+1}$$

Differentiating it with respect to x n times,

$$y_n = 0 + (-1)^n D^n \left(\frac{1}{x+1}\right)$$

or $y_n = (-1)^n \frac{(-1)^n\, n!}{(x+1)^{n+1}}$

or $y_n = (-1)^{2n} \frac{n!}{(x+1)^{n+1}}$

$\therefore$ $y_n = \frac{n!}{(x+1)^{n+1}}$

Ex. 3: If $y = (x^2-1)^n$, then $(x^2 - 1)\, y_{n+2} + 2x\, y_{n+1} - n(n+1)\, y_n = 0$ 2059 B.E.

Solution:

Here $y = (x^2 - 1)^n$

Differentiating it with respect to x, we get

$$y_1 = n(x^2 - 1)^{n-1} \cdot 2x$$

or $y_1 = \frac{n(x^2 - 1)^n}{(x^2 - 1)}\, 2x = \frac{ny}{x^2 - 1} \cdot 2x$

or $(x^2 - 1)\, y_1 = 2nxy$

Again, differentiating it with respect to x, we get

$$(x^2 - 1)\, y_2 + 2xy_1 = 2nxy_1 + 2ny$$
$$(x^2 - 1)\, y_2 + (2x - 2nx)\, y_1 = 2ny$$

Differentiating it n times using Leibnitz's theorem

$$(x^2-1)\,y_{n+2} + \frac{n}{1!}\,y_{n+1}\,.\,2x + \frac{n(n-1)}{2!}\,y_n.2 + (2x-2nx)\,y_{n+1} + \frac{n}{1!}\,y_n\,(2-2n) = 2ny_n$$

or $$(x^2-1)\,y_{n+2} + (2nx + 2x - 2nx)\,y_{n+1} + (n^2 - n + 2n - 2n^2 - 2n)\,y_n = 0$$

$\therefore$ $$(x^2-1)\,y_{n+2} + 2xy_{n+1} - n(n+1)\,y_n = 0$$

Ex. 4: If $y = \sin mx + \cos mx$, then prove $y_n = m^n\,\{1 + (-1)^n \sin 2mx\}^{1/2}$

Solution:

Given function is

$$y = \sin mx + \cos mx$$

Using the formula for n^{th} derivative

$$y_n = m^n\left\{\sin\left(\frac{n\pi}{2} + mx\right) + \cos\left(\frac{n\pi}{2} + mx\right)\right\}$$

$$= m^n\left[\left\{\sin\left(\frac{n\pi}{2} + mx\right) + \cos\left(\frac{n\pi}{2} + mx\right)\right\}^2\right]^{1/2}$$

$$= m^n\left\{1 + 2\sin\left(\frac{n\pi}{2} + mx\right)\cos\left(\frac{n\pi}{2} + mx\right)\right\}^{1/2}$$

$$= m^n\,[1 + \sin(n\pi + 2mx)]^{1/2}$$

or $y_n = m^n\,[1 + \sin 2mx \cos n\pi + \cos 2mx \sin n\pi]^{1/2}$

Using $\sin n\pi = 0$, $\cos n\pi = (-1)^n$

$$y_n = m^n[1 + (-1)^n \sin 2mx]^{1/2}$$

Ex. 5: If $y = (x + \sqrt{1+x^2})^m$, then show that $(1 + x^2)\,y_2 + xy_1 - m^2y = 0$ and hence prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)\,y_n = 0$

Solution:

Given function is

$$y = (x + \sqrt{1+x^2})^m \quad \text{.........(1)}$$

Differentiating it with respect to x

$$y_1 = m(x + \sqrt{1+x^2})^{m-1}\left(1 + \frac{1}{2\sqrt{1+x^2}}\,2x\right)$$

or $$y_1 = m\,(x + \sqrt{1+x^2})^{m-1}\,\frac{(x + \sqrt{1+x^2})}{\sqrt{1+x^2}}$$

or $$y_1 = \frac{m(x + \sqrt{1+x^2})^m}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}} \quad \text{[from (1)]}$$

or $(1+x^2)\,y_1^2 = m^2y^2$

Again differentiating it

$$(1+x^2)\,2y_1y_2 + 2xy_1^2 = 2ym^2y_1$$

or $(1+x^2)y_2 + xy_1 - m^2y = 0$ ....(2)

Differentiating it n times by using Leibnitz's theorem

$$(1+x^2)y_{n+2} + \frac{n}{1!}\,y_{n+1}\,2x + \frac{n(n-1)}{2!}\,2 + xy^{n+1} + \frac{n}{1!}\,y_n - m^2y_n = 0$$

$\therefore$ $$(1+x^2)y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - m^2)y_n = 0$$

Ex. 6: If $y^{1/m} + y^{-1/m} = 2x$, show that

$(x^2-1)y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - m^2)y_n = 0$ 2062 B.E.

Solution:

Given function is

$$y^{1/m} + y^{-1/m} = 2x$$

Differentiating it with respect to x,

$$\frac{1}{m}\,y^{(1/m)-1}\,y_1 + \frac{(-1)}{m}\,y^{-(1/m)-1}\,y_1 = 2$$

or $(y^{1/m} - y^{-1/m})\,y_1 = 2my$

Squaring,

$$(y^{1/m} - y^{-1/m})^2\,y_1^2 = 4m^2y^2$$

or $[(y^{1/m} + y^{-1/m})^2 - 4]\,y_1^2 = 4m^2y^2$

or $(4x^2 - 4)y_1^2 = 4m^2y^2$

or $(x^2-1)\,y_1^2 = m^2y^2$

Again differentiating with respect to x.

$$(x^2-1)2y_1y_2 + 2x.y_1^2 = 2m^2yy_1$$

or $(x^2-1)y_2 + xy_1 - m^2y = 0$

Differentiating it n times by using Leibnitz's theorem

$$(x^2-1)y_{n+2} + \frac{n}{1!}\,2x\,.y_{n+1} + \frac{n(n-1)}{2!}\,2.y_n + xy_{n+1} + \frac{n}{1!}\,y_n - m^2y_n = 0$$

$$(x^2-1)\,y_{n+2} + (2nx + x)\,y_{n+1} + (n^2 - n^2 + n^2 - m^2)y_n = 0$$

$\therefore$ $$(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2 - m^2)y_n = 0$$

Ex. 7: If $y = \log\left(x + \sqrt{a^2 + x^2}\right)$, then show that $(a^2 + x^2)\, y_2 + xy_1 = 0$ and hence show that $(a^2 + x^2)\, y_{n+2} + (2n + 1)\, xy_{n+1} + n^2 y_n = 0$

Solution:

Given function is

$$y = \log\left(x + \sqrt{a^2 + x^2}\right)$$

Differentiating it with respect to x,

$$y_1 = \frac{1}{\left(x + \sqrt{a^2 + x^2}\right)}\left(1 + \frac{1}{2}\cdot\frac{2x}{\sqrt{a^2 + x^2}}\right)$$

or $\quad y_1 = \dfrac{1}{\sqrt{a^2 + x^2}}$

Again differentiating it with respect to x

$$y_2 = -\frac{1}{2(a^2 + x^2)^{3/2}}\,.\,2x,$$

or $\quad (a^2+x^2)\, y_2 = -\dfrac{x}{\sqrt{a^2 + x^2}} = -\,xy_1$

or $\quad (a^2 + x^2)\, y_2 + xy_1 = 0$

Using Leibnitz's theorem

$$(a^2 + x^2)y_{n+2} + \frac{n}{1!}2x.y_{n+1} + \frac{n(n-1)}{2!}2.y_n + xy_{n+1} + \frac{n}{1!}\, y_n = 0$$

$$\therefore \quad (a^2 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2 y_n = 0.$$

Exercise-2

1. Find the n^{th} derivative of the following:

   i. $y = (a - bx)^m$ ii. $y = (2 - 3x)^n$

   iii. $y = \dfrac{1}{\sqrt{x}}$ iv. $y = x^{2n}$

   v. $y = \dfrac{1}{a - x}$ vi. $y = \dfrac{x^n}{x - 1}$

   vii. $y = \dfrac{x}{a + bx}$ viii. $y = \dfrac{x}{x^2 + a^2}$

   (ix) $y = \tan^{-1}\left(\dfrac{1 + x}{1 - x}\right)$ (x) $y = \tan^{-1}\dfrac{\sqrt{1 + x^2} - 1}{x}$

2. If $y = \dfrac{1}{x^3}$, then show that $y_n(1) = 0$

3. Find y_n if   i. $y = e^{ax} x^3$ ii. $y = x^3 \log x$

4. If $y = \tan^{-1}x$, then prove that $(1+ x^2)\, y_{n+1} + 2nxy_n + n(n - 1)\, y_{n-1} = 0$

5. If $y = e^{a\tan^{-1}x}$, then prove that
   $(1 + x^2)\, y_{n+2} + (2nx + 2x - a)\, y_{n+1} + n(n + 1)\, y_n = 0$ [2061 B.E.]

6. If $y = ae^{mx} + be^{-mx}$, then prove that $y_2 - m^2 y = 0$

7. If $y = e^{ax}\sin bx$, then prove that
   $y_{n+1} = 2a\, y_n - (a^2 + b^2)\, y_{n-1}$

8. If $y = x^{n-1}\log x$, then prove that $xy_n = (n - 1)!$.

9. If $y = a\cos(\log x) + b\sin(\log x)$, then prove that
   $x^2 y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 + 1)\, y_n = 0$ [2060 B.E.]

10. If $y = \sin^{-1}x$, then prove that
    $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2\, y_n = 0$

11. If $y = e^{a\sin^{-1}x}$, then prove that
    $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + a^2)\, y_n = 0$

12. If $y = e^{x^2}$, then show that $y_{n+1} - 2xy_n - 2ny_{n-1} = 0$

13. If $y = (\sin^{-1}x)^2$, then show that
    $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0$

14. If $y = \sin(m\sin^{-1}x)$, then show that
    $(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} + (m^2 - n^2)\, y_n = 0$

Answers

1. i. $\dfrac{m!\,(-1)^n b^n}{(m - n)!}(a - bx)^{m-n}$ ii. $(-1)^n 3^n n!$

iii. $(-1)^n \dfrac{1.3.5 \ldots\ldots (2n-1)}{2^n x^{n+\frac{1}{2}}}$ iv. $2^n \{1.3.5 \ldots\ldots (2n-1)\}x$

v. $\dfrac{n!}{(a-x)^{n+1}}$ vi. $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$

vii. $\dfrac{(-1)^{n+1} a.b^{n-1} n!}{(a+bx)^{n+1}}$

viii. $\dfrac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \cos(n+1)\theta$, where $\theta = \tan^{-1}\dfrac{a}{x}$.

ix. $(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$, where $\theta = \tan^{-1}\left(\dfrac{1}{x}\right)$

x. $(-1)^{n-1}\dfrac{1}{2}(n-1)! \sin^n\theta \sin\theta$ where $\theta = \tan^{-1}\left(\dfrac{1}{x}\right)$

3. i. $e^{ax} a^{n-3}\{a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)\}$.

ii. $(-1)^n \dfrac{6(n-4)!}{x^{n-3}}$

✧✧✧

Chapter - 3

Mean Value Theorems

3.1 Rolle's Theorem

If a function is continuous in [a, b], differentiable in (a, b) and f(a) = f(b), then there exist a point c ∈(a, b) such that f '(c) = 0

Proof:

Since f(x) is differentiable in (a, b), then there exist a point c∈ (a, b) such that

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h}$$

Since f(a) = f(b)

If f(x) is constant throughout the interval, then f(x) = f(a) = f(b) and hence f '(x) = 0 for all x ∈(a, b).

If f(x) is not constant throughout the interval [a, b], it is somewhere positive and also being continuous on [a, b] then it must have maximum value at some interior point say, x=c ∈(a, b). Thus

$$f(c-h) \le f(c) \quad \Rightarrow f(c-h) - f(c) \le 0$$

$$\Rightarrow \frac{f(c-h) - f(c)}{-h} \ge 0$$

Taking limit as h → 0 on both sides

$$\Rightarrow \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h} \ge 0$$

$$\therefore \quad f'(c) \ge 0 \qquad \ldots\ldots\ldots(1)$$

and $f(c+h) \le f(c) \quad \Rightarrow f(c+h) - f(c) \le 0$

$$\Rightarrow \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \le 0$$

$$\therefore \quad f'(c) \le 0 \qquad \ldots\ldots\ldots(2)$$

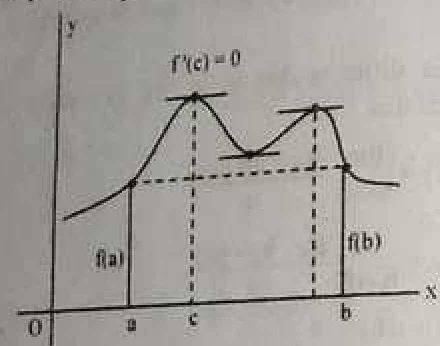
From (1) and (2),

$$f'(c) = 0$$

Hence the theorem.

The *Geometrical interpretation* of the theorem is given in th following figure. If the function is continuous in [a, b differentiable in (a, b) and f(a) = f(b) then the theorem asserts th there is at least one point in the interval (a, b) where the *tangent parallel to the x-axis*. So, the slope of the tangent at that point mu be zero.

i.e. Slope $m = f'(x) = 0$



3.2 Lagrange's Mean Value Theorem

If f(x) is continuous in [a, b] and differentiable in (a, b), then the exist at least one point c ∈(a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. [2061 BE

Proof:

Given that f(x) is continuous in [a, b] and differentiable i [a, b].

Let us consider a function defined by

$$F(x) = f(x) + Kx \qquad \text{..........(}$$

where f(x) and Kx are continuous in [a, b] and differentiable i (a, b). It, therefore, follows that the sum of continuous functions continuous and sum of differentiable functions is differentiable.

So, F(x) is continuous in [a, b] and differentiable in (a, b). Th constant K is to be chosen in such a way that

$$F(a) = F(b).$$

or $\quad f(a) + Ka = f(b) + Kb$

$\therefore \quad K = -\dfrac{f(b) - f(a)}{b - a}$.

Thus, F(x) satisfies all three conditions of Rolle's theorem. Hence there exist at least one point c∈(a, b) such that

$$F'(c) = 0 \qquad \text{..........(2)}$$

So from (1),

$$F(x) = f(x) + Kx$$

Differentiate with respect to x,

$$F'(x) = f'(x) + K$$

or $\quad F'(c) = f'(c) + K$

Using (2),

$$0 = f'(c) + K$$

or $\quad f'(c) = -K$

$\therefore \quad f'(c) = \dfrac{f(b) - f(a)}{b - a}$, $a < c < b$ which proves the theorem.

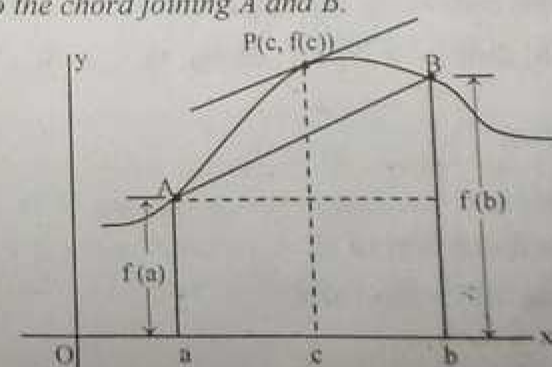
Note:

Writing b = a + h so that b - a = h, then the above result can be expressed as

$$f(a + h) = f(a) + h\,f'(a + \theta h),\ 0 < \theta < 1$$

This form of *Mean Value Theorem* is often found more useful in practical applications.

The *Geometrical meaning* of the theorem is interpreted from the following figure. If the function f(x) is continuous in [a, b] and differentiable in (a, b) then the theorem asserts that there is always a point c∈(a, b) such that the *tangent line at the point (c, f(c)) is parallel to the chord joining A and B.*



In general, for every such chord AB there always exist a tangent *line parallel to chord AB.*

Slope of the tangent line = f'(c)

Slope of the chord AB = $\frac{f(b) - f(a)}{b - a}$

Then c is such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

It is interesting to keep in mind that the function F(x) represents the difference in ordinates of the curve APB and the line AB at any point $x \in (a, b)$.

3.3 Cauchy's Mean Value Theorem

If f(x) and g(x) are continuous in [a, b] and differentiable in (a, b), then there exist at least one point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \text{ where } g'(x) \neq 0, \text{ for all } x \in (a, b).$$

Proof:

Given that f(x) and g(x) are continuous in [a, b] and differentiable in (a, b).

Let us consider a function defined by,

$$F(x) = f(x) + Kg(x) \qquad \text{..........(1)}$$

where K is constant which will be determined by choosing F(a) = F(b).

or $f(a) + K g(a) = f(b) + Kg(b)$

$$\therefore \quad K = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Since f(x) and Kg(x) are continuous in [a, b] and differentiable in (a,b), F(x) is continuous in [a, b], differentiable in (a, b) and

F(a) = F(b), then by Rolle's Theorem, there exist a point $c \in (a, b)$ such that

$$F'(c) = 0$$

Thus from (1)

$$F(x) = f(x) + K g(x)$$

Differentiate with respect to x

$$F'(x) = f'(x) + Kg'(x)$$

or $F'(c) = f'(c) + Kg'(c)$

or $0 = f'(c) + Kg'(c)$

or $\frac{f'(c)}{g'(c)} = -K$

$\therefore \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$ which proves the theorem.

Note:

If we put b = a + h, c = a + θh where 0 < θ < 1, then the *Cauchy's Mean Value Theorem* can be written as

$$\frac{f'(a + \theta h)}{g'(a + \theta h)} = \frac{f(a + h) - f(a)}{g(a + h) - g(a)}, \quad 0 < \theta < 1.$$

3.4 Taylor's Series in Finite form

(Generalized Mean Value Theorem)

If $f^{n-1}(x)$ is continuous in [a, b] and $f^n(x)$ is differentiable in (a, b), then there exist a point $c \in (a, b)$ such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \ldots$$
$$\ldots\ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c)$$

Proof:

Consider a function F(x) defined in (a, b) by

$$F(x) = \phi(x) - \frac{(b-x)^n}{(b-a)^n} \phi(a) \qquad \text{..........(1)}$$

Where,

$$\phi(x) = f(b) - f(x) - \frac{(b-x)}{1!} f'(x) - \frac{(b-x)^2}{2!} f''(x) - \frac{(b-x)^3}{3!} f'''(x) - \ldots$$
$$\ldots\ldots - \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) \qquad \text{........(2)}$$

Since $(b-x)^{n-1}$ and f(x) are continuous in [a, b] and differentiable in (a, b), φ (x) is continuous in [a, b] and differentiable in (a, b), then evidently, F(x) is continuous in [a, b] and differentiable in (a, b).

So differentiating with respect to x to (1) and (2) ,we get

$$F'(x) = \phi'(x) + \frac{n(b-x)^{n-1}}{(b-a)^n}\ \phi(a) \qquad \text{.............(3)}$$

and $\phi'(x) = -f'(x) + f'(x) - (b-x)f''(x) + (b-x)f''(x)$

$$-\frac{(b-x)^2}{2!}f'''(x) + \ldots\ldots + \frac{(b-x)^{n-1}}{(n-2)!}f^{n-1}(x) - \frac{(b-x)^{n-1}}{(n-1)!}f^n(x)$$

or $\phi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!}f^n(x)$ ....(4)

Clearly, $\phi(b) = 0$, and then from (1) F(b) = 0 and F(a) = 0 and hence F(x) is continuous in [a, b] differentiable in (a, b) and F(a) = F(b). By Rolle's Theorem there exist a point $c \in (a, b)$ such that

$F'(c) = 0$ ....(5)

From (3) and (4)

$$F'(x) = \phi'(x) + \frac{n(b-x)^{n-1}}{(b-a)^n}\ \phi(a)$$

or $F'(c) = \phi'(c) + \frac{n(b-c)^{n-1}}{(b-a)^n}\ \phi(a)$

or $0 = -\frac{(b-c)^{n-1}}{(n-1)!}f^n(c) + \frac{n(b-c)^{n-1}}{(b-a)^n}\ \phi(a)$

$\therefore\ \phi(a) = \frac{(b-a)^n}{n!}f^n(c)$ ....(6)

From (2)

$$\phi(a) = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f'(a) - \ldots\ldots\ldots - \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$

Putting the value of $\phi(a)$ from (6) in it

$$\frac{(b-a)^n}{n!}f^n(c) = f(b) - f(a) - \frac{(b-a)}{1!}f'(a) - \frac{(b-a)^2}{2!}f'(a) - \ldots\ldots\ldots - \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$

$$\therefore\ f(b) = f(a) + \frac{(b-a)}{!}f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f'''(a) + \ldots$$

$$\ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n}{n!}f^n(c) \qquad \text{.........(7)}$$

Note 1:

If we put $b = a + h$ and $c = a + \theta h$, $0 < \theta < 1$,

We get

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^n(a) + \frac{h^n}{n!}f^n(a+\theta h) \qquad \text{...... (8)}$$

Note 2:

If we put $a = x$, then we get

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f'(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots$$

$$\ldots\ldots + \frac{x^{n-1}}{(n-1)!}f^n(x) + \frac{h^n}{n!}f^n(x+\theta h) \qquad \text{......(9)}$$

The series of (7), (8) and (9) are called *Taylor's Series* with the remainder in Lagrange's form,

$$\frac{(b-a)^n}{n!}f^n(c) \quad \text{or} \quad \frac{h^n}{n!}f^n(a+\theta h) \quad \text{or} \quad \frac{h^n}{n!}f^n(x+\theta h)$$

It is generally denoted by R_n .

$\therefore\ R_n = \frac{h^n}{n!}f^n(a+\theta h)$

or $R_n = \frac{(b-a)^n}{n!}f^n(c)$

or $R_n = \frac{h^n}{n!}f^n(x+\theta h)$

3.5 Maclaurin's Series

We have from Taylor's Series in finite form

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h),\ 0 < \theta < 1$$

Writing a = 0 and h = x in the series we get

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

This is called *Maclaurin's Series* of the function f(x) in *Finite form*. Thus the *Maclaurin's* series sates that "If a function f(x) defined on [0, x], $f^{n-1}(x)$ is continuous in [0, x] and $f^n(x)$ is differentiable in (0, x), then there exists a number θ∈ (0, 1) such that

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

Where the remainder term

$$R_n = \frac{x^n}{n!} f^n(0)".$$

If R_n tends to zero as n tends to infinity, then the Maclaurin's series extended to infinity is valid and we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots \text{ to } \infty$$

This is the *Maclaurin's Series* of f(x) in *infinite form.*

3.6 Determination of the Coefficients in Expansions of f(x)

It is the alternative method to expand the f(x) by *Maclaurin's Series in infinite form*. We know that the Maclaurin's Series can be expanded if we can show that R_n tends to zero as n tends to infinity. If R_n does not tends to zero as n tends to infinity then the f(x) fails to expand by Maclaurins Series. In this method, we assume that R_n tends to zero as n tends to infinity. By assuming the validity of expansions of f(x) and by successively differentiation expansion of f(x), we can get the coefficient of various powers of x.

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots\ldots$

where $a_0, a_1, a_2 \ldots\ldots$ are constants which are to be determined in following ways:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots$$

or $$f''(x) = 2a_2 + 3.2\,a_3x + 4.3a_4x^2 + \ldots\ldots$$

or $$f'''(x) = 3.2\,a_3 + 4.3.2\,a_4x + \ldots\ldots\ldots$$

: : :

: : :

Put x = 0, we get

$f(0) = a_0$

$f'(0) = a_1 = 1!\,a_1$

$f''(0) = 2a_2 = 2!\,a_2$

$f'''(0) = 3.2a_3 = 3!\,a_3$

: : :

: : :

$f^n(0) = n!\,a_n$

Worked Out Examples

Ex. 1: Verify Rolle's theorem for the function

$\psi(x) = (x - a)^m (x - b)^n$, m, n being positive integers and $x \in [a, b]$

Solution:

Given function is

$$\psi(x) = (x - a)^m (x - b)^n$$

Since ψ(a) = 0 = ψ(b), ψ(x) is continuous in [a, b] and differentiable in (a, b), there exist a number c∈ (a, b) such that

$$\psi'(c) = 0$$

Now,

$$\psi'(x) = m(x - a)^{m-1}(x - b)^n + n(x - b)^{n-1}(x - a)^m$$

or $$\psi'(c) = m(c - a)^{m-1}(c - b)^n + n(c - b)^{n-1}(c - a)^m$$

or $$0 = m(c - a)^{-1} + n(c - b)^{-1}$$

or $$m(c - b) + n(c - a) = 0$$

or $$(m + n)c = mb + na$$

$$\therefore \quad c = \frac{mb + na}{m + n}$$

This value of c represents a point dividing the interval (a, b) in the ratio m:n internally and therefore belongs to the interval (a, b), hence Rolle's theorem is verified.

Ex. 2: Verify Lagrange's Mean value theorem for

$$f(x) = x(x-1)(x-2) \text{ for } x \in \left[0, \frac{1}{2}\right]$$

Solution:

Given function is

$$f(x) = x(x-1)(x-2),\ x \in \left[0, \frac{1}{2}\right]$$

Here

$$f(0) = 0$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) = \frac{3}{8}$$

$$f'(x) = 3x^2 - 6x + 2.$$

Since f(x) be continuous in $\left[0, \frac{1}{2}\right]$ and differentiable in $\left(0, \frac{1}{2}\right)$, there exist a number $c \in \left(0, \frac{1}{2}\right)$ such that

$$f'(c) = \frac{f(1/2) - f(0)}{(1/2) - 0} = \frac{(3/8) - 0}{1/2} = \frac{3}{4}$$

or $3c^2 - 6c + 2 = \frac{3}{4}$

or $12c^2 - 24c + 5 = 0$

or $c = \frac{24 \pm \sqrt{576 - 240}}{2 \times 12} = \frac{24 \pm 4\sqrt{21}}{24} = \frac{6 \pm \sqrt{21}}{6}$

$\therefore c = \frac{6 \pm \sqrt{21}}{6}$

Clearly, the point $\frac{6 - \sqrt{21}}{6}$ belongs to the interval $\left(0, \frac{1}{2}\right)$. Hence Lagrange's Mean value theorem is verified.

Ex. 3: In the Mean value theorem $f(x + h) = f(x) + h f'(x + \theta h)$, find θ if $f(x) = Ax^2 + Bx + C, A \neq 0$.

Solution:

Given function is $f(x) = Ax^2 + Bx + C, \quad A \neq 0$

Now,

$$f'(x) = 2Ax + B,$$

$$f'(x + \theta h) = 2A(x + \theta h) + B$$

$$= 2Ax + 2A\theta h + B,$$

$$f(x + h) = A(x + h)^2 + B(x + h) + C$$

Then,

$$f(x + h) = f(x) + h f'(x + \theta h)$$

$$Ax^2 + 2Axh + Ah^2 + Bx + Bh + C$$

$$= Ax^2 + Bx + C + h(2Ax + 2A\theta h + B)$$

or $Ah^2 + 2Axh + Bh = 2Ahx + 2A\theta h^2 + Bh$

or $Ah^2 = 2A\theta h^2$

$\therefore \theta = \frac{1}{2}.$

Ex. 4: Find the value of θ in the Mean value theorem

$$f(x + h) = f(x) + h f'(x + \theta h), \text{ if } f(x) = \frac{1}{x}$$

Solution:

Given function is

$$f(x) = \frac{1}{x}$$

Differentiate with respect to x

$$f'(x) = -\frac{1}{x^2}$$

Now, $f(x + h) = \frac{1}{x + h}$

$$f'(x + \theta h) = -\frac{1}{(x + \theta h)^2}$$

We have

$$f(x + h) = f(x) + h f'(x + \theta h)$$

or $\frac{1}{x + h} = \frac{1}{x} + h\left\{-\frac{1}{(x + \theta h)^2}\right\}$

or $\frac{h}{(x+\theta h)^2} = \frac{1}{x} - \frac{1}{x+h}$

or $\frac{h}{(x+\theta h)^2} = \frac{1}{x} - \frac{1}{x+h}$

or $\frac{h}{(x+\theta h)^2} = \frac{h}{x(x+h)}$

or $(x+\theta h)^2 = x(x+h)$

or $x + \theta h = \sqrt{x(x+h)}$

$\therefore \quad \theta = \frac{\sqrt{x(x+h)} - x}{h}$

Ex. 5: Verify Cauchy's Mean value theorem for the functions $f(x) = x^3$ and $g(x) = x^4$, $x \in [1, 2]$.

Solution:

Here $f(x) = x^3$, $\quad g(x) = x^4$

$f(2) = 8$, $\quad g(2) = 16$

$f(1) = 1$, $\quad g(1) = 1$

Now,

$f'(x) = 3x^2$, $\quad g'(x) = 4x^3$

Since f(x) and g(x) are continuous in [1, 2] and differentiable in (1, 2) then by Cauchy's Mean value theorem, there exist a point $c \in (1, 2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(c)}$$

or $\frac{3c^2}{4c^3} = \frac{8-1}{16-1}$

or $\frac{3}{4c} = \frac{7}{15}$

$\therefore \quad c = \frac{45}{28}$

Clearly, $c = \frac{45}{28} \in (1, 2)$.

Hence Cauchy's Mean value theorem is verified.

Ex. 6: Assuming the validity of expansion, expand the following functions in ascending power of x.

i. Cosx, $\quad$ ii. Sinx $\quad$ iii. e^x

iv. log (1 - x) $\quad$ v. $(1 + x)^m$

i. Cosx

Solution:

Given function is

$f(x) = \cos x$, $\quad f(0) = 1$

$f'(x) = -\sin x$, $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$f^{iv}(x) = \cos x$ $\quad f^{iv}(0) = 1$

$f^{v}(x) = -\sin x$ $\quad f^{v}(0) = 0$

$f^{vi}(x) = -\cos x$ $\quad f^{vi}(0) = -1$.

We have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots\ldots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots$$

ii. Sinx

Solution:

Here $f(x) = \sin x$, $\quad f(0) = 0$

Differentiating it with respect to x,

$f'(x) = \cos x$, $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$, $\quad f'''(0) = -1$

$f^{iv}(x) = \sin x$, $\quad f^{iv}(0) = 0$

$f^{v}(x) = \cos x$, $\quad f^{v}(0) = 1$

We have the Maclaurin's Series of the f(x) is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\ldots$$

iii. e^x

Solution:

Here, $f(x) = e^x$, $\quad f(0) = 1$

Differentiating it with respect to x

$f'(x) = e^x$, $\quad f'(0) = 1$,

$f''(x) = e^x$, $\quad f''(0) = 1$,

$f'''(x) = e^x$, $\quad f'''(0) = 1$,

$f^{iv}(x) = e^x$ $\quad f^{iv}(0) = 1$

We have the Maclaurin's series of the function f(x) is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

$$\therefore \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots$$

iv. **log (1 - x)**

Solution:

Given function is

$f(x) = \log(1 - x)$

Differentiating it with respect to x

Here $f'(x) = \frac{(-1)}{(1-x)}$, $\quad f''(x) = \frac{(-1)}{(1-x)^2}$

$f'''(x) = \frac{(-1)\,1.2}{(1-x)^3}$, $\quad f^{iv}(x) = \frac{(-1)\,1.2.3}{(1-x)^4}$

$f^{v}(x) = \frac{(-1)\,1.2.3.4}{(1-x)^5}$

: : : : : :

$f^n(x) = -\frac{(n-1)!}{(1-x)^n}$, $\quad f^n(\theta x) = -\frac{(n-1)!}{(1-\theta x)^n}$

$$R_n = \frac{x^n}{n!} f^n(\theta x) = -\frac{x^n}{n!}\frac{(n-1)!}{(1-\theta x)^n} = -\frac{1}{n}\left(\frac{x}{1-\theta x}\right)^n$$

Case 1

When $-1 \le x \le 0$, $|R_n| = \left|-\frac{1}{n}\left(\frac{x}{1-\theta x}\right)^n\right|$

$$= \left|\frac{1}{n}\right| \left|\left(\frac{x}{1-\theta x}\right)^n\right| \le \left|\frac{1}{n}\right|$$

Since $\frac{x}{1-\theta x} < 1$ for $-1 \le x \le 0$, $0 < \theta < 1$, $\frac{1}{n} \to 0$ as $n \to \infty$

$\therefore R_n \to 0$ as $n \to \infty$

Case 2

When $0<x<1$, then $\frac{x}{1-\theta x}$ may or may not be numerically less than 1 we fail to draw any definite conclusion from Lagarange's form of remainder in Maclaurin's series.

Now, Cauchy form of remainder term is

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

or $\quad R_n = -\frac{x^n}{(n-1)!}\cdot\frac{(n-1)!.(1-\theta)^{n-1}}{(1-\theta x)^n}$

$$= -\left(\frac{x}{1-\theta x}\right)^n (1-\theta)^{n-1} = -\left(\frac{1-\theta}{1-\theta x}\right)^{n-1}\cdot\frac{x^n}{1-\theta x}$$

or $\quad |R_n| = \left|\left(\frac{1-\theta}{1-\theta x}\right)\right|^{n-1}\left|\left(\frac{x^n}{1-\theta x}\right)\right|$

Now,

$0 < x < 1$, then $x < 1$

or $\theta x < \theta$ $\quad$ or $-\theta x > -\theta$

or $\quad 1 - \theta x > 1 - \theta$

or $1 > \frac{1-\theta}{1-\theta x}$, $\quad$ or $\frac{1-\theta}{1-\theta x} < 1$

Thus,

$$|R_n| = \left|\left(\frac{1-\theta}{1-\theta x}\right)^{n-1}\right| \left|\frac{x^n}{1-\theta x}\right| \le \left|\frac{x^n}{1-\theta x}\right|$$

$R_n \to 0$ as $n \to \infty$

Hence the Maclaurin's Series in infinite form is valid.

$f(0) = 0$, $\quad f'(0) = -1$, $\quad f''(0) = -1$,

$f'''(0) = -2!$, $\quad f^{iv}(0) = -3!$, We have Maclaurin's Series in infinite form is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots\ldots$$

v. $(1 + x)^m$

Solution:

Given function is

$$f(x) = (1 + x)^m$$

Since the Maclaurin's Series infinite form of this function is valid.

Now,

$f(x) = (1 + x)^m$, $\quad f(0) = 1$,

$f'(x) = m(1 + x)^{m-1}$, $\quad f'(0) = m$

$f''(x) = m(m - 1)x^{m-2}$, $\quad f''(0) = m(m - 1)$

$f'''(x) = m(m - 1)(m - 2)(1+x)^{m-3}$, $\quad f'''(0) = m(m-1)(m-2)$

$f^{iv}(x) = m(m - 1)(m - 2)(m - 3)(1 + x)^{m-4}$,

$f^{iv}(0) = m(m-1)(m-2)(m-3)$

We have Maclaurin's Series in infinite form is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$(1 + x)^m = 1 + \frac{m}{1!}x + \frac{m(m - 1)}{2!} x^2 + \frac{m(m - 1)(m - 2)}{3!} x^3 + \ldots$$

Ex. 7: Assuming the validity of expansion, prove the following series by using Maclaurin's theorem.

i. $\log(1 + \tan x) = x - \frac{x^2}{2!} + \frac{4}{3!} x^3 - \ldots$

ii. $e^x \sec x = 1 + x + x^2 + \frac{2}{3} x^3 + \ldots$

iii. $e^x \log(1 + x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \ldots$

iv. $e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5}{4!} x^4 + \ldots$

i. $\log(1 + \tan x) = x - \frac{x^2}{2!} + \frac{4}{3!} x^3 - \ldots$

Solution:

Given function is

$f(x) = \log(1 + \tan x)$, $\quad f(0) = 0$,

$f'(x) = \frac{\sec^2 x}{1 + \tan x}$, $\quad f'(0) = 1$

$$f''(x) = \frac{(1 + \tan x)\, 2\sec^2 x \tan x - \sec^2 x . \sec^2 x}{(1 + \tan x)^2}$$

$= \frac{2\sec^2 x \tan x}{(1 + \tan x)} - \frac{\sec^4 x}{(1 + \tan x)^2}$, $\quad f''(0) = -1$

$$\text{or } f'''(x) = \frac{(1 + \tan x)\{4\sec^2 x \tan^2 x + 2\sec^4 x\} - 2\sec^2 x \tan x . \sec^2 x}{(1 + \tan x)^2}$$

$- \frac{\{(1 + \tan x)^2\, 4\sec^4 x \tan x - 2(1 + \tan x)\sec^6 x\}}{(1 + \tan x)^4}$, $\quad f'''(0) = 4$

:::::: ::::::

We have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

$$\therefore \log(1 + \tan x) = x - \frac{x^2}{2!} + \frac{4}{3!} x^3 - \ldots\ldots$$

ii. $e^x \sec x = 1 + x + x^2 + \frac{2}{3} x^3 + \ldots$

Given function is

$f(x) = e^x \sec x$, $\quad f(0) = 1$

Its expansion is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$f'(x) = e^x \sec x \tan x + e^x \sec x$ $\quad f'(0) = 1$

$= e^x \sec x (1 + \tan x) = f(x)(1 + \tan x)$

or $f''(x) = f(x)\sec^2 x + f'(x)(1 + \tan x)$, $\quad f''(0) = 2$

or $f'''(x) = 2f(x)\sec^2 x \tan x + f'(x)\sec^2 x + f'(x)\sec^2 x + f''(x)(1 + \tan x)$, $\quad f'''(0) = 4$

We have the Maclaurin's Series of the function f(x) is given by

$$f(x) = f(0) + \frac{x}{1!} + f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$\therefore e^x \sec x = 1 + x + x^2 + \frac{2}{3}x^3 + \ldots\ldots$$

iii. $e^x \log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots$

Solution:

Here,

$f(x) = e^x \log(1+x)$, $\quad f(0) = 0$

$f'(x) = \frac{e^x}{1+x} + e^x \log(1+x)$, $\quad f'(0) = 1$

or $f'(x) = \frac{e^x}{1+x} + f(x)$.

or $(1+x) f'(x) = e^x + (1+x) f(x)$

or $(1+x) f''(x) + f'(x) = e^x + (1+x) f'(x) + f(x)$

or $(1+x) f''(x) = e^x + x f'(x) + f(x)$, $\quad f''(0) = 1$

or $(1+x) f'''(x) + f''(x) = e^x + x f''(x) + f'(x) + f''(x)$.

or $(1+x) f'''(x) + f''(x) = e^x + x f''(x) + 2f'(x)$, $\quad f'''(0) = 2$

or $(1+x) f'''(x) + f''(x) = e^x + x f''(x) + 2f'(x)$,

or $(1+x) f^{iv}(x) + 2f'''(x) = e^x + x f'''(x) + 3f''(x)$, $\quad f^{iv}(0) = 0$

We have Maclaurin's Series of f(x) is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$\therefore \quad e^x \log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots$

iv. $e^{\sin^{-1}x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5}{4!} x^4 + \ldots\ldots\ldots$

Solution:

Given function is

$y = f(x) = e^{\sin^{-1}x}$, $\quad f(0) = 1$.

Differentiating

$y_1 = f'(x) = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$, $\quad f'(0) = 1$

$= \frac{f(x)}{\sqrt{1-x^2}} = \frac{y}{\sqrt{1-x^2}}$

or $(1-x^2) y_1^2 = y^2$

Again differentiating

$2(1-x^2) y_1 y_2 + y_1^2(-2x) = 2yy_1$

or $(1-x^2) y_2 - xy_1 - y = 0$, $\quad (y_2)_0 = f''(0) = 1$

Differentiating

$(1-x^2) y_3 - 2xy_2 - xy_2 - y_1 - y_1 = 0$

or $(1-x^2) y_3 - 3xy_2 - 2y_1 = 0$, $\quad (y_3)_0 = f'''(0) = 2$

Differentiating

$(1-x^2) y_4 - 5x y_3 - 5y_2 = 0$, $(y_4) = f^{iv}(x)$, $(y_4)_0 = f^{iv}(0) = 5$

We have the Maclaurin's Series is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

$\therefore \quad e^{\sin^{-1}x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \ldots\ldots\ldots\ldots$

Ex. 8: If $y = e^{m\tan^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots$, then show that

i. $(1+x^2) y_1 = my$

ii. $(n+1) a_{n+1} + (n-1) a_{n-1} = m\, a_n$

and hence obtain the expansion of $e^{m\tan^{-1}x}$

Solution:

Given function is

$y = e^{m\tan^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\ldots$

Differentiating,

$y_1 = \frac{me^{m\tan^{-1}x}}{1+x^2} = \frac{my}{1+x^2}$

or $(1+x^2) y_1 = my$ $\quad\ldots\ldots\ldots\ldots(1)$

Differentiating (1) n times by using Leibnitz's theorem

$(1+x^2) y_{n+1} + \frac{n}{1!} y_n \times 2x + \frac{n(n-1)}{2!} y_{n-1} \times 2 = my_n$

or $(1+x^2) y_{n+1} + 2nxy_n + n(n-1) y_{n-1} = my_n$

Putting x = 0

$(y_{n+1})_0 + n(n-1)(y_{n-1})_0 = m(y_n)_0$ $\quad\ldots\ldots\ldots(2)$

Also we have

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots$

Differentiating

$y_1 = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \ldots\ldots\ldots$,

$y_2 = 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots\ldots\ldots\ldots$,

$y_3 = 3.2.1\, a_3 + 4.3.2.\, a_4 x + \ldots\ldots\ldots\ldots\ldots$,

:::::: $\quad$ ::::::

Thus

$(y_1)_0 = a_1$

$(y_2)_0 = 2!a_2$

$(y_3)_0 = 3!a_3$

::::::

$(y_{n-1})_0 = (n-1)!\, a_{n-1}$

$(y_n)_0 = n!\, a_n$

$(y_{n+1})_0 = (n+1)!\, a_{n+1}$

Then (2) gives

$(n+1)!\, a_{n+1} + n(n-1)(n-1)!\, a_{n-1} = m\, n!\, a_n$

$(n+1)\, n!\, a_{n+1} + n!(n-1)a_{n-1} = m\, n!\, a_n$

$\therefore \quad (n+1)a_{n+1} + (n-1)a_{n-1} = m a_n.$

Putting $x = 0$ in given function

$(y)_0 = f(0) = 1.$

Putting $x = 0$ in (1)

$(y_1)_0 = f'(0) = m.$

Putting $n = 1, 2, 3 \ldots\ldots n$, in (2)

We get

$(y_2)_0 = f''(0) = m^2$

$(y_3)_0 = f'''(0) = m^3 - 2m$

$(y_4)_0 = f^{iv}(0) = m^4 - 2m^2 - 6m^2 = m^4 - 8m^2$

$\vdots \qquad\qquad \vdots$

We have Maclaurin's Series is given by

$$e^{m\tan^{-1}x} = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$\therefore \quad e^{m\tan^{-1}x} = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m(m^2-2)}{3!}x^3 + \ldots\ldots$$

Exercise-3

1. Verify Rolle's theorem for the function
   - i. $f(x) = x^2, x \in [-1, 1]$
   - ii. $f(x) = e^{-x} \sin x, x \in [0, \pi]$
   - iii. $f(x) = \left(\dfrac{\sin x - \cos x}{e^{-x}}\right), x \in \left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$
   - iv. $f(x) = \log\left\{\dfrac{x^2 + ab}{(a+b)x}\right\}, x \in [a, b]$
   - v. $f(x) = \dfrac{x(x+3)}{e^{x/2}}, x \in [-3, 0]$ [2060 B]
2. Verify Lagrange's Mean value theorem for the function.
   - i. $f(x) = e^x, \quad x \in [1, 2]$
   - ii. $f(x) = x^2, \quad x \in [1, 2]$

   - iii. $f(x) = \log x, \quad x \in [1, e]$
   - iv. $f(x) = Ax^2 + Bx + C, \quad x \in [a, b]$
   - v. $f(x) = \dfrac{x^2+1}{x}, \quad x \in \left[\dfrac{1}{2}, 2\right]$
   - vi. $f(x) = x^2 + 3x + 2, \quad x \in [0, 2]$ [2059 B.E.]
3. Verify Cauchy's Mean value theorem for
   - i. $f(x) = x^4, g(x) = 3x^2, \quad x \in [1, 2]$
   - ii. $f(x) = x^2, g(x) = x^3, \quad x \in [0, 1]$
4. In the Mean value theorem $f(x+h) = f(x) + hf'(x + \theta h)$ find θ if
   - i. $f(x) = \dfrac{1}{x}$
   - ii. $f(x) = e^x$
   - iii. $f(x) = \log x$
5. In the Mean value theorem, $f(a+h) = f(a) + h f'(a + \theta h)$ find θ if $f(x) = \sqrt{x}$, $a = 1$, $h = 3$.
6. Assuming the validity of expansion, find the expansion of the following functions ~~very~~ by Maclaurin's theorem
   - i. $\log(1+x)$
   - ii. $\sinh x$
   - iii. $\cosh x$
   - iv. $\tanh x$
   - v. $\tan x$
   - vi. $e^x \sin x$
   - vii. $\log(1 + \sin x)$
   - viii. $\log \sec x$
   - ix. $\dfrac{e^x}{1+e^x}$
   - x. $\log(1 + e^x)$ [2062 B.E.]
   - xi. $e^{\sin x}$
   - xii. $x \cot x$
7. Assuming the validity of expansion, prove the following series using Maclaurin's Series
   - i. $\sin^{-1} x = x + \dfrac{x^3}{3!} + \dfrac{9x^5}{5!} + \ldots\ldots$

     and hence $\cos^{-1} x = \dfrac{\pi}{2} - x - \dfrac{x^3}{3!} - \ldots\ldots$
   - ii. $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5}{24}x^4 + \ldots\ldots$
   - iii. $\log(1 + x + x^2) = x + \dfrac{x^2}{2} - \dfrac{2}{3}x^3 + \dfrac{x^4}{4} + \ldots\ldots$

iv. $x \operatorname{cosec} x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \ldots\ldots$

v. $\frac{1}{\sqrt{x^2+1}} = 1 - \frac{x^2}{2!} + \frac{9}{4!}x^4 - \ldots\ldots$

vi. $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \ldots\ldots$

8. If $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, then show that

i. $(1 - x^2)y_2 - xy_1 - a^2y = 0$

ii. $(n+1)(n+2)a_{n+2} - (n^2 + a^2)a_n = 0$

and hence obtain the expansion of $e^{a\sin^{-1}x}$.

Answers

4. i. $\frac{\sqrt{x^2 + xh} - x}{h}$ ii. $\frac{1}{h}\log\frac{e^h - 1}{h}$

iii. $\frac{1}{\log\left(1 + \frac{h}{x}\right)} - \frac{x}{h}$

5. $\frac{5}{12}$

6. i. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ ii. $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

iii. $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$ iv. $x - \frac{2}{3!}x^3 + \frac{16}{5!}x^5 - \ldots\ldots$

v. $x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$ vi. $x + x^2 + \frac{x^3}{3} + \ldots\ldots$

vii. $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$ viii. $\frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{16x^6}{6!} + \ldots$

ix. $\frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \ldots$ xi. $1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \ldots\ldots$

xii. $1 - \frac{x^2}{3} - \frac{1}{45}x^4 + \ldots$

8. $1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1+a^2)}{3!}x^3 + \frac{a^2(2^2+a^2)}{4!}x^4 + \ldots\ldots$

Chapter - 4

Indeterminate Forms

4.1 Introduction

If the limits of quotients of function tend to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as x approaches to 'a' which are meaning less, then these are called *Indeterminate Forms*. Other indeterminate forms are

$0 \times \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$.

If the indeterminate forms $0 \times \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$ are reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then L' Hospital's rule is applicable. Statement of the L' Hospital's rule is as follows:

L' Hospital's Rule

If f(x) and g(x) are two functions such that their derivatives f '(x) and g'(x) are continuous at x = a and f(a) = g(a) = 0, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \text{ provided } g'(a) \neq 0.$$

Proof:

Given that f(x) and g(x) are derivable at x = a it implies that f(x) and g(x) are both continuous at x = a.

i.e. $\lim_{x\to a} f(x) = f(a)$ and

$\lim_{x\to a} g(x) = g(a)$.

Since, $f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}$

Using f(a) = 0

$$f'(a) = \lim_{x\to a}\frac{f(x)}{x - a}$$

and $g'(a) = \lim_{x\to a}\frac{f(x) - g(a)}{x - a}$

Using g (a) = 0

$$g'(a) = \lim_{x\to a} \frac{g(x)}{x-a}$$

Hence $\frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f(x)}{x-a} \cdot \frac{x-a}{g(x)}$

$$\therefore \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g(x)}$$

If however f '(a) = g'(a) = 0, then we repeat the process and obtain

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

The process is repeated until the quotients cease to be a indeterminate form.

4.2 Some Standard Forms

Form $\frac{\infty}{\infty}$

If f(x) and g(x) are two functions such that their derivative are continuous at x = a and f (a) = ∞, g(a) = ∞.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to a} \frac{f'(a)}{g'(a)} \text{ provided } g'(a) \neq 0$$

To find $\lim_{x\to a} \frac{f(x)}{g(x)}$

We write

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$$

This is of the form $\frac{0}{0}$, and then it can be evaluated by L' Hospital rule.

Form 0 × ∞

If the limit of a function is of the form $\lim_{x\to a} [f(x)\, g(x)]$ where

$\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \infty$, then this can be written as

$$\lim_{x\to a} [f(x)\, g(x)] = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}}$$

This is of the form $\frac{0}{0}$ and therefore can be evaluated by using L' Hospital's rule.

Form ∞ - ∞

If the limit of a function is of the form $\lim_{x\to a} [f(x) - g(x)]$ where

$\lim_{x\to a} g(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then this can be written as

$$\lim_{x\to a} [(f(x) - g(x)] = \lim_{x\to a} \frac{\left(\frac{1}{g(x)} - \frac{1}{f(x)}\right)}{\frac{1}{f(x)\, g(x)}}$$

Which is of the form $\frac{0}{0}$ and therefore can be evaluated by using L' Hospital's rule.

Form 0^0, ∞^0 and 1^∞

If the limit of a function is of the form $\lim_{x\to a} [f(x)]^{g(x)}$ where,

i. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

ii. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$

iii. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then this can be put as

$$A = \lim_{x\to a} [f(x)]^{g(x)}$$

$$\log A = g(x) \log f(x)$$

Which is of the form 0 × ∞ and thus can be evaluated by reducing it to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Worked Out Examples

Ex. 1: Evaluate $\lim\limits_{x\to 0} \dfrac{\log(1-x^2)}{\log \cos x}$

Solution:

Given limit $= \lim\limits_{x\to 0} \dfrac{\log(1-x^2)}{\log(\cos x)}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{(-2x/1-x^2)}{(-\sin x/\cos x)}$

$= \lim\limits_{x\to 0} \dfrac{2x(1-x^2)^{-1}}{\tan x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{-2x(1-x^2)^{-2}.(-2x)+2(1-x^2)^{-1}}{\sec^2 x}$

$= \dfrac{0+2}{1} = 2$

$\therefore \lim\limits_{x\to 0} \dfrac{\log(1-x^2)}{\log \cos x} = 2$

Ex. 2: Evaluate $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$

Solution:

Given limit $= \lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{xe^x + e^x - \dfrac{1}{1+x}}{2x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{xe^x + e^x + e^x + \dfrac{1}{(1+x)^2}}{2}$

$= \dfrac{0+1+1+1}{2} = \dfrac{3}{2}$

$\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2} = \dfrac{3}{2}$

Ex. 3: Evaluate $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x}$ **2060/062 B.E.**

Solution:

Given $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x}$

Since $\lim\limits_{x\to 0} (1+x)^{1/x} = e$

Therefore it is of the form $\dfrac{0}{0}$

So.

$\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x} = \lim\limits_{x\to 0} \dfrac{\dfrac{d}{dx}(1+x)^{1/x}}{1}$

$= \lim\limits_{x\to 0} \dfrac{(1+x)^{1/x}\{x-(1+x)\log(1+x)\}}{x^2(1+x)}$

$= e.\lim\limits_{x\to 0} \dfrac{x-(1+x)\log(1+x)}{x^2(1+x)}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= e.\lim\limits_{x\to 0} \dfrac{1-(1+x)\dfrac{1}{1+x} - \log(1+x)}{2x+3x^2}$

$= e.\lim\limits_{x\to 0} \dfrac{-\log(1+x)}{(2x+3x^2)}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= e.\lim\limits_{x\to 0} \dfrac{\dfrac{-1}{1+x}}{2+6x} = -e.\dfrac{1}{2} = -\dfrac{e}{2}$

$\therefore \lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x} = -\dfrac{e}{2}$

Ex. 4: Evaluate $\lim\limits_{x\to 0} \dfrac{\log \sin x}{\cot x}$

Solution:

Given $\lim\limits_{x\to 0} \dfrac{\log \sin x}{\cot x}$ $\left(\dfrac{\infty}{\infty}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{\dfrac{1}{\sin x}\cos x}{-\operatorname{cosec}^2 x}$

$\therefore \lim\limits_{x\to 0} \dfrac{\log \sin x}{\cot x} = 0$

Ex. 5: Evaluate $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$

Solution:

Given $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$

$= \lim\limits_{x\to 0} \dfrac{\sin^2 x - x^2}{x^2 \sin^2 x}$ $\left(\dfrac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to 0} \dfrac{\sin^2 x - x^2}{x^4 (\sin x/x)^2}$

$$= \lim_{x\to0} \frac{\sin^2 x - x^2}{x^4} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to0} \frac{2\sin x \cos x - 2x}{4x^3} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to0} \frac{2\cos 2x - 2}{12x^2} = \lim_{x\to0} \frac{-4\sin 2x}{24x}$$

$$= -\frac{1}{3} \lim_{x\to0} \frac{\sin 2x}{2x} = -\frac{1}{3}$$

$$\therefore \lim_{x\to0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = -\frac{1}{3}$$

Ex. 6: Evaluate $\lim_{x\to0} \sin x . \log x^2$

Solution:

Here $\lim_{x\to0} \sin x \log x^2$

This is of the form $0 \times \infty$,

So, $\lim_{x\to0} \sin x \log x^2 = \lim_{x\to0} \frac{\log x^2}{\operatorname{cosec} x}$

$$= \lim_{x\to0} \frac{2\log x}{\operatorname{cosec} x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to0} \frac{\frac{2}{x}}{-\operatorname{cosec} x \cot x}$$

$$= -\lim_{x\to0} \sin x . \cos x = 0.1 = 0$$

$$= -\lim_{x\to0} \frac{2\sin^2 x}{x\cos x} = -2\lim_{x\to0} \left(\frac{\sin x}{x}\right) \tan x$$

$$= -2.1.0 = 0$$

$$\therefore \lim_{x\to0} \sin x . \log x^2 = 0$$

Ex. 7: Evaluate the following limits

i. $\lim_{x\to0} x^x$ ii. $\lim_{x\to0} (\cot x)^{\sin 2x}$ iii. $\lim_{x\to1} x^{\frac{1}{1-x}}$

i. $\lim_{x\to0} x^x$

Solution:

Let $A = x^x$

So, $\lim_{x\to0} A = \lim_{x\to0} x^x$

$$\log\left(\lim_{x\to0} A\right) = \lim_{x\to0} x \log x = \lim_{x\to0} \frac{\log x}{(1/x)} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to0} \frac{(1/x)}{-(1/x^2)} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= -\lim_{x\to0} x = 0.$$

or $\lim_{x\to0} A = e^0 = 1$ $\therefore \lim_{x\to0} x^x = 1$

ii. $\lim_{x\to0} (\cot x)^{\sin 2x}$

Solution:

Let $A = (\cot x)^{\sin 2x}$

So, $\lim_{x\to0} A = \lim_{x\to0} (\cot x)^{\sin 2x}$

or $\log\left(\lim_{x\to0} A\right) = \lim_{x\to0} \sin 2x . \log(\cot 2x)$

$$= \lim_{x\to0} \frac{\log(\cot 2x)}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to0} \frac{\frac{-2\operatorname{cosec}^2 2x}{\cot 2x}}{-2\operatorname{cosec} 2x . \cot 2x} = \lim_{x\to0} \frac{\operatorname{cosec} 2x}{\cot^2 2x}$$

$$= \lim_{x\to0} \tan 2x . \sec 2x = 0 \times 1 = 0.$$

or $\lim_{x\to0} A = e^0 = 1$ $\therefore \lim_{x\to0} (\cot x)^{\sin 2x} = 1$

iii. $\lim_{x\to1} x^{\frac{1}{1-x}}$

Solution:

Let $\lim_{x\to1} A = \lim_{x\to1} x^{1/(1-x)}$

or $\log\left(\lim_{x\to1} A\right) = \lim_{x\to1} \frac{\log x}{1-x} \quad \left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to1} \frac{1/x}{-1} = -1$$

or $\lim\limits_{x\to 1} A = e^{-1}$

$\therefore \lim\limits_{x\to 1} x^{\frac{1}{1-x}} = e^{-1}$

Ex. 8: Find the values of 'a' and 'b' so that $\lim\limits_{x\to 0} \dfrac{x(1 + a\cos x) - b \sin x}{x^3} = 1$

Solution:

Given $\lim\limits_{x\to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3} = 1$ $\quad \left(\dfrac{0}{0} \text{ form}\right)$

or $\lim\limits_{x\to 0} \dfrac{-ax\sin x + 1 + a\cos x - b\cos x}{3x^2} = 1$ ....(1)

or $\lim\limits_{x\to 0} \dfrac{a - b + 1}{0} = 1$

or $a - b + 1 = 0$

$\therefore$ $b = a + 1$ ....(2)

So the limit (1) becomes

$\lim\limits_{x\to 0} \dfrac{-ax\sin x + a\cos x - (a+1)\cos x}{3x^2} = 1$

or $\lim\limits_{x\to 0} \dfrac{-ax\sin x + 1 - \cos x}{3x^2} = 1$ $\quad \left(\dfrac{0}{0} \text{ form}\right)$

or $\lim\limits_{x\to 0} \dfrac{-ax\cos x - a\sin x + \sin x}{6x} = 1$ $\quad \left(\dfrac{0}{0} \text{ form}\right)$

or $\lim\limits_{x\to 0} \dfrac{ax\sin x - a\cos x - a\cos x + \cos x}{6} = 1$

or $-2a + 1 = 6 \quad \therefore a = -\dfrac{5}{2}$

Then from (2) $b = -\dfrac{3}{2}$

Exercise-4

1. Evaluate the following limits

i. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ ii. $\lim\limits_{x\to 2} \dfrac{x^3 - 2x^2 + 2x - 4}{x^2 - 5x + 6}$

iii. $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$ iv. $\lim\limits_{x\to 0} \dfrac{x\cos x - \log(1 + x)}{x^2}$

v. $\lim\limits_{x\to 0} \dfrac{x - \sin^{-1}x}{\sin^3 x}$ vi. $\lim\limits_{x\to 0} \dfrac{2\sin x - \sin 2x}{\tan^3 x}$

vii. $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^2 \tan x}$ viii. $\lim\limits_{x\to 1} \dfrac{1 + \log x - x}{1 - 2x + x^2}$

ix. $\lim\limits_{x\to 0} \dfrac{\cosh x - \cos x}{x \sin x}$ x. $\lim\limits_{x\to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$

xi. $\lim\limits_{x\to 0} \dfrac{x\cos x - \sin x}{x^3}$ xii. $\lim\limits_{x\to 0} \dfrac{x\sin x - \sin^2 x}{x^4}$

xiii. $\lim\limits_{x\to 0} \dfrac{x e^x - (1 + x)\log(1 + x)}{x^2}$ [2059 B.E.]

2. Evaluate the following limits

i. $\lim\limits_{x\to 0} \dfrac{\log \tan x}{\cot x}$ ii. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\tan 5x}{\tan x}$

iii. $\lim\limits_{x\to 0} x \log(\tan x)$ iv. $\lim\limits_{x\to a} (a - x) . \tan\left(\dfrac{\pi x}{2a}\right)$

v. $\lim\limits_{x\to 0} x \log x$ vi. $\lim\limits_{x\to 0} x \log \sin^2 x$

vii. $\lim\limits_{x\to 0} \log_{\tan x}(\tan 2x)$

3. Evaluate the following limits

i. $\lim\limits_{x\to \frac{\pi}{2}} (\sec x - \tan x)$ ii. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$

iii. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \cot^2 x\right)$ iv. $\lim\limits_{x\to 0} \left[\dfrac{1}{x} - \dfrac{1}{x^2}\log(1 + x)\right]$

v. $\lim\limits_{x\to 1} \left(\dfrac{1}{x - 1} - \dfrac{1}{\log x}\right)$

4. Evaluate

i. $\lim\limits_{x\to \pi} (\sin x)^{\tan x}$ ii. $\lim\limits_{x\to 0} x^{2\sin x}$

iii. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2}\right)^{\tan x}$ iv. $\lim\limits_{x\to 0} (\cot x)^{1/\log x}$

v. $\lim\limits_{x\to \infty} x^{1/x}$ vi. $\lim\limits_{x\to 0} (\cot^2 x)^{\sin x}$

5. Evaluate

i. $\lim_{x\to 0} (\cos x)^{1/x^2}$

ii. $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$

iii. $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

iv. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

v. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

vi. $\lim_{x\to a} \left(2 - \frac{x}{a}\right)^{\tan \frac{\pi x}{2a}}$ 2061 B...

vii. $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x}$

6. i. If $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ be finite, then find the value of ... and the limit.

ii. If $\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x}$ be finite, then find the value of 'a' and the limit.

iii. If $\lim_{x\to 0} \frac{(a + b\cos x)\,x - c\sin x}{x^5} = 1$ then find the value of 'a', ... and c.

Answers

1. i. 2 ii. -6 iii. 2 iv. $\frac{1}{2}$ v. $\frac{1}{6}$ vi. 1
   vii. $\frac{1}{3}$ viii. $-\frac{1}{2}$ ix. 1 x. 1 xi. $-\frac{1}{3}$ xii. $\frac{1}{6}$ xiii. ...
2. i. 0 ii. $\frac{1}{5}$ iii. 0 iv. $\frac{2a}{\pi}$ v. 0 vi. 0
   vii. 1
3. i. 0 ii. $\frac{1}{2}$ iii. $\frac{2}{3}$ iv. $\frac{1}{2}$ v. $-\frac{1}{2}$
4. i. 1 ii. 1 iii. 1 iv. e^{-1} v. 1 vi. 1
5. i. $e^{-1/2}$ ii. 1 iii. $e^{-1/2}$ iv. $e^{1/3}$ v. 1 vi. $e^{2/\pi}$ vii. 1
6. i. a = -2, limit = -1 ii. a = 2, limit = 1
   iii. a = 120, b = 60, c = 180

✧✧✧

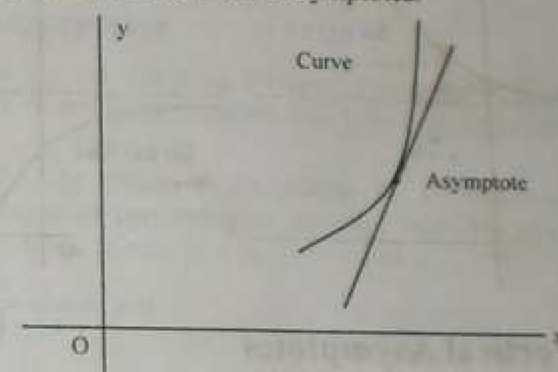
Chapter - 5

Asymptotes

5.1 Introduction

An *Asymptote* is the straight line which touches the curve at infinity but does not lie altogether at infinity. It asserts that the perpendicular distance of the straight line from a point on the curve approaches to zero as the curve moves to infinity along the curve.

Thus, the only unbounded curve has asymptotes.



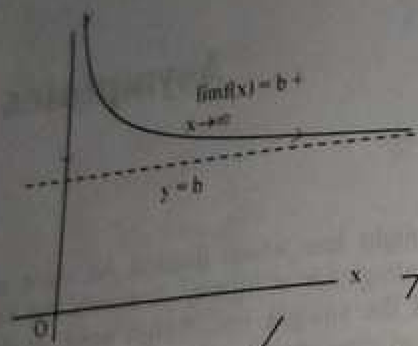
Since a straight line may be either parallel to x-axis or parallel to y-axis; or neither parallel to x-axis nor parallel to y-axis. Thus *Asymptotes* may also be

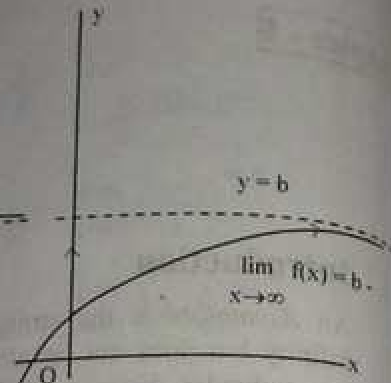
i. Parallel to x-axis (*Horizontal Asymptotes*)
ii. Parallel to y-axis (*Vertical Asymptotes*)
iii. Parallel to neither (*Oblique Asymptotes*)

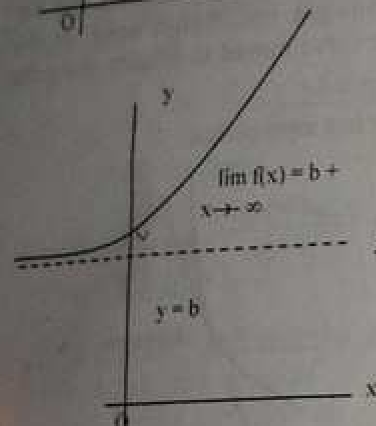
5.2 Horizontal Asymptotes

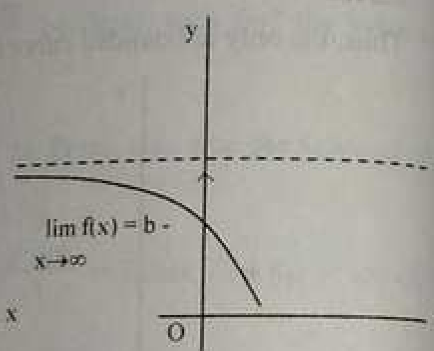
A straight-line y = b is said to be a *Horizontal Asymptote* of the curve y = f(x) if $\lim_{x\to\infty} f(x) = b$.

The position of the curve and horizontal asymptotes are mentioned as follows:







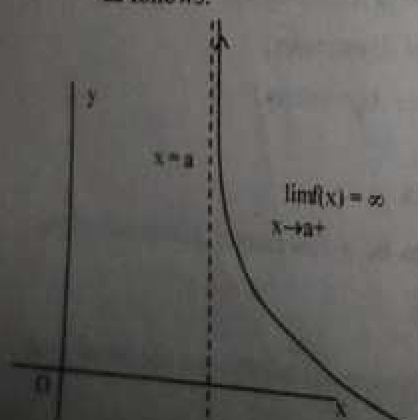


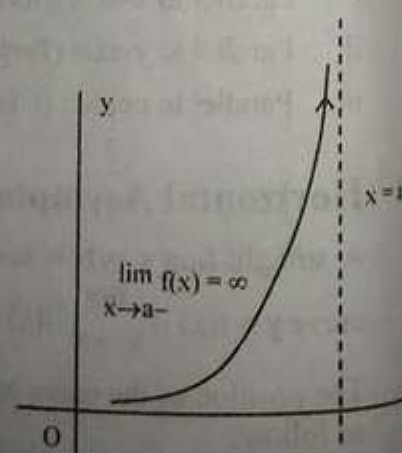
5.3 Vertical Asymptotes

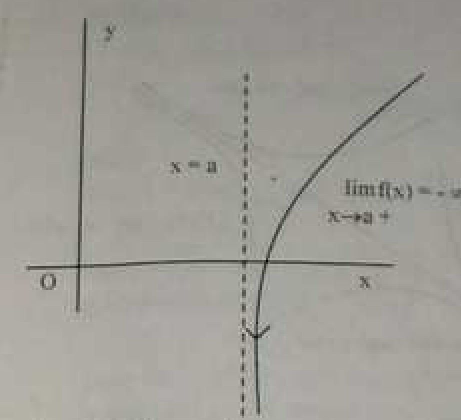
A straight line x = a is said to be a *Vertical Asymptote* of the cur

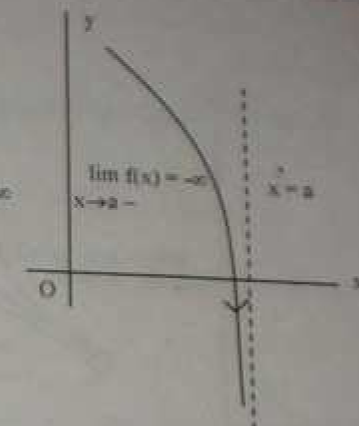
$y = f(x)$ if $\lim_{x \to a} f(x) = \infty$.

The position of the curves and vertical asymptotes are mentione as follows.









5.4 Oblique Asymptotes

A straight-line y = mx + c is an *Oblique Asymptote* of the curve y = f(x) where m and c are constant which will be determined as follows.

Let P(x, y) be any point on the curve, by definition of the asymptotes, length of perpendicular distance from (x, y) on the line y = mx + c tends to zero as x tends to infinity.

i.e. $\lim_{x \to \infty} \dfrac{mx - y + c}{\sqrt{m^2 + 1}} = 0$

or $\lim_{x \to \infty} (mx - y + c) = 0$

or $\lim_{x \to \infty} \left(m - \dfrac{y}{x} + \dfrac{c}{x}\right) = 0$

or $m = \lim_{x \to \infty} \dfrac{y}{x} + 0$

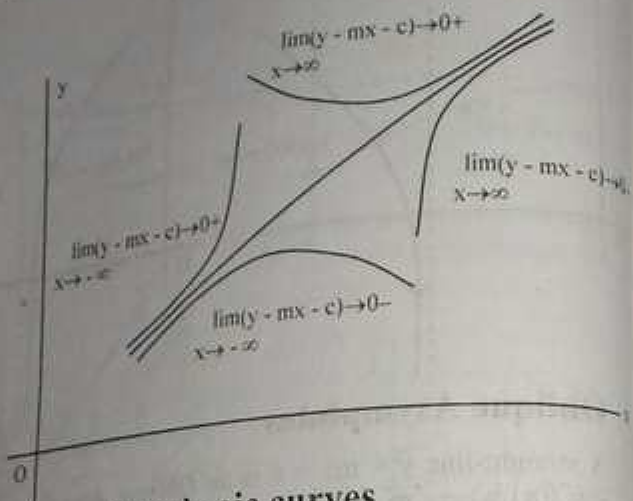
$\therefore \quad m = \lim_{x \to \infty} \dfrac{y}{x}$ ....(1)

and $\lim_{x \to \infty} (mx - y + c) = 0$

$\therefore \quad c = \lim_{x \to \infty} (y - mx)$ ....(2)

Determining the values of m and c from (1) and (2) and thus the *Oblique Asymptote* y = mx + c can be obtained.

The position of the curve and *Oblique Asymptotes* are mentioned as follows.



5.5 Asymptotes of Algebraic curves

If an algebraic curve f (x, y) = 0 is of degree n ,then it has at mo[st] n asymptotes. The curve may have horizontal, vertical and obliq[ue] asymptotes. We find three types of asymptotes as follows.

Asymptotes parallel to the axes

(Vertical and Horizontal Asymptotes)

Let f(x, y) = 0 be the equation of any algebraic curve of n^{th} degr[ee] in x and y.

So, it can be written as in the form.

$$(a_0 x^n + a_1 x^{n-1} y + \ldots\ldots\ldots\ldots + a_{n-1} x\, y^{n-1} + a_n y^n)$$
$$+ (b_0 x^{n-1} + b_1 x^{n-2} y + \ldots\ldots + b_{n-2} x y^{n-2} + b_{n-1} y^{n-1})$$
$$+ (c_0 x^{n-2} + c_1 x^{n-3} y + \ldots + c_{n-3} x y^{n-3} + c_{n-2} y^{n-2}) + \ldots = [0]$$

For the vertical asymptote, arranging the terms in descendi[ng] powers of y

$$a_n y^n + (a_{n-1} x + b_{n-1}) y^{n-1} + (a_{n-2} x^2 + b_{n-2} x + c_{n-2}) y^{n-2} + \ldots = [0]$$

This will have a vertical asymptote of the type x = a where a [is] finite, provided y→∞ when x→a + 0, or x→a -0, must have a_n [the] coefficient of highest powers of y i.e. y^n) equal to zero.

Therefore, the necessary condition for the vertical asymptote is

$$a_n = 0 \text{ (i.e. } y^n \text{ is absent).}$$

So (1) becomes

$$(a_{n-1} x + b_{n-1}) y^{n-1} + (a_{n-2} x^2 + b_{n-2} x + c_{n-2}) y^{n-2} + \ldots = 0 \ldots (2)$$

Dividing by y^{n-1} on both sides and making y→∞

We have

$$a_{n-1} x + b_{n-1} = 0 \qquad \ldots\ldots\ldots(3)$$

$$x = -\frac{b_{n-1}}{a_{n-1}},\ a_{n-1} \neq 0$$ is the required an *Vertical Asymptote.*

Note:

If $a_{n-1} = 0$ along with $a_n = 0$, then from (2) we have

$$b_{n-1} = 0.$$

Hence from (2), dividing by y^{n-2} and making y→∞, we get the asymptotes parallel to y-axis.

$$a_{n-2} x^2 + b_{n-2} x + c_{n-2} = 0$$

provided the value of x exists and so on.

It shows that the *Vertical Asymptote* of the curve of degree n exists only when y^n is absent and obtained by equating to zero the coefficient of highest degree terms in y.

Similarly, *Horizontal Asymptote* of the curve of degree n exists only when x^n is absent and obtained by equating to zero the coefficient of highest degree terms in x.

Oblique asymptotes

Let f(x, y) = 0 be the equation an algebraic curve of n^{th} degree in x and y.

So it can be written in the form

$$(a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots + a_n y^n)$$
$$+ (b_0 x^{n-1} + b_1 x^{n-2} y + b_2 x^{n-3} y^2 + \ldots\ldots + b_{n-1} y^{n-1})$$
$$+ c_0 x^{n-2} + c_1 x^{n-3} y + c_2 x^{n-4} y^2 + \ldots\ldots + c_{n-2} y^{n-2}) + \ldots = 0$$

where $a_0, a_1 \ldots\ldots a_n;\ b_1, b_1 \ldots\ldots b_{n-1};\ c_0, c_1, \ldots\ldots c_{n-2}$ are constants.

$$\text{or}\quad x^n \left\{ a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \ldots\ldots + a_n \left(\frac{y}{n}\right)^n \right\}$$
$$+ x^{n-1} \left\{ b_0 + b_1 \left(\frac{y}{x}\right) + b_2 \left(\frac{y}{x}\right)^2 + \ldots\ldots b_{n-1} \left(\frac{y}{n}\right)^{n-1} \right\}$$
$$+ x^{n-2} \left\{ c_0 + c_1 \left(\frac{y}{x}\right) + c_2 \left(\frac{y}{x}\right)^2 + \ldots\ldots c_{n-2} \left(\frac{y}{n}\right)^{n-2} \right\} + \ldots\ldots = 0$$

$$\text{or}\quad x^n \phi_n \left(\frac{y}{x}\right) + x^{n-1} \phi_{n-1} \left(\frac{y}{x}\right) + x^{n-2} \phi_{n-2} \left(\frac{y}{x}\right) + \ldots\ldots\ldots = 0 \quad \ldots(1)$$

Let y = mx + c ..(2)

where m and c are constants be an *Oblique Asymptote* of the cur
(1).

Solving (1) and (2), we get

$$x^n\phi_n\left(\frac{mx+c}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{mx+c}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{mx+c}{x}\right)+\ldots=0$$

or $$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\ldots=0$$

Using Taylor's theorem

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{1}{2!}\frac{c^2}{x^2}\phi_n''(m)+\frac{1}{3!}\frac{c^3}{x^3}\phi_n'''(x)+\ldots\ldots\right]$$

$$+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi_{n-1}'(m)+\frac{1}{2!}\frac{c^2}{x^2}\phi_{n-1}''(m)+\frac{1}{3!}\frac{c^3}{x^3}\phi_{n-1}'''(m)+\ldots\right]$$

$$+x^{n-2}\left[\phi_{n-2}(m)+\frac{c}{x}\phi_{n-2}'(m)+\frac{1}{2!}\frac{c^2}{x^2}\phi_{n-2}''(m)+\frac{1}{3!}\frac{c^3}{x^3}\phi_{n-2}'''(m)+\ldots\right]$$

$$+\ldots\ldots\ldots\ldots=0$$

For making $x\to\infty$, $y - mx - c$ tends to zero, we must have the highest coefficient of x^n, x^{n-1} to zero.

$\therefore\quad \phi_n(m)=0$ ....(2)

This gives the value of m.

and $c\phi_n'(m)+\phi_{n-1}(m)=0$

$c=-\frac{\phi_{n-1}(m)}{\phi_n'(m)}$, $\quad \phi_n(m)\neq 0.$ ....(3)

Substituting the values of m and c obtained from (2) and (3) in $y = mx + c$, we get the equation of *Oblique Asymptotes* to the given curve.

Note:

If $\phi_n'(m)=0$ then from (3) the values of c cannot be determined. The values of m obtained from $\phi_n(m) = 0$ makes $\phi_n'(m) = 0$ and $\phi_{n-1}(m)=0$ i.e. *the two values of m are equal*, then equating the coefficient of x^{n-2} to zero.

$$\frac{c^2}{2!}\phi_n''(m)+\frac{c}{1!}\phi_{n-1}'(m)+\phi_{n-2}(m)=0,\qquad \phi_n''(m)\neq 0.$$

If $\phi_n''(m) = 0$, then from (4) the values of c can not also be determined. The values of m obtained from $\phi_n(m) = 0$ make

$\phi_n''(m)=0$, $\phi_{n-1}'(m)=0$ and $\phi_{n-2}(m)=0$ i.e. *the three values of m are equal*, then equating the coefficient of x^{n-3} to zero.

$$\frac{c^3}{3!}\phi_n'''(m)+\frac{c^2}{2!}\phi_{n-1}''(m)+\phi_{n-2}'(m)+\phi_{n-3}(m)=0.$$

provided $\phi_n'''(m)\neq 0$

5.6 Equation of Asymptotes on a Polar Curve

Let $r = f(\theta)$ be the equation of polar curve, then it can be written as $\frac{1}{r}=\frac{1}{f(\theta)}=F(\theta)$ say.

Writing $F(\theta) = 0$ give the values θ. So the equation of the asymptote in polar form is $r\sin(\theta-\alpha)=\frac{1}{F'(\alpha)}$.

Worked Out Examples

Ex. 1: Find the asymptotes of the curves

i. $y=\frac{x}{(x-1)^2(x-2)}$ ii. $y=\frac{2x-3}{x^2-3x+2}$

iii. $y=\frac{x^2-4x+5}{x^2+4x+3}$ iv. $x^3+y^3=3axy.$

i. $y=\frac{x}{(x-1)^2(x-2)}$

Solution:

Given curve is

$$y=\frac{x}{(x-1)^2(x-2)}$$

For the vertical asymptote

$$\lim_{x\to a} f(x)=\infty$$

Here $\lim_{x\to 1} y=\lim_{x\to 1}\frac{x}{(x-1)^2(x-2)}=\frac{1}{0}=\infty$

and $\lim_{x\to 2} y=\lim_{x\to 2}\frac{x}{(x-1)^2(x-2)^2}=\frac{2}{0}=\infty$

Therefore, there are two vertical asymptotes, $x = 1$ and $x = 2$

For the horizontal asymptote

$$\lim_{x\to\infty} f(x) = b$$

Here $$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x}{(x-1)^2(x-2)} = \lim_{x\to\infty} \frac{x}{(x-1)^2 x\left(1-\frac{2}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{(x-1)^2\left(1-\frac{2}{x}\right)} = \frac{1}{\infty} = 0$$

$\therefore$ $y = 0$

Hence the asymptotes are $x = 1, x = 2$ and $y = 0$

ii. $y = \dfrac{2x-3}{x^2-3x+2}$

Solution:

Given curve is

$$y = \frac{2x-3}{x^2-3x+2}.$$

For the vertical asymptote,

Here $$\lim_{x\to 1} y = \lim_{x\to 1} \frac{(2x-3)}{(x-1)(x-2)} = -\frac{1}{0} = \infty$$

and $$\lim_{x\to 2} y = \lim_{x\to 2} \frac{(2x-3)}{(x-1)(x-2)} = \frac{+1}{0} = \infty$$

Hence the vertical asymptote are $x = 1$ and $x = 2$.

For the horizontal asymptote

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{(2x-3)}{(x^2-3x+2)}$$

$$= \lim_{x\to\infty} \frac{x\left(2-\frac{3}{x}\right)}{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)} = \lim_{x\to\infty} \frac{\left(2-\frac{3}{x}\right)}{x\left(1-\frac{3}{x}+\frac{2}{x^2}\right)} = \frac{2-0}{\infty} = 0$$

Thus, horizontal asymptote is $y = 0$.

Hence the asymptotes are $x = 1, x = 2, y = 0$

iii. $y = \dfrac{x^2-4x+5}{x^2+4x+3}$

Solution:

Here, the equation of curve is

$$y = \frac{x^2-4x+5}{x^2+4x+3}$$

or $$y = \frac{x^2-4x+5}{(x+1)(x+3)}$$

Now,

$$\lim_{x\to -3} = \lim_{x\to -3} \frac{x^2-4x+5}{(x+1)(x+3)} = \frac{9+12+5}{0} = \frac{26}{0} = \infty$$

and $$\lim_{x\to -1} y = \lim_{x\to -1} \frac{x^2-4x+5}{(x+1)(x+3)} = \frac{1+4+5}{0} = \frac{10}{0} = \infty$$

Therefore there are two vertical asymptotes

$x = -3, x = -1$

Also, $$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x^2-4x+5}{(x^2+4x+3)} = \lim_{x\to\infty} \frac{1-\frac{4}{x}+\frac{5}{x^2}}{1+\frac{4}{x}+\frac{3}{x^2}}$$

$$\therefore \lim_{x\to\infty} y = 1$$

So the horizontal asymptote is $y = 1$

Hence the three asymptotes are

$x + 1 = 0, x + 3 = 0, y - 1 = 0$

iv. $x^3 + y^3 = 3axy$.

Solution:

Given curve is

$$x^3 + y^3 - 3axy = 0.$$

This is the equation of 3rd degree. It has at most three asymptotes. Here, x^3, y^3 are both present, so, there is no horizontal and vertical asymptotes.

For the oblique asymptote, putting 1 for x and m for y in 3rd, 2nd, 1st degree terms to obtain $\phi_3(m)$, $\phi_2(m)$ and $\phi_1(m)$ respectively.

$\phi_3(m) = 1 + m^3$, $\quad \phi_2(m) = -3am$, $\quad \phi_1(m) = 0$,

$\phi_3'(m) = 3m^2$

Now,

$\phi_3(m) = 0$ gives

$1 + m^3 = 0 \quad \therefore\ m = -1$

Since, $\phi'_3(-1) = 3 \neq 0$,

$\therefore \quad c = -\frac{\phi_2(-1)}{\phi'_3(-1)} = -\frac{3a}{3} = -a$

We have $y = mx + c$ is oblique asymptote.

Putting the values of m and c in this equation, we get

$y = -x - a$

Hence the asymptote is $x + y + a = 0$

Ex. 2: Find the asymptotes of the following curves:

i. $x^2y + xy^2 = a^3$ ii. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$

iii. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ iv. $y^2 = \frac{(a-x)^2}{a^2 + x^2} x^2$

i. $x^2y + xy^2 = a^3$

Solution:

Given curve is

$$x^2y + xy^2 = a^3$$

This is the equation of 3[rd] degree. It has at most three asymptotes. x^3 and y^3 are absent, so there is horizontal and vertical asymptotes. *For the horizontal asymptote*, equating the coefficient of highest degree term in x with zero

i.e. $y = 0$.

For the vertical asymptote, equating the coefficient of highest degree term in y with zero

i.e. $x = 0$

For the oblique asymptotes,

Putting 1 for x and m for y in 4[th], 3[rd], 2[nd], 1[st] degree terms to the curve to obtain $\phi_3(m)$, $\phi_2(m)$, $\phi_1(m)$, respectively.

$\phi_3(m) = m + m^2$, $\quad \phi_2(m) = 0$, $\quad \phi_1(m) = 0$

$\phi'_3(m) = 1 + 2m$

Now, $\phi_3(m) = 0$ gives,

$m(1 + m) = 0$, $\quad \therefore m = 0, -1$

Since $\phi'_3(0) = 1 \neq 0$ $\quad \therefore c = -\frac{\phi_2(0)}{\phi'_3(0)} = -\frac{0}{1} = 0$

$\phi'_3(-1) = -1 \neq 0$, $\quad \therefore c = -\frac{\phi_2(-1)}{\phi'_3(-1)} = -\frac{0}{-1} = 0$

Thus the oblique asymptote is $y = mx + c$,

$y = 0$, $y = -x$

Hence the asymptotes are $x = 0$, $y = 0$, $y = -x$.

ii. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$

Solution:

Given curve is

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$$

It is the equation of 4[th] degree. It has at most four asymptotes and the terms x^4 and y^4 are absent, so there are horizontal and vertical asymptotes.

For the horizontal asymptote, equating the coefficient of highest degree terms in x with zero,

$y^2 - y = 0 \quad \therefore y = 0, \ y = 1$

For the vertical asymptote, equating the coefficient of highest degree term in y with zero,

$x^2 - x = 0 \quad \therefore x = 0, x = 1$

Clearly, so there is no oblique asymptote.

Hence the asymptotes are $x = 0$, $y = 0$, $x = 1$, $y = 1$.

iii. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution:

Given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or $\quad b^2x^2 - a^2y^2 - a^2b^2 = 0$

This is the equation of the curve of degree 2. It has at most two asymptotes, and the terms x^2 and y^2 are both present, there are no horizontal and vertical asymptotes.

For the oblique asymptote, putting 1 for x and m for y in 2[nd], 1[st] degree terms to obtain $\phi_2(m)$, $\phi_1(m)$, respectively.

$\phi_2(m) = b^2 - a^2m^2$, $\quad \phi_1(m) = 0$

$\phi'_2(m) = -2a^2m$

Now, $\phi_2(m) = 0$ gives

$b^2 - a^2m^2 = 0$, $\therefore m = \pm b/a$

Since $\phi'_2(\pm b/a) = -2ab \neq 0$

$\therefore$ $c = -\dfrac{\phi_1(\pm b/a)}{\phi_2'(\pm b/a)} = -\dfrac{0}{-2ab} = 0$

Thus the oblique asymptote is $y = mx + c$

or $y = \pm\dfrac{b}{a}x + 0$

Hence the asymptote are $y = \pm\dfrac{b}{a}x$.

iv. $y^2 = \dfrac{(a-x)^2}{a^2+x^2}x^2$

Solution:

Given curve is

$$y^2 = \frac{(a-x)^2}{a^2+x^2}x^2$$

or $a^2y^2 + x^2y^2 - a^2x^2 + 2ax^3 - x^4 = 0$

This is the 4^{th} degree equation in x and y. It has at most four asymptotes and x^4 is present so there is no horizontal asymptote but y^4 is absent so there is vertical asymptote.

For this, equating the coefficient of highest degree term in y with zero

i.e. $a^2 + x^2 = 0$,

or $x = \pm\sqrt{-a^2}$

This is imaginary, so there is also no vertical asymptote.

For oblique asymptote, putting 1 for x and m for y in 4^{th}, 3^{rd}, 2^{nd} and 1^{st} degree terms of the given equation to obtain $\phi_4(m)$, $\phi_3(m)$, $\phi_2(m)$, $\phi_1(m)$, respectively.

$\phi_4(m) = m^2 - 1$, $\phi_3(m) = 2a$, $\phi_2(m) = a^2m^2 - a^2$,

$\phi_4'(m) = 2m$ $\phi_1(m) = 0$,

Now $\phi_4(m) = 0$ gives

$m^2 - 1 = 0$, $\therefore m = \pm 1$

Since $\phi_4'(1) = 2 \neq 0$, $\therefore c = -\dfrac{\phi_3(1)}{\phi_4'(1)} = -\dfrac{2a}{2} = -a$

and $\phi'_4(-1) = -2, \neq 0$ $\therefore c = -\dfrac{\phi_3(-1)}{\phi_4'(-1)} = -\dfrac{2a}{-2} = a$

The oblique asymptote is $y = mx + c$

i.e. $y = 1.x - a$, and $y = -1.x + a$

Hence the asymptotes are $y - x + a = 0$, $y + x - a = 0$.

Ex. 3: Find the asymptotes of the curve $x^2 - y^2 + 2x + 2y + 3 = 0$

Solution:

Here, the equation of the curve is

$$x^2 - y^2 + 2x + 2y + 3 = 0 \quad ...(1)$$

Which is second-degree equation in x and y, so it has at most two asymptotes. Since x^2 and y^2 both present, there are no horizontal and vertical asymptotes.

For the oblique asymptote, put x = 1 and y = m in 2^{nd} and 1^{st} degree terms in (1)

$\phi_2(m) = 1 - m^2$, $\phi_1(m) = 2 + 2m$

$\phi_2'(m) = -2m$

For the values of m, put $\phi_2(m) = 0$

or $1 - m^2 = 0$

or $(m-1)(m+1) = 0$

$\therefore$ $m = 1, -1$

Now, we have $c = -\dfrac{\phi_1(m)}{\phi_2'(m)}$

When m = 1, $c = -\dfrac{\phi_1(1)}{\phi_2'(1)} = -\dfrac{4}{-2} = 2$

$\therefore$ $c = -2$

When m = -1, $c = -\dfrac{\phi_1(-1)}{\phi_2'(-1)} = -\dfrac{0}{2} = 0$

The asymptotes are $y = x + 2$, $y = -x$.

Ex. 4: Find the asymptotes of the curve $(a + x)^2(b^2 + x^2) = x^2y^2$

Solution:

Given curve is

$(a + x)^2(b^2 + x^2) = x^2y^2$

or $a^2b^2 + 2ab^2x + b^2x^2 + a^2x^2 + 2ax^3 + x^4 - x^2y^2 = 0$

This is 4^{th} degree equation, so it has four asymptotes.

Here, x^4 is present, so there is no horizontal asymptote and y^4 is absent, so there is vertical asymptote.

For the vertical asymptote, equating the coefficient of highest degree, terms in y with zero

i.e. $-x^2 = 0$, $\therefore x = 0$

For oblique asymptote, putting 1 for x and m for y in 4^{th}, 3^{rd}, 2^{nd} and 1^{st} degree terms of the curve to obtain $\phi_4(m)$, $\phi_3(m)$, $\phi_2(m)$ and $\phi_1(m)$, respectively.

$\phi_4(m) = 1 - m^2$, $\phi_3(m) = 2a$, $\phi_2(m) = b^2 + a^2$, $\phi_1(m) = 2ab^2$

$\phi_4'(m) = -2m$

Now, $\phi_4(m) = 0$ gives

$1 - m^2 = 0, m = \pm 1$

Since $\phi_4'(1) = -2 \neq 0$ $\quad \therefore c = -\dfrac{\phi_3(1)}{\phi'_4(1)} = -\dfrac{2a}{-2} = a$

and $\phi'_4(-1) = 2 \neq 0$ $\quad \therefore c = -\dfrac{\phi_3(-1)}{\phi'_4(-1)} = -\dfrac{2a}{2} = -a$

Let $y = mx + c$ is oblique asymptote

Putting the values of m and c in this equation, we get,

$y = 1.x + a$, and $\quad y = -1.x - a$

Hence the asymptotes are

$x = 0, \quad y - x - a = 0, \quad y + x + a = 0.$

Ex. 5: Find the asymptotes of the curve $(x + y)^2 (x+2y + 2) = x + 9y - 2$

Solution:

Here, the equation of the given curves is

$(x + y)^2 (x + 2y + 2) = x + 9y - 2$

or $(x + y)^2 (x + 2y) + 2(x + y)^2 = x + 9y - 2$

or $(x^2 + 2xy + y^2)(x + 2y) + 2(x^2 + 2xy + y^2) = x + 9y - 2$

or $x^3 + 2y^3 + 4x^2y + 5xy^2 + 2x^2 + 2y^2 + 4xy - 9y + 2 = 0$

Which is third degree equation in x and y, so it has at most three asymptotes. Since x^3 and y^3 are both present, there are no horizontal and vertical asymptotes.

For the oblique asymptotes,

Let $y = mx + c$ be an asymptote,

For this, put x = 1, y = m in 3^{rd}, 2^{nd} and 1^{st} degree terms of the given curve, then

$\phi_3(m) = 1 + 2m^3 + 4m + 5m^2$,

$\phi_2(m) = 2 + 2m^2 + 4m$

$\phi_1(m) = -1 - 9m$

$\phi_3'(m) = 6m^2 + 4 + 10m$

$\phi_3''(m) = 12m + 10$ and $\phi_2'(m) = 4m + 4.$

For the value of m, put $\phi_3(m) = 0$

$2m^3 + 5m^2 + 4m + 1 = 0$

or $2m^3 + 2m^2 + 3m^2 + 3m + m + 1 = 0$

or $(m + 1)(2m^2 + 3m + 1) = 0$

or $(m + 1)(m + 1)(2m + 1) = 0$

$\therefore m = -1, -1, -1/2$

Also, $c = -\dfrac{\phi_2(m)}{\phi_3'(m)}$

or $c = -\dfrac{2 + 2m^3 + 4m}{6m^2 + 4 + 10m}$

When $m = -\dfrac{1}{2}$, $c = -1$,

Here, m = - 1 is repeated root and $\phi_3'(-1) = 0$

So using

$\dfrac{c^2}{2!}\phi_3''(m) + \dfrac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$

or $\dfrac{c^2}{2!}(12m + 10) + c(4m + 4) + (-1 - 9m) = 0$

When, m = - 1

$\dfrac{c^2}{2!} \times -2 + 0 + 8 = 0$

or $-c^2 + 8 = 0$

$\therefore c = \pm 2\sqrt{2}$

Hence the asymptotes are

$x + 2y + 2 = 0, \ x + y \pm 2\sqrt{2} = 0.$

Ex. 6: Find the asymptotes of the curve $x(x - y)^2 - 3(x^2 - y^2) + 8y = 0$

Solution:

Given curve is

$x(x - y)^2 - 3(x^2 - y^2) + 8y = 0$

or $x^3 - 2x^2y + xy^2 - 3x^2 + 3y^2 + 8y = 0$

This is the equation of 3^{rd} degree, it has at most three asymptotes and x^3 is present, so there is no horizontal asymptote but y^3 is absent, there is vertical asymptote.

For the vertical asymptotes, equating the coefficient of highest degree terms in y with zero.

i.e. $x + 3 = 0$

For oblique asymptote, putting 1 for x and m for y in 3rd, 2nd and 1st degree terms to obtain $\phi_3(m)$, $\phi_2(m)$ and $\phi_1(m)$, respectively.

$\phi_3(m) = 1 - 2m + m^2$, $\quad \phi_2(m) = -3 + 3m^2$, $\quad \phi_1(m) = 8m$

$\phi_3'(m) = 2m - 2$, $\quad \phi_2'(m) = 6m$

$\phi_3''(m) = 2$,

Now, $\phi_3(m) = 0$ gives

$m^2 - 2m + 1 = 0$, $\quad \therefore m = 1, 1$

Since $\phi_3'(1) = 0$, m has repeated roots, $\quad \phi_3''(1) = 2 \neq 0$

Thus we have

$$\frac{c^2}{2!}\phi_3''(1) + \frac{c}{1!}\phi_2'(1) + \phi_1(1) = 0$$

or $\frac{c^2}{2!}.2 + \frac{c}{1}.6 + 8 = 0$

or $c^2 + 6c + 8 = 0$

or $(c + 4)(c + 2) = 0$, $\quad \therefore c = -2, -4$

Thus, the oblique asymptote is

$y = mx + c$

Putting the values of m and c in this equation, we get

$y = 1.x - 2$ and $y = 1.x - 4$

Hence the asymptotes are

$x + 3 = 0$, $x - y = 2$, $\quad x - y = 4$.

Ex. 7: Find the asymptotes of the curve $(x^2 - y^2)^2 - 2(x^2 + y^2) + x - 1 = 0$

Solution:

Here, the equation of the curve is

$(x^2 - y^2)^2 - 2(x^2 + y^2) + x - 1 = 0$

or $x^4 - 2x^2y^2 + y^4 - 2x^2 - 2y^2 + x - 1 = 0$

Which is fourth degree equation in x and y, so it has at most four asymptotes.

Since x^4 and y^4 are both present, there are no horizontal and vertical asymptotes.

For the oblique asymptote,

Let $y = mx + c$ be the oblique asymptotes

For this, put $x = 1$, and $y = m$ in 4th, 3rd, 2nd and 1st degree terms of the given equation to get $\phi_4(m)$, $\phi_3(m)$, and $\phi_2(m)$ respectively.

Then,

$\phi_4(m) = 1 - 2m^2 + m^4$, $\quad \phi_3(m) = 0$, $\quad \phi_2(m) = -2 - 2m^2$

$\phi_4'(m) = -4m + 4m^3$, $\quad \phi_3'(m) = 0$

$\phi_4''(m) = -4 + 12m^2$

Now, $\phi_4(m) = 0$ gives

$1 - 2m^2 + m^4 = 0$

$\therefore \quad m = 1, 1, -1, -1$

So, we have

$$\frac{c^2}{2!}\phi_4''(m) + \frac{c}{1!}\phi_3'(m) + \phi_2(m) = 0$$

or $\frac{c^2}{2!}(-4 + 12m^2) + c.0 + (-2 - 2m^2) = 0$

When $m = \pm 1$

$\frac{c^2}{2}.8 - 4 = 0$

or $4c^2 - 4 = 0$

or $c^2 - 1 = 0$ $\quad \therefore c = \pm 1$

Therefore, the four asymptotes are

$y = x \pm 1$ and $y = -x \pm 1$.

Exercise-5

1. Find the asymptotes of the following curves

i. $y = \frac{3x}{x - 2}$ $\quad$ ii. $y = \frac{4x^2 + 4x - 3}{x^2 - 4x + 3}$

iii. $y = \frac{x^2 + 2x - 1}{x}$ $\quad$ iv. $y = \frac{x^2}{x^2 + 1}$

v. $y = \frac{(x + 2)^2(x - 3)}{x - 1}$ $\quad$ vi. $x^3 - y^3 = 3ax^2$

vii. $y(y - 1)^2 - x^2y = 0$ $\quad$ viii. $xy^2 - a^2(x - a) = 0$

ix. $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ $\quad$ x. $(y - a)^2(x^2 - a^2) = x^4 + a^4$

xi. $x^3 + xy^2 - ay^2 = 0$ $\quad$ xii. $x^2y^2 = a^2(x^2 + y^2)$

2. Find the asymptotes of the following curves

i. $y^3 - x^2y + 2y^2 + 4y + x = 0$

ii. $y^3 + x^2y + 2xy^2 - y + 1 = 0$

iii. $x^2y^2 - 4(x - y)^2 + 2y - 3 = 0$

iv. $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$

v. $x^3 + y^3 - xy^2 - x^2y + x^2 - y^2 = 1$ 2059/060/061/062 B.E.

vi. $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$ 2059 B.E.

vii. $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$

viii. $x^3 - 4y^3 + 3x^2y + y - x + 3 = 0$

ix. $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$

x. $y^3 + 2xy^2 + x^2y - y + 1 = 0$

3. Find the asymptotes of the following curves

   i. $r\cos\theta = a\sin\theta$ ii. $r\theta = a$

   iii. $2r^2 = \tan 2\theta$.

Answers

1. i. $x = 2, y = 3$ ii. $x = 1, x = 3, y = 4$

   iii. $x = 0, x - y + 2 = 0$ iv. $y = 1$

   v. $x = 1$ vi. $x - y - a = 0$

   vii. $y = 0, x + y - 1 = 0, x - y + 1 = 0$ viii. $x = 0, y = \pm a$,

   ix. $x = \pm a, y = \pm b$ x. $x = \pm a, y \pm x = a$,

   xi. $x = a$ xii. $x = \pm a, y = \pm a$.

2. i. $y = 0, x - y - 1 = 0, x + y + 1 = 0$

   ii. $y = 0, x + y \pm 1 = 0$

   iii. $x = \pm 2, y = \pm 2$.

   iv. $x = \pm a, x - y = \pm a\sqrt{2}$

   v. $x - y = 0, x + y = 0, x - y + 1 = 0$

   vi. $y = x, y = 2x, y = 3x$

   vii. $x = 0, x - y = 0, x - y + 1 = 0$

   viii. $y = x, x + 2y - 1 = 0, x + 2y + 1 = 0$

   ix. $x - y = 0, x + y = 0, x + 2y + 1 = 0$

   x. $y = 0, y + x - 1 = 0, x + y + 1 = 0$

3. i. $r\cos\theta = \pm a$ ii. $r\sin\theta = a$

   iii. $\theta = \pm\frac{\pi}{4}$

❖❖❖

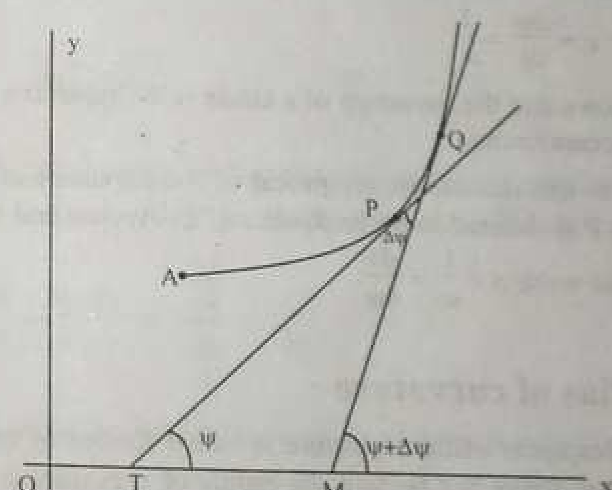
Chapter - 6

Curvature

6.1 Introduction

The rate of change of direction of curves with respect to the arc is called *Curvature* at the point P along the curve.

Let AP = s be an arc of the curve $y = f(x)$, and the tangent at P makes an angle ψ with x-axis. The slope at a point to the curve determines the direction of the curve at that point but the measure of the rapidity with which the curve is running or bending at the point is given by the rate of change of ψ with respect to s, the arc-length.



Let Q be near to P on the curve so that the tangent at Q makes an angle $\psi + \Delta\psi$ with x-axis.

So, $AP = s$, $AQ = s + \Delta s$, $PQ = \Delta s$,

$\angle QTX = \psi$, $\angle QMX = \psi + \Delta\psi$,

Then, the rate of change of ψ with respect to s is defined by

$$\frac{d\psi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s}$$

where $\frac{\Delta\psi}{\Delta s}$ is called *Average Curvature* of the arc PQ.

Thus, the curvature of the curve at the point P is denoted by κ(kappa) and We write

$$\kappa = \frac{\Delta\psi}{\Delta s}$$

For example,

The equation of circle with center (0, 0) and radius 'a' is

$$s = a\psi$$

Differentiating it with respect to s

$$\frac{ds}{d\psi} = a$$

Curvature of the circle,

$$\kappa = \frac{d\psi}{ds} = \frac{1}{a}$$

It shows that the curvature of a circle is the same at every point of its circumference.

Due to this reason, the reciprocal of the curvature of a curve at point P is defined to be its *Radius of Curvature* and denoted by ρ and we write $\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$

6.2 Radius of curvature

The reciprocal of the curvature is called *Radius of curvature*. We can now derive for finding the radius of curvature ρ to the given curves.

Radius of Curvature for the Cartesian equation

Let the tangent at P(x, y) of the curve y = f(x) makes an angle ψ with x-axis, then

$$\frac{dy}{dx} = \tan\psi, \quad \frac{dx}{ds} = \cos\psi$$

Differentiating the first relation with respect to 's', we get,

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{ds}(\tan\psi)$$

or $$\frac{d^2y}{dx^2}\frac{ds}{dx} = \sec^2\psi\frac{d\psi}{ds}$$

or $$\frac{d^2y}{dx^2}\cos\psi = \sec^2\psi\frac{d\psi}{ds}$$

or $$\frac{d^2y}{dx^2} = \sec^2\psi.\sec\psi\frac{d\psi}{ds}$$

or $$\frac{d^2y}{dx^2}\frac{ds}{d\psi} = (1+\tan^2\psi)\sqrt{(1+\tan^2\psi)}$$

or $$\frac{d^2y}{dx^2}.\rho = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}$$

$$\therefore \quad \rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where } y_2 \neq 0.$$

In case $\frac{dy}{dx}$ becomes infinite at any point, we can derive the formula for

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}, \quad \text{where } x_1 = \frac{dx}{dy}, x_2 = \frac{d^2x}{dy^2} \neq 0.$$

Radius of Curvature for the parametric equation

Let x = x(t) and y = y(t) be the equation of parametric curve then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$$

Where dashes denote differentiation with respect to 't'.

Also, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx} = \frac{x'y''-y'x''}{x'^2}.\frac{1}{x'}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{x'y''-y'x''}{x'^3}$$

We have

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}}{\frac{x'y''-y'x''}{x'^3}}$$

$$\therefore \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}.$$

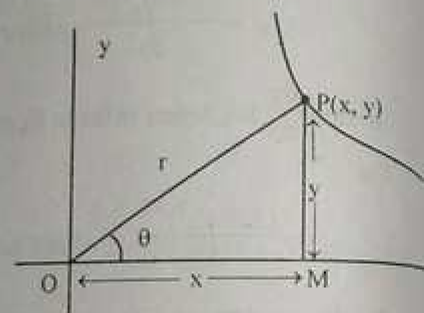
6.3 Polar Equation

The equation of the form $r = f(\theta)$ where r is radius vector measu[red] from origin to the point on the curve and θ be the angle made b[y] the radius vector with x-axis is called *Polar Equation.*

To change polar to Cartesian coordinates, let P(x, y) be any poi[nt] on the curve. Draw PM perpendicular to OX so that,

$$OM = x,\ PM = y$$

In the figure,



$$\angle POX = \theta$$

$$OP = r.$$

$$\cos\theta = \frac{OM}{OP} = \frac{x}{r}$$

$$\therefore \quad x = r\cos\theta$$

and $\sin\theta = \frac{PM}{OP} = \frac{y}{r}$

$$\therefore \quad y = r\sin\theta$$

Also, $OP^2 = OM^2 + PM^2$

or $r^2 = x^2 + y^2$

or $r^2 = x^2 + y^2$ and

$$\tan\theta = \frac{PM}{OM}$$

$$\therefore \quad x^2 + y^2 = r^2, \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

Theorem:

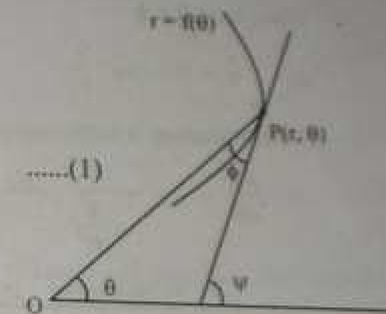
If ϕ be the angle between the tangent and radius vector at a[ny] point the curve $r = f(\theta)$ then $\tan\phi = r\dfrac{d\theta}{dr}$.

Proof:

Let P(r, θ) be the point on the curve $r = f(\theta)$ and the tangent [at] P(r, θ) makes an angle ψ with the initial line OX and the radi[us] vector OP makes and angle θ with OX.

Let ϕ be the angle between radius vector and tangent at P to the given curve $r = f(\theta)$.

From the figure, we have



$$\psi = \theta + \phi$$

or $\tan\psi = \tan(\theta + \phi)$

or $\tan\psi = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$ ......(1)

Also, we know that

$$\tan\psi = \frac{dy}{dx}$$

To change polar coordinates, we have the relation

$$x = r\cos\theta,\ y = r\sin\theta,$$

$$\text{So, } \tan\psi = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

$$\text{So, } \tan\psi = \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - r\frac{d\theta}{dr}\tan\theta} \qquad \text{..............(2)}$$

Comparing (1) and (2), we get

$$\tan\phi = r\frac{d\theta}{dr}.$$

Radius of Curvature for the Polar Equation

Let PT be the tangent at P(r, θ) to the given equation of the curve $r = f(\theta)$ and the tangent at P makes an angle ψ with x-axis.

Let θ be the angle made by the tangential line with radius vector and θ be vectorial angle.

We have

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\cdot\frac{d\theta}{d\psi} = \frac{ds}{d\theta} \Big/ \frac{d\psi}{d\theta} \qquad \text{..............(1)}$$

and $\tan\phi = r\dfrac{d\theta}{dr} = r\Big/\dfrac{dr}{d\theta} = \dfrac{r}{r_1}$

$$\therefore \phi = \tan^{-1}\frac{r}{r_1}$$

Now, $\psi = \theta + \phi$

or $\psi = \theta + \tan^{-1}\frac{r}{r_1}$

Differentiating it with respect to θ,

$$\frac{d\psi}{d\theta} = 1 + \frac{1}{1+\frac{r^2}{r_1^2}} \cdot \frac{r_1 r_1 - r r_2}{r_1^2} = 1 + \frac{r_1^2 - r r_2}{r_1^2 + r^2}$$

$$= \frac{r_1^2 + r^2 + r_1^2 - r r_2}{r_1^2 + r^2} = \frac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2}$$

$$\therefore \frac{d\psi}{d\theta} = \frac{r^2 + 2r_1^2 - r r_2}{r^2 + r_1^2} \qquad \text{.............(2)}$$

Again, $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$$\therefore \frac{ds}{d\theta} = \sqrt{r^2 + r_1^2} \qquad \text{.............(3)}$$

From (1), (2) and (3),

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

6.4 Pedal Equation of a Curve

The equation is of the form p = f(r), where p, length of perpendicular from the origin to the tangent at any point P on the curve and r, the radius vector of the point of contact P from some given point O is called *Pedal equation* of the curve with respect to the origin O.

Pedal equation deduced from Cartesian equation

Let y = f(x) be the equation of the curve.

The tangent at P(x, y) to the curve is

$$Y - y = \frac{dy}{dx}(X - x)$$

or $Y - y = f'(x)(X - x)$

or $f'(x)X - Y - x f'(x) + y = 0$

Let p be the length of perpendicular from origin on it then

$$p = \frac{y - x f'(x)}{\sqrt{1 + \{f'(x)\}^2}}$$

Squaring,

$$p^2 = \frac{\{y - x f'(x)\}^2}{1 + \{f'(x)\}^2} \qquad \text{———(1)}$$

Also, $r^2 = x^2 + y^2$ ———(2)

and $y = f(x)$ ———(3)

Eliminating x and y from (1), (2) and (3), we get, the required *Pedal equation*.

Pedal equation deduced from polar equation

Let r = f(θ) be the equation of the curve and p be the length of perpendicular from the pole on the tangent at P(r, θ) to the curve r = f(θ)

Then, we have

$p = r\sin\phi$ ———(1)

$\tan\phi = r\frac{d\theta}{dr}$ ———(2)

and $r = f(\theta)$ ———(3)

Eliminating θ and ϕ from the relation (1), (2) and (3) we get the required *Pedal equation* of the curve.

Radius of Curvature for the Pedal Equation

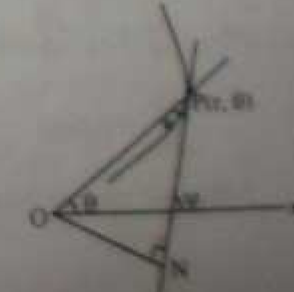
The tangent at P(r, θ) to the given equation of curve p = f(r) makes an angle ψ with initial line OX and θ be vectorial angle.

Let ϕ be the angle between radius vector and tangential line. Draw ON perpendicular to PN such that ON = p.

We have

$$\sin\phi = r\frac{d\theta}{ds}, \qquad \cos\phi = \frac{dr}{ds},$$

$$\psi = \theta + \phi$$



We know that

$$p = r\sin\phi$$

Differentiating it with respect to r,

$$\frac{dp}{dr} = r\cos\phi\frac{d\phi}{dr} + \sin\phi$$

$$= r\frac{dr}{ds}\frac{d\phi}{dr} + r\frac{d\theta}{ds} = r\frac{d\phi}{ds} + r\frac{d\theta}{ds}$$

$$= r\left(\frac{d\phi}{ds} + \frac{d\theta}{ds}\right) = r\frac{d}{ds}(\phi + \theta) = r\frac{d\psi}{ds}$$

or $\frac{dp}{dr} = r\frac{d\psi}{ds} = r.\frac{1}{\rho}$

$\therefore \rho = r\frac{dr}{dp}.$

Radius of Curvature for the Tangential Polar Equation

Let the tangential polar equation of the curve is $p = f(\psi)$ where p is the length of perpendicular from the pole to the tangent at P and ψ be the angle made by the tangent line at P with initial line.

Then, we have

$$\cos\phi = \frac{dr}{ds}$$

Radius of curvature at $P(r, \theta)$ to the curve $p = f(r)$ is given by

$$\rho = r\frac{dr}{dp}$$

Also, we have

$$p = r\sin\phi \quad \text{.........(1)}$$

Differentiating it with respect to ψ,

$$\frac{dp}{d\psi} = \frac{dp}{dr}\frac{dr}{ds}\frac{ds}{d\psi}$$

or $\frac{dp}{d\psi} = \frac{dp}{dr}\cos\phi.\rho$

or $\frac{dp}{d\psi} = \frac{1}{r}\frac{dp}{dr}.r\cos\phi.\rho = \frac{1}{\rho}.r\cos\phi.\rho$

$\therefore \frac{dp}{d\psi} = r\cos\phi$ ....(2)

Squaring and adding (1) and (2), we get

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2(\sin^2\phi + \cos^2\phi) = r^2$$

Differentiating it with respect to p,

$$2p + 2\frac{dp}{d\psi}\frac{d^2p}{d\psi^2}\frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

or $p + \frac{d^2p}{d\psi^2} = r\frac{dr}{dp}$

or $p + \frac{d^2p}{d\psi^2} = \rho.$

$\therefore \rho = p + \frac{d^2p}{d\psi^2}.$

6.5 Curvature at the origin

The radius of curvature at the origin is obtained by using the following methods.

Method of substitution

We know that the radius of curvature for the equation of curve $y = f(x)$ is

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \quad \text{..............(1)}$$

The curvature at origin is obtained by putting the value y_1 and y_2 at the origin (0, 0) in the equation (1)

If the value of $y_1 = \infty$ or $y_2 = 0$ at the origin (0, 0) then the radius of curvature at origin is obtained by putting the value of x_1 and x_2 at (0, 0) in the relation

$$\rho = \frac{(1 + x_1^2)^{3/2}}{x_2}$$

In some cases, the method of substitution fails or become laborious to find the radius of curvature at the origin. If the curve $y = f(x)$ passes through the origin then the expansion of f(x) is used to find the values of y_1 and y_2 at the origin.

So, $y = f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ...$

Since the curve passes through the origin, $f(0) = 0$,

$$\therefore \quad y = xp + \frac{x^2}{2!} q + \ldots\ldots$$

where $p = f'(0) = y_1$, $q = f''(0) = y_2$

Substituting the value of y in terms of p and q in the given equation of the curve and then equating coefficient of like powers of x, we will obtain the values of p and q. After substituting the values of p and q in the formula

$$\rho = \frac{(1+p^2)^{3/2}}{q}.$$

We get the radius of curvature at origin.

Newton's method

This is a fundamental method of determining the *Radius of Curvature* at the origin but has a limited application because it is used only for the given equation of the curve must passes through the origin and has either x-axis or y-axis as the tangent at the origin.

Let the equation of the curve y = f(x) can be expanded by Maclaurin's theorem, so, we have

$$y = f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots$$

Since the curve passes through origin, $f(0) = 0$

$$\text{So, } y = \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots$$

If the equation of tangent at origin is x-axis, then

$$p = f'(0) = 0.$$

$$\therefore \quad y = \frac{x^2}{2!} q + \ldots\ldots\ldots \quad \text{where } q = f''(0)$$

$$q = \frac{2y}{x^2} + \ldots$$

When $x \to 0$, $y \to 0$,

$$q = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{2y}{x^2}$$

Putting the values of p and q in the formula

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(1+0)^{3/2}}{\frac{2y}{x^2}} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y}$$

$$\therefore \quad \rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y}$$

Instead of axis of x, if the axis of y is the tangent at the origin, then

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x}$$

Note

If the equation of the curve is polar form, then the *Radius of Curvature* at pole is

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y} = \lim_{\theta \to 0} \frac{r^2 \cos^2\theta}{2r \sin\theta}$$

$$= \lim_{\theta \to 0} \left(\frac{r}{2} \cos^2\theta . \frac{\theta}{\sin\theta} . \frac{1}{\theta} \right)$$

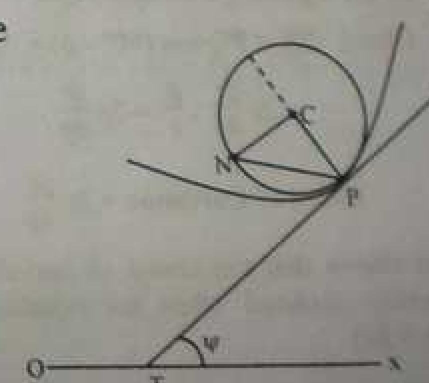
$$= \lim_{\theta \to 0} \left(\frac{r}{2} . 1.1. \frac{1}{\theta} \right)$$

$$\therefore \quad \rho = \lim_{\theta \to 0} \frac{r}{2\theta}$$

6.6 Chord of Curvature

If ρ be the radius of curvature of a curve at point P measured from P along the positive direction of the normal, then the point C is called the *Center of Curvature* at P.

The circle with center C and radius ρ i.e. (CP) is called *Circle of Curvature.*



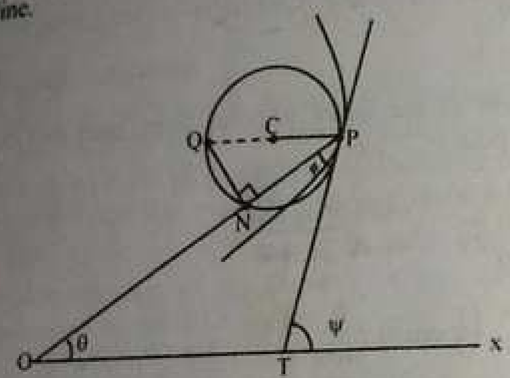
Any chord of the circle of curvature through the point of contact is called a *Chord of Curvature*.

In the figure, PN is a *Chord of Curvature*.

Chord of Curvature through the pole (origin)

Let C be the center of the circle of curvature at the point P on the given curve, the chord PN is a *Chord of Curvature* through the origin O.

Let PT be the tangent at P which makes an angle ψ with initial line.



Join PC and produce it to Q, then

$\angle OPT = \phi, \quad \angle NPQ = 90° - \phi.$

We have

In the right angle triangle PQN,

Chord $PN = PQ\cos(90° - \phi) = 2\rho\sin\phi$

$$= 2r\frac{dr}{dp}\cdot\frac{p}{r} = 2p\frac{dr}{dp}$$

$\therefore$ Chord of Curvature $= 2p\dfrac{dr}{dp}$

It shows that the chord of curvature through the origin can be easily obtained when the equation of curve is pedal equation $p = f(r)$.

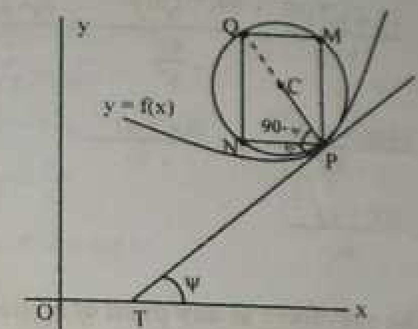
Chord of Curvature parallel to coordinate axes

Let C be the center of circle of curvature and ρ be the radius of curvature at the point P on the given equation of curve $y = f(x)$.

The chords PN and PM are parallel to the coordinate axes. Thus, the chords PN and PM are *Chord of Curvature* parallel to the coordinate axes.

Join PC, produce it to Q, and join NQ and MN. The tangent at P makes an angle ψ with x-axis, so that

$\angle NPT = \psi, \quad \angle NPQ = 90° - \psi$



Chord of Curvature parallel to x-axis,

$PN = PQ\cos(90° - \psi) = 2\rho\sin\psi$

$\therefore$ Chord of Curvature parallel to x-axis $= 2\rho\sin\psi$

Chord of Curvature parallel to y-axis,

$PM = PQ\cos\psi = 2\rho\cos\psi$

$\therefore$ Chord of Curvature parallel to y-axis $= 2\rho\cos\psi$.

Worked out Examples

Ex. 1: Find the radius of curvature at the point (x, y) for the curve $y^2 = 4ax$.

Solution:

Given curve is

$y^2 = 4ax,$

Differentiating it with respect to x,

$$2y\frac{dy}{dx} = 4a,$$

or $y_1 = \dfrac{2a}{y},$

or $y_2 = -\frac{2a}{y^2}y_1 = -\frac{2a}{y^2}\cdot\frac{2a}{y} = -\frac{4a^2}{y^3}$

Therefore, Radius of curvature at any point (x, y) to the given curve is

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{4a^2}{y^2}\right)^{3/2}}{-\frac{4a^2}{y^3}} = \frac{(y^2+4a^2)^{3/2}}{-4a^2}$$

$$= \frac{(4ax+4a^2)^{3/2}}{-4a^2} = \frac{2(x+a)^{3/2}}{-\sqrt{a}}$$

$$\therefore \rho = \frac{2(x+a)^{3/2}}{\sqrt{a}}.$$

Ex. 2: Show that the curvature at any point of a circle is constant.

Solution:

Here, the equation of the circle is

$$x^2 + y^2 = a^2.$$

Differentiating it with respect to x,

$$2x + 2y\frac{dy}{dx} = 0$$

or $\frac{dy}{dx} = -\frac{x}{y}$

or $\frac{d^2y}{dx^2} = -\frac{\left(y - x\frac{dy}{dx}\right)}{y^2} = -\frac{\left(y + \frac{x^2}{y}\right)}{y^2} = -\frac{(y^2+x^2)}{y^3} = -\frac{a^2}{y^3}$

We have

Radius of Curvature at any point (x, y)

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{x^2}{y^2}\right)^{3/2}}{-\frac{a^2}{y^3}} = \frac{(x^2+y^2)^{3/2}}{y^3}\cdot\frac{y^3}{a^2} = -a.$$

or $|\rho| = a$

$\therefore$ Curvature $= \frac{1}{a}$ which is constant.

Ex. 3: Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where it cuts the line y = x.

Solution:

Here, the equation of line is

$$y = x \qquad \text{.........(1)}$$

and the equation of curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \qquad \text{.........(2)}$$

Solving (1) and (2),

$$x = \frac{a}{4}, y = \frac{a}{4}$$

So, the point of contact is

$$\left(\frac{a}{4}, \frac{a}{4}\right)$$

Differentiating (2) with respect to x,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

or $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

So, $\left(\frac{dy}{dx}\right)_{at\left(\frac{a}{4},\frac{a}{4}\right)} = -\frac{\sqrt{\frac{a}{4}}}{\sqrt{\frac{a}{4}}} = -1$

Also, $\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}}{2\sqrt{y}}\frac{dy}{dx} - \frac{\sqrt{y}}{2\sqrt{x}}\right]\cdot\frac{1}{x}$

$$= -\frac{1}{2x}\left[\frac{\sqrt{x}}{\sqrt{y}}\cdot\left(-\sqrt{\frac{y}{x}}\right) - \sqrt{\frac{y}{x}}\right] = \frac{1}{2x}\left(1+\sqrt{\frac{y}{x}}\right)$$

So, $\left(\frac{dy}{dx}\right)_{at\left(\frac{a}{4},\frac{a}{4}\right)} = \frac{1}{2.(a/4)}\left(1+\sqrt{\frac{a/4}{a/4}}\right) = \frac{4}{a}$

Radius of curvature at $\left(\frac{a}{4}, \frac{a}{4}\right)$

$$\rho = \frac{(1+1)^{3/2}}{4/a} = \frac{a}{4}\cdot 2^{3/2} = \frac{a}{4}\cdot 2\sqrt{2} = \frac{a}{\sqrt{2}}$$

$\therefore \quad \rho = \frac{a}{\sqrt{2}}$

Ex. 4: Show that the radius of curvature at the point (r, θ) for the curve $r = ae^{\theta\cot\alpha}$ is $\rho = r\,\text{cosec}\alpha$. [2060 B.E.]

Solution:

Given curve is

$$r = ae^{\theta\cot\alpha}$$

Differentiating it with respect to θ,

$$r_1 = ae^{\theta\cot\alpha}\cot\alpha = r\cot\alpha$$

Again, differentiating it with respect to θ,

$$r_2 = r_1\cot\alpha = r\cot^2\alpha$$

We have radius of curvature at any point (r, θ) to the given curve is

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha - r^2\cot^2\alpha}$$

$$= \frac{r^3\text{cosec}^3\alpha}{r^2(1+\cot^2\alpha)} = \frac{r^3\text{cosec}^3\alpha}{r^2\text{cosec}^2\alpha} = r\,\text{cosec}\alpha,$$

$$\therefore \rho = r\,\text{cosec}\,\alpha$$

Ex. 5: Show that radius of curvature at the point θ = 0 for the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $\rho = 4a$.

Solution:

Given curve is

$$x = a(\theta + \sin\theta), \qquad y = a(1 - \cos\theta)$$

Differentiating it with respect θ,

$$x' = a(1 + \cos\theta), \qquad y' = a\sin\theta$$

$$x'' = -a\sin\theta \qquad y'' = a\cos\theta$$

$$(x')_{at\,(\theta=0)} = 2a, \qquad (y')_{at\,(\theta=0)} = 0$$

$$(x'')_{at\,(\theta=0)} = 0 \qquad (y'')_{at\,(\theta=0)} = a$$

We have the radius of curvature at (x, y) to the given curve is

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-x''y'} = \frac{(4a^2+0)^{3/2}}{2a^2-0} = \frac{8a^3}{2a^2},$$

$\therefore \quad \rho = 4a$

Ex. 6: Find the radius of curvature at any point (p, r) of the pedal equation $r^{m+1} = a^m p$.

Solution:

Given curve is

$$r^{m+1} = a^m p,$$

Differentiating it with respect to p,

$$(m+1)\,r^m\frac{dr}{dp} = a^m$$

or $\quad \frac{dr}{dp} = \frac{a^m}{(m+1)r^m}$

or $\quad r\frac{dr}{dp} = \frac{a^m r}{(m+1)\,r^m} = \frac{a^m}{(m+1)r^{m-1}}$

We have the radius of curvature at the point (p, r) is

$$\rho = r\frac{dr}{dp}$$

$$\therefore \rho = \frac{a^m}{(m+1)r^{m-1}}$$

Ex. 7: Show that chord of curvature parallel to y-axis for the curve $y = a\log\sec\left(\frac{x}{a}\right)$ is constant.

Solution:

Here, the equation of curve is

$$y = a\log\sec\left(\frac{x}{a}\right)$$

Differentiating it with respect to x

$$\frac{dy}{dx} = a\frac{1}{\sec\left(\frac{x}{a}\right)} \times \sec\left(\frac{x}{a}\right)\tan\left(\frac{x}{a}\right).\frac{1}{a}$$

or $\quad \frac{dy}{dx} = \tan\left(\frac{x}{a}\right)$

Again, differentiating it with respect to x,

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\left(\frac{x}{a}\right)$$

We have

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\tan^2\left(\frac{x}{a}\right)\right)^{3/2}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)}$$

$$= a\frac{\sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)} = a\sec\left(\frac{x}{a}\right)$$

or $\rho = a\sec\left(\frac{x}{a}\right)$

And, $\tan\psi = \frac{dy}{dx} = \tan\frac{x}{a}$

$\therefore\ \psi = \frac{x}{a}$

The chord of curvature parallel to y-axis   For x

$= 2\rho\cos\psi = 2a\sec\left(\frac{x}{a}\right)\cos\left(\frac{x}{a}\right)$ $2\rho\sin\psi$

= 2a which is constant.

Ex. 8: Find the chord of curvature through the pole for the curve $r^2 = a^2\cos 2\theta$.

Solution:

Given curve is

$$r^2 = a^2\cos 2\theta$$

Differentiating it with respect to θ

$$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta,$$

or $\frac{d\theta}{rdr} = -\frac{1}{a^2\sin 2\theta}$

or $\frac{rd\theta}{dr} = -\frac{r^2}{a^2\sin 2\theta}$

or $\tan\phi = -\frac{a^2\cos 2\theta}{a^2\sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2}+2\theta\right)$

$\therefore\ \phi = \frac{\pi}{2} + 2\theta.$

Also, we have

$$p = r\sin\phi$$

or $p = r\sin\left(\frac{\pi}{2}+2\theta\right) = r\cos 2\theta$

or $p = r\cdot\frac{r^2}{a^2}$

or $a^2p = r^3$

Differentiating it with respect to p

$$3r^2\cdot\frac{dr}{dp} = a^2$$

or $2p\frac{dr}{dp} = \frac{2p\,a^2}{3r^2}$

or $2p\frac{dr}{dp} = \frac{2r^3}{3r^2} = \frac{2}{3}r$

$\therefore$ Chord of curvature through the pole $= 2p\frac{dr}{dp} = \frac{2}{3}r$

Ex. 9: For any curve prove that

i. $\frac{r}{\rho} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right)$

ii. $\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2$

i. $\frac{r}{\rho} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right)$ 2059 B.E.

Solution:

Here, $\sin\phi\left(1+\frac{d\phi}{d\theta}\right) = \sin\phi + \sin\phi\frac{d\phi}{d\theta} = r\frac{d\theta}{ds} + r\frac{d\theta}{ds}\frac{d\phi}{d\theta}$

$$= r\frac{d\theta}{ds} + r\frac{d\phi}{ds} = r\frac{d}{ds}(\theta+\phi)$$

$$= r\frac{d\psi}{ds} = \frac{r}{\frac{ds}{d\psi}} = \frac{r}{\rho}$$

or $\quad \sin\phi\left(1+\frac{d\phi}{d\theta}\right)=\frac{r}{\rho}$

$\therefore \quad \frac{r}{\rho}=\sin\phi\left(1+\frac{d\phi}{d\theta}\right)$

ii. $\quad \frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2$

Solution:

Now, we have

$$\frac{dx}{ds}=\cos\psi$$

Differentiating it with respect to s,

$$\frac{d^2x}{ds^2}=-\sin\psi\frac{d\psi}{ds}$$

$$\therefore \quad \frac{ds}{d\psi}\left(\frac{d^2x}{ds^2}\right)=-\sin\psi \quad \text{.........(1)}$$

and $\quad \frac{dy}{ds}=\sin\psi$

Differentiating it with respect to s

$$\frac{d^2y}{ds^2}=\cos\psi\frac{d\psi}{ds}$$

$$\therefore \quad \frac{ds}{d\psi}\left(\frac{d^2y}{ds^2}\right)=\cos\psi \quad \text{.........(2)}$$

Squaring and adding (1) and (2),

$$\left(\frac{ds}{d\psi}\right)^2\left[\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2\right]=1$$

or $\quad \rho^2\left[\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2\right]=1$

$$\therefore \quad \frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2$$

Ex. 10: If ρ_1 and ρ_2 be the radii of the curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, prove that $\rho_1^{-2/3}+\rho_2^{-2/3}=(2a)^{-2/3}$

Solution:

The equation of the parabola is

$$y^2 = 4ax$$

Its ends of the focal chords are (a, - 2a) and (a, 2a).

Differentiating it with respect x

$$2y\frac{dy}{dx} = 4a,$$

or $\quad \frac{dy}{dx} = \frac{2a}{y}$

or $\quad (y_1)_{at\,(a,\,-2a)} = \frac{2a}{-2a} = -1 \quad$ and $\quad (y_1)_{at\,(a,\,2a)} = \frac{2a}{2a} = 1$

Again, differentiating with respect to x

$$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{y^2}\frac{2a}{y} = -\frac{4a^2}{y^3}$$

$\therefore (y_2)_{at\,(a,\,-2a)} = \frac{-4a^2}{-8a^3} = \frac{1}{2a}$ and $(y_2)_{at\,(a,\,2a)} = -\frac{4a}{8a^3} = \frac{1}{-2a}$

Let ρ_1 be radius of curvature at (a, - 2a)

$$\rho_1 = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1/2a} = 2^{3/2}\,2a$$

and $\quad \rho_2$ be radius of curvature at (a, 2a)

$$\rho_2 = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{-1/2a} = -2^{3/2}\,.\,2a$$

Thus,

$$\rho_1^{-2/3}+\rho_2^{-2/3} = 2^{-1}(2a)^{-2/3}+2^{-1}(2a)^{-2/3} = (2a)^{-2/3}.$$

$$\therefore \rho_1^{-2/3}+\rho_2^{-2/3} = (2a)^{-2/3}.$$

Ex. 11: Find the radius of curvature of the following curves at the origin

i. $\quad x^3+y^3=3axy$

ii. $\quad y^2=\frac{a+x}{a-x}\,.\,x^2$

iii. $\quad r = a\sin n\theta$ (at $\theta = 0°$)

i. $\quad x^3+y^3=3axy$

Solution:

Here, the equation of the curve is

$$x^3+y^3=3axy \quad \text{.............(1)}$$

Since the curve passes through origin, we use the method of expansion to find the value of ρ.

Let $y = px + \frac{qx^2}{2} + ...$

Substituting the value of y in equation (1)

$$x^3 + \left(px + \frac{qx^2}{2} + ...\right)^3 = 3ax\left(px + \frac{qx^2}{2} + ...\right)$$

Equating the coefficient of like powers of x, we have

$3ap = 0$ $\therefore\ p = 0$

and $1 + p^3 - \frac{3aq}{2} = 0$

or $1 + 0 - \frac{3aq}{2} = 0$

or $1 - \frac{3aq}{2} = 0$ $\therefore\ q = \frac{2}{3a}$

We have

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+0)^{3/2}}{2/3a}$$

$\therefore\ \rho = \frac{3a}{2}$

Alternative method (Newton's method)

Since the curve passes through origin, so the tangent at origin is

$3axy = 0$ i.e. $x = 0, y = 0$

When $y = 0$, the tangent is the axis of x

We have the radius of curvature is defined as

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

Here, the equation of curve is

$x^3 + y^3 = 3axy$ ....(1)

Dividing each term by 2xy and taking the limits

$$\lim_{\substack{x\to 0\\ y\to 0}} \left(\frac{x^2}{2y} + \frac{y^2}{2x}\right) = \frac{3a}{2}$$

or $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y} + \lim_{\substack{x\to 0\\ y\to 0}} \frac{1}{4}\cdot xy \frac{1}{\left(\frac{x^2}{2y}\right)} = \frac{3a}{2}$$

or $\rho + \frac{1}{4\rho}.0.0 = \frac{3a}{2}$

$\therefore\ \rho = \frac{3a}{2}$

When the tangent is the axis of y i.e. $x = 0$, then

$$\rho = \lim_{\substack{y\to 0\\ x\to 0}} \frac{y^2}{2x}$$

Dividing (1) by 2xy and taking limit on both sides,

$$\lim_{\substack{y\to 0\\ x\to 0}} \left(\frac{x^2}{2y} + \frac{y^2}{2x}\right) = \frac{3a}{2}$$

or $$\lim_{\substack{y\to 0\\ x\to 0}} \frac{1}{4}\cdot xy \cdot \frac{1}{(y^2/2x)} + \lim_{\substack{y\to 0\\ x\to 0}} \frac{y^2}{2x} = \frac{3a}{2}$$

or $\frac{1}{4}.0.0.\frac{1}{\rho} + \rho = \frac{3a}{2}$

$\therefore\ \rho = \frac{3a}{2}$

ii. $y^2 = \frac{a+x}{a-x}.x^2$

Solution:

Here, the equation of curve is

$y^2 = \frac{a+x}{a-x}.x^2$ ......(1)

Since the curve passes through the origin, we use by method of expansion to find the value of ρ.

Putting $y = px + \frac{qx^2}{2} + ...$ in (1),

$$\left(px + \frac{q}{2}x^2 + ...\right)^2 = \frac{a+x}{a-x}.x^2$$

or $$(a-x)\left(px + \frac{q}{2}x^2 + ...\right)^2 = ax^2 + x^3$$

Equating the coefficient of x^2 and x^3,

$ap^2 = a$ $\therefore\ p = \pm 1$

and $-p^2 + apq = 1$

or $-1 \pm aq = 1$

$\therefore \quad q = \pm\frac{2}{a}$

or $\quad \pm aq = 2$

We have

$$\rho = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{\pm\frac{2}{a}} = \pm\frac{a}{2}.(2)^{3/2} = \pm\frac{a}{2}.2\sqrt{2} = \pm a\sqrt{2}$$

$\therefore \quad \rho = \pm a\sqrt{2}$

iii. $r = a \sin n\theta$ (at $\theta = 0^\circ$)

Solution:

Here, the equation of curve is

$$r = a \sin n\theta$$

We know,

$$\rho = \lim_{\theta\to 0}\frac{r}{2\theta} = \lim_{\theta\to 0}\frac{a \sin n\theta}{2\theta} = \frac{a}{2}\lim_{\theta\to 0}\left(\frac{\sin n\theta}{n\theta}\right).n = \frac{an}{2}$$

$\therefore \quad \rho = n\frac{a}{2}$

Ex. 12: Find the pedal equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Solution:

Here, the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

Differentiating it, with respect to x

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx} = 0$$

or $\quad \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

Any point of the ellipse is ($a \cos\theta$, $b \sin\theta$)

Slope of tangent at ($a \cos\theta$, $b \sin\theta$)

$$= -\frac{b^2}{a^2}\frac{a\cos\theta}{b\sin\theta} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$$

The equation of tangent at ($a \cos\theta$, $b \sin\theta$) to the ellipse is

$$y - b\sin\theta = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}(x - a\cos\theta)$$

or $\quad ay \sin\theta - ab \sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

or $\quad bx\cos\theta + ay\sin\theta = ab.$

If p be the length of perpendicular from origin to the tangent, then

$$p = \pm\frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

or $\quad p^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$ ....(1)

Also,

$$x^2 + y^2 = r^2$$

or $\quad (a\cos\theta)^2 + (b\sin\theta)^2 = r^2$

or $\quad a^2\cos^2\theta + b^2\sin^2\theta = r^2$ ....(2)

Eliminating θ from (1) and (2),

From (1),

$$\frac{a^2b^2}{p^2} = b^2\cos^2\theta + a^2\sin^2\theta = b^2 - b^2\sin^2\theta + a^2 - a^2\cos^2\theta$$

$$= a^2 + b^2 - (b^2\sin^2\theta + a^2\cos^2\theta) = b^2 + a^2 - r^2$$

$\therefore \quad \frac{a^2b^2}{p^2} = b^2 + a^2 - r^2$ is the required pedal equation.

Ex. 13: Find the pedal equation of the curve $c^2(x^2+y^2) = x^2y^2$

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Solution:

Given curve is

$c^2(x^2 + y^2) = x^2y^2$ ....(1)

Differentiating,

$$c^2\left(2x + 2y.\frac{dy}{dx}\right) = x^2\,2y\frac{dy}{dx} + 2xy^2$$

or $\quad \frac{dy}{dx} = \frac{xy^2 - c^2x}{c^2y - x^2y} = -\frac{x(c^2-y^2)}{y(c^2-x^2)}$

Equation of tangent at (x, y) to the curve is

$$Y - y = -\frac{x(c^2-y^2)}{y(c^2-x^2)}(X - x)$$

or $\quad x(c^2-y^2)X + y(c^2-x^2)Y = c^2(x^2+y^2) - 2x^2y^2$

Using (1),

$$x(c^2-y^2)X+y(c^2-x^2)Y=x^2y^2-2x^2y^2$$

or $$x(c^2-y^2)X+y(c^2-x^2)Y+x^2y^2=0 \quad \text{.............(2)}$$

Perpendicular length from (0, 0) to the tangent (2) is

$$p=\pm\frac{x(c^2-y^2).\,0+y(c^2-x^2).\,0+x^2y^2}{\sqrt{x^2(c^2-y^2)^2+y^2(c^2-x^2)^2}}$$

$$=\frac{x^2y^2}{\sqrt{x^2(c^2-y^2)^2+y^2(c^2-x^2)^2}}$$

Squaring,

$$p^2=\frac{x^4y^4}{c^4(x^2+y^2)-4c^2x^2y^2+x^2y^4+x^4y^2}$$

or $$p^2=\frac{c^4(x^2+y^2)^2}{c^4(x^2+y^2)-4c^2(x^2y^2)+x^2y^2(x^2+y^2)}$$

$$=\frac{c^4(x^2+y^2)^2}{c^4(x^2+y^2)-4c^4(x^2+y^2)+c^2(x^2+y^2)^2}$$

Also using, $r^2=x^2+y^2$, we get,

$$p^2=\frac{c^4r^4}{c^4r^2-4c^4r^2+c^2r^4}=\frac{c^4r^4}{c^2r^4-3c^4r^2}=\frac{c^4r^4}{c^2r^2(r^2-3c^2)}$$

or $$p^2=\frac{c^2r^2}{(r^2-3c^2)}$$

or $$p^2(r^2-3c^2)=c^2r^2$$

$\frac{1}{p^2}+\frac{3}{r^2}=\frac{1}{c^2}$, is the required pedal equation.

Ex. 14: Find the pedal equation of the curve $r=\frac{2a}{1-\cos\theta}$

Solution:

Given curve is

$$r=\frac{2a}{1-\cos\theta}=\frac{2a\times 1}{2\sin^2(\theta/2)}=a\operatorname{cosec}^2\left(\frac{\theta}{2}\right)$$

Differentiating,

$$\frac{dr}{d\theta}=2a\operatorname{cosec}\left(\frac{\theta}{2}\right)\left[-\operatorname{cosec}\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)\right]\times\frac{1}{2}$$

or $$r\frac{d\theta}{dr}=-\frac{r}{a.\operatorname{cosec}^2(\theta/2)\cot(\theta/2)}=-\frac{a\operatorname{cosec}^2(\theta/2)}{a\operatorname{cosec}^2(\theta/2)\cot(\theta/2)}$$

or $$\tan\phi=-\tan\left(\frac{\theta}{2}\right)=\tan\left[\pi-\left(\frac{\theta}{2}\right)\right]$$

or $$\tan\phi=\tan\left[\pi-\left(\frac{\theta}{2}\right)\right]$$

$$\therefore\ \phi=-\left(\frac{\theta}{2}\right)$$

Also, we have

$$p=r\sin\phi$$

or $$p=r\sin\left(\pi-\frac{\theta}{2}\right)=r\sin\left(\frac{\theta}{2}\right)$$

Squaring,

$$p^2=r^2\sin^2\left(\frac{\theta}{2}\right)$$

or $$p^2=r^2\frac{a}{r},$$

$\therefore$ $p^2=a\,r$ is the required pedal equation.

Ex. 15: Find the pedal equation of the curve $r^2=a^2\cos2\theta$

Solution:

Here, $r^2=a^2\cos2\theta$ ....(1)

Differentiating,

$$2r\frac{dr}{d\theta}=-2a^2\sin2\theta$$

or $$\frac{d\theta}{dr}=-\frac{r}{a^2\sin2\theta}$$

or $$r\frac{d\theta}{dr}=-\frac{r^2}{a^2\sin2\theta}=-\frac{a^2\cos2\theta}{a^2\sin2\theta}$$

or $$\tan\phi=-\cot2\theta=\tan\left(\frac{\pi}{2}+2\theta\right)$$

$$\therefore\ \phi=\frac{\pi}{2}+2\theta \quad \text{.............(2)}$$

We have

$$p=r\sin\phi$$

Using (2),

$$p = r\sin\left(\frac{\pi}{2} + 2\theta\right)$$

or $p = r\cos 2\theta$

Using (1),

$$p = r.\frac{r^2}{a^2}$$

$\therefore$ $r^3 = a^2p$ is the required pedal equation.

Exercise - 6

1. Find the radius of curvature at any point (x, y) of the following curves
   i. $ay^2 = x^3$
   ii. $xy = c^2$
   iii. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
   iv. $x^{2/3} + y^{2/3} = a^{2/3}$
   v. $y^2 = 4x$ at the vertex (0, 0)
   vi. $y = x^3 - 2x^2 + 7x$ at (0, 0)
2. Find the radius of curvature at any point (r, θ) for the curves
   i. $r^3 = a^3\cos 3\theta$
   ii. $r = a\cos\theta$
   iii. $r^2\cos 2\theta = a^2$
   iv. $r^2 = a^2\cos 2\theta$ at $\theta = 0$
   v. $r = a(\theta + \sin\theta)$ at $\theta = 0$
3. Find the radius of curvature at the point (r, θ) on the Cardiode $r = a(1 - \cos\theta)$ and show that it varies as $\sqrt{r}$. **2061 B.E**
4. Show that the radius of curvature at any point (r, θ) of the curve $r^m = a^m\cos m\theta$ is $\frac{a^m}{(m+1)r^{m-1}}$ **2062 B.E**
5. Find the radius of curvature at any point for the curves
   i. $x = at^2$, $y = 2at$
   ii. $x = a\cos\theta$, $y = a\sin\theta$

   iii. $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = 0°$
   iv. $x = a\cos\theta$, $y = b\sin\theta$
6. Find the radius of curvature at any point (p, r) on the following curves whose pedal equation are
   i. $r^2 = 2ap$
   ii. $r^3 = a^2p$
   iii. $p^2 = ar$
7. Find the radius of curvature at the origin for the following curves:
   i. $4x^2 - 3xy + y^2 - 3y = 0$
   ii. $x^3 + y^3 - 2x^2 + 6y = 0$
8. Find the radius of curvature of the curve $y = x^2(x - 3)$ at the points where the tangent is parallel to x-axis.
9. Show that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the radius of curvature at the extremity of the major axis is equal to half the latus rectum.
10. Find the chord of curvature through the pole for the following curves
    i. $r = a(1 + \cos\theta)$
    ii. $r^2\cos 2\theta = a^2$
    iii. $r = ae^{\theta\cot\alpha}$
11. Show that the chord of curvature through the pole of the curve $r^m = a^m\cos m\theta$ is $\frac{2r}{m+1}$
12. Show that the chord of curvature through the pole for the curve $p = f(r)$ is given by $\frac{2f(r)}{f'(r)}$
13. Show that the chord of curvature parallel to y-axis for the curve $y = c\cosh\left(\frac{x}{c}\right)$ is double ordinate.
14. Find the pedal equation of the following curves:
    i. $y^2 = 4a(x + a)$
    ii. $y^2 = 4ax$.
    iii. $x^{2/3} + y^{2/3} = a^{2/3}$
15. Find the pedal equation of the following curves
    i. $r = a(1 - \cos\theta)$

ii. $r = a(1 + \cos\theta)$

iii. $r^2 = a^2 \cos 2\theta$

iv. $r^2 \cos 2\theta = a^2$ [2059 B.E.]

v. $r^m = a^m \cos m\theta$

vi. $r = ae^{\theta\cot\alpha}$

Answers

1. i. $\frac{(4a+9x)^{3/2}x^{1/2}}{6a}$ ii. $\frac{(x^2+y^2)^{3/2}}{2c^2}$
   iii. $\frac{(b^4x^2+a^4y^2)^{3/2}}{a^4b^4}$ iv. $3(axy)^{1/3}$ v. 2 vi. $\frac{1}{2}$ $125\sqrt{2}$
2. i. $\frac{a^2}{4r^2}$ ii. $\frac{1}{2}a$ iii. $\frac{r^3}{a^2}$ iv. $\frac{a}{3}$ v. a
3. $\frac{2}{3}\sqrt{2ar}$ 5. i. $2a(t^2+1)^{3/2}$ ii. a iii. 4a iv. $\frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$
6. i. a ii. $\frac{a^2}{3r}$ iii. $2\sqrt{\frac{r^3}{a}}$
7. i. $\frac{3}{8}$ ii. $\frac{3}{2}$ 8. $\frac{1}{6}, -\frac{1}{6}$
10. i. $\frac{4}{3}r$. ii. 2r iii. 2r.
14. i. $p^2 = ar$ ii. $(p^2r^2 + 4a^2p^2)(4a^2 + p^2) = (ar^2 - ap^2)^2$
    iii. $r^2 + 3p^2 = a^2$
15. i. $2ap^2 = r^3$, ii. $r^3 = 2ap^2$ iii. $r^3 = pa^2$
    iv. $pr = a^2$ v. $pa^m = r^{m+1}$ vi. $p = r\sin\alpha$

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Unit – II

Integration and its Applications

Chapter - 7

Indefinite Integral

7.1 Introduction

Let f(x) be differentiable function and its derivative with respect to x is F(x)

i.e. $\frac{d}{dx}$ [f(x)] = F(x), then

the integration of F(x) with respect to x is f(x).

i.e. $\int F(x)\,dx = f(x)$

Also, we have

$$\frac{d}{dx}[f(x) + c] = F(x)$$

$$\therefore \int F(x)\,dx = f(x) + c$$

where c is called the constant of integration.

The constant of integration is usually omitted in general practice but it is always kept in mind that this constant exists in every case of integration.

Thus, the integral $\int F(x)\,dx$ is called *Indefinite Integral* of F(x) with respect to x.

For example,

$$\frac{d}{dx}(\sin x) = \cos x$$

The integration of cosx with respect to x is sinx,

i.e. $\int \cos x\,dx = \sin x$

To evaluate the integrals, the students must be thoroughly familiar with the following formulae. These formulae are known as *Fundamental Integrals.*

7.2 Fundamental Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$ $(n \neq -1)$
2. $\int \frac{1}{x} dx = \log x$, for $x > 0$.
3. $\int e^x dx = e^x$
4. $\int a^x dx = \frac{a^x}{\log a}$
5. $\int \sin x \, dx = -\cos x$
6. $\int \cos x \, dx = \sin x$
7. $\int \sec^2 x \, dx = \tan x$
8. $\int \operatorname{cosec}^2 x \, dx = -\cot x$
9. $\int \sec x \tan x \, dx = \sec x$
10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$
11. $\int \sinh x \, dx = \cosh x$
12. $\int \cosh x \, dx = \sinh x$

7.3 Standard Integrals

13. $\int \frac{f'(x)}{f(x)} dx = \log f(x)$
14. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)}$
15. $\int \tan x \, dx = \log \sec x$
16. $\int \cot x \, dx = \log \sin x$
17. $\int \sec x \, dx = \log(\sec x + \tan x) = \log \tan \frac{x}{2}$
18. $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$
20. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a + x}{a - x}$
21. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a}$
22. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$
23. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log\left(x + \sqrt{a^2 + x^2}\right) = \sin h^{-1} \frac{x}{a}$
23. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right) = \cos h^{-1} \frac{x}{a}$
24. $\int uv \, dx = u\int v dx - \int\left[\frac{du}{dx} \cdot \int v dx\right] dx$ (Integration by parts)

25. $\int \sqrt{a^2 + x^2} \, dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2} \log\left(x + \sqrt{a^2 + x^2}\right)$
26. $\int \sqrt{a^2 - x^2} \, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$
27. $\int \sqrt{x^2 - a^2} \, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right)$
28. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$
29. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$
30. $\int_{-a}^{a} f(x) \, dx = 2\int_0^a f(x) \, dx$, if f(x) is an even function.
$= 0$, if f(x) is odd function.

7.4 Some Standard Forms

Standard Form I: $\int \frac{1}{px^2 + qx + r} dx$

Let $I = \int \frac{1}{px^2 + qx + r} dx$

In this form, if $px^2 + qx + r$ can be factorized into two linear factors, then it can be written as

$$I = \int \frac{1}{(ax + b)(cx + d)} dx$$

$$= \frac{1}{(ad - bc)} \int \left(\frac{a}{ax + b} - \frac{c}{cx + d}\right) dx$$

If $px^2 + ax + r$ can not be factorized as two linear factors then it can be written as

$$I = \int \frac{1}{px^2 + qx + r} dx = \frac{1}{p} \int \frac{1}{x^2 + \frac{q}{p}x + \frac{r}{p}} dx$$

$$= \frac{1}{p}\int \frac{1}{(x)^2 + 2.x.\frac{q}{2p} + \left(\frac{q}{2p}\right)^2 + \frac{r}{p} - \left(\frac{q}{2p}\right)^2}\, dx$$

$$= \frac{1}{p}\int \frac{1}{\left(x + \frac{q}{2p}\right)^2 + \frac{4pr - q^2}{4p^2}}\, dx$$

This form is reduced to the standard formula for

$\int \frac{1}{x^2 - a^2}\, dx$ if $4pr < q^2$ and

$\int \frac{1}{x^2 + a^2}\, dx$ if $4pr > q^2$

For example, $\int \frac{dx}{2x^2 + x + 1}$

$$\text{Let } I = \int \frac{dx}{2x^2 + x + 1} = \frac{1}{2}\int \frac{dx}{\left(x^2 + \frac{1}{2}x + \frac{1}{2}\right)}$$

$$= \frac{1}{2}\int \frac{dx}{(x)^2 + 2x\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \frac{1}{2} - \frac{1}{16}} = \frac{1}{2}\int \frac{dx}{\left(x + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}$$

$$= \frac{1}{2}\,\frac{1}{\frac{\sqrt{7}}{4}}\tan^{-1}\frac{(x + 1/4)}{\frac{\sqrt{7}}{4}} = \frac{2}{\sqrt{7}}\tan^{-1}\frac{4x + 1}{\sqrt{7}}$$

Standard Form II: $\int \frac{1}{\sqrt{px^2 + qx + r}}\, dx$

$$\text{Let } I = \int \frac{1}{\sqrt{px^2 + qx + r}}\, dx$$

In this form, if $px^2 + qx + r$ can be factorized into two linear factors, then it can be written as

$$I = \int \frac{1}{\sqrt{(ax + b)(cx + d)}}\, dx$$

It can be solved by putting either $ax + b = t^2$ or $cx + d = t^2$.

If $px^2 + qx + r$ can not be factorized into two linear factors, then

$$I = \int \frac{1}{\sqrt{px^2 + qx + r}}\, dx$$

$$= \frac{1}{\sqrt{p}}\int \frac{1}{\sqrt{x^2 + \frac{q}{p}x + \frac{r}{p}}}\, dx$$

$$= \frac{1}{\sqrt{p}}\int \frac{1}{\sqrt{(x)^2 + 2.x.\frac{q}{2p} + \left(\frac{q}{2p}\right)^2 + \frac{r}{p} - \left(\frac{q}{2p}\right)^2}}\, dx$$

$$= \frac{1}{\sqrt{p}}\int \frac{1}{\sqrt{\left(x + \frac{q}{2p}\right)^2 + \frac{4pr - q^2}{4p^2}}}\, dx$$

This form is reduced to the standard formula for

$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx$ if $4pr < q^2$ and

$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx$ if $4pr > q^2$

For example, $\int \frac{dx}{\sqrt{x^2 + x - 2}}$

$$\text{Let } I = \int \frac{dx}{\sqrt{x^2 + x - 2}} = \int \frac{dx}{\sqrt{(x - 1)(x + 2)}}$$

Put $x + 2 = t^2$,

$dx = 2tdt$

$$= \int \frac{2tdt}{\sqrt{(t^2 - 3)}\ t} = 2\int \frac{dt}{\sqrt{t^2 - 3}}$$

$$= 2\log\left(t + \sqrt{t^2 - 3}\right)$$

$$= 2\log\left(\sqrt{x + 2} + \sqrt{x - 1}\right)$$

Standard Form III: $\int \frac{ax+b}{px^2+qx+r}dx$

Let $I = \int \frac{ax+b}{px^2+qx+r}dx$

In this form, the numerator must be derivative of denominator.

$$= \int \frac{\frac{a}{2p}(2px+q)+b-\frac{aq}{2p}}{px^2+qx+r}dx$$

$$= \frac{a}{2p}\int \frac{(2px+q)}{px^2+qx+r}dx + \frac{2bp-aq}{2p}\int \frac{1}{px^2+qx+r}dx$$

$$= \frac{a}{2p}\log(px^2+qx+r) + \frac{2bp-aq}{2p} \quad \text{(Standard Form I)}$$

For example, $\int \frac{3x+1}{2x^2-2x+3}dx$

Let I $= \int \frac{3x+1}{2x^2-2x+3}dx = \int \frac{\frac{3}{4}(4x-2)+\frac{5}{2}}{2x^2-2x+3}dx$

$$= \frac{3}{4}\int \frac{(4x-2)}{(2x^2-2x+3)}dx + \frac{5}{2}\int \frac{dx}{2x^2-2x+3}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{x^2-x+\frac{3}{2}}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{x^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2+\frac{3}{2}-\frac{1}{4}}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$$

Standard Form IV: $\int \frac{ax+b}{\sqrt{px^2+qx+r}}dx$

Let $I = \int \frac{ax+b}{\sqrt{px^2+qx+r}}dx$

$$= \int \frac{\frac{a}{2p}(2px+q)+b-\frac{aq}{2p}}{\sqrt{px^2+qx+r}}dx$$

$$= \frac{a}{2p}\int \frac{(2px+q)}{\sqrt{px^2+qx+r}}dx + \frac{2bp-aq}{2p}\int \frac{1}{\sqrt{px^2+qx+r}}dx$$

Put $px^2+qx+r = t^2$

$(2px+q)\,dx = 2t\,dt$

$$= \int \frac{2t\,dt}{\sqrt{t^2}} + \frac{2bp-aq}{2p} \quad \text{(Standard Form II)}$$

$$= 2\int dt + \frac{2bp-aq}{2p} \quad \text{(Standard Form II)}$$

$$= 2t + \frac{2bp-aq}{2p} \quad \text{(Standard Form II)}$$

$$= 2\sqrt{px^2+qx+r} + \frac{2bp-aq}{2p} \quad \text{(Standard Form II)}$$

For example, $I = \int \frac{(x-2)\,dx}{\sqrt{2x^2-8x+5}}$

Let $I = \int \frac{(x-2)\,dx}{\sqrt{2x^2-8x+5}} = \int \frac{\frac{1}{4}(4x-8)\,dx}{\sqrt{(2x^2-8x+5)}}$

Put $2x^2-8x+5 = t^2$,

$(4x-8)\,dx = 2t\,dt$

$\therefore \quad I = \frac{1}{4}\int \frac{2t\,dt}{t} = \frac{1}{2}\int dt = \frac{1}{2}t = \frac{1}{2}\sqrt{2x^2-8x+5}$

Standard Form V: $\int \frac{ax^2+bx+c}{px^2+qx+r}dx$

$$\text{Let } I = \int \frac{ax^2 + bx + c}{px^2 + qx + r}\,dx$$

$$= \int \frac{\frac{a}{p}(px^2 + qx + r) + bx + c - \frac{aq}{p}x - \frac{ar}{p}}{px^2 + qx + r}\,dx$$

$$= \frac{a}{p}\int \frac{px^2 + qx + r}{px^2 + qx + r}\,dx + \int \frac{\left[\left(\frac{pb - aq}{p}\right)x + \frac{pc - ar}{p}\right]}{px^2 + qx + r}\,dx$$

$$= \frac{a}{p}\int dx + \quad \text{(Standard Form III)}$$

$$= \frac{a}{p}x + \quad \text{(Standard Form III)}$$

For example, $\int \frac{x^2 + 2x - 1}{2x^2 + 3x + 1}\,dx$

$$\text{Let } I = \int \frac{x^2 + 2x - 1}{2x^2 + 3x + 1}\,dx$$

$$= \int \frac{\frac{1}{2}(2x^2 + 3x + 1) + \frac{1}{2}(x - 3)}{(2x^2 + 3x + 1)}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{(x - 3)dx}{2x^2 + 3x + 1}$$

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{\left(\frac{1}{4}(4x + 3) - \frac{15}{4}\right)}{2x^2 + 3x + 1}\,dx$$

$$= \frac{1}{2}x + \frac{1}{8}\int \frac{(4x + 3)}{2x^2 + 3x + 1}\,dx - \frac{15}{8}\int \frac{dx}{2x^2 + 3x + 1}$$

$$= \frac{1}{2}x + \frac{1}{8}\log(2x^2 + 3x + 1) - \frac{15}{8}\int \frac{dx}{(x)^2 + 2.x.\frac{3}{4} + \left(\frac{3}{4}\right)^2 + \frac{1}{2} - \frac{9}{16}}$$

$$= \frac{1}{2}x + \frac{1}{8}\log(2x^2 + 3x + 1) - \frac{15}{8}\int \frac{dx}{\left(x + \frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2}$$

$$= \frac{1}{2}x + \frac{1}{8}\log(2x^2 + 3x + 1) - \frac{15}{8}\log\frac{x + \frac{1}{2}}{x + 1}$$

Standard Form VI : $\int \frac{ax^2 + bx + c}{\sqrt{px^2 + qx + r}}\,dx$

$$\text{Let } I = \int \frac{ax^2 + bx + c}{\sqrt{px^2 + qx + r}}\,dx$$

$$= \int \frac{\frac{a}{p}(px^2 + qx + r) + bx + c - \frac{aq}{p}x - \frac{ar}{p}}{\sqrt{px^2 + qx + r}}\,dx$$

$$= \frac{a}{p}\int \frac{px^2 + qx + r}{\sqrt{px^2 + qx + r}}\,dx + \int \frac{\left[\left(\frac{pb - aq}{p}\right)x + \frac{pc - ar}{p}\right]}{\sqrt{px^2 + qx + r}}\,dx$$

$$= \frac{a}{p}\int \sqrt{px^2 + qx + r}\,dx + \quad \text{(Standard Form IV)}$$

$$= \frac{a}{p}\int \sqrt{p}\sqrt{x^2 + \frac{q}{p}x + \frac{r}{p}}\,dx + \quad \text{(Standard Form IV)}$$

$$= \frac{a}{\sqrt{p}}\int \sqrt{(x)^2 + 2.x.\frac{q}{2p} + \left(\frac{q}{2p}\right)^2 + \frac{r}{p} - \frac{q^2}{4p^2}}\,dx + \quad \text{(Standard Form IV)}$$

$$= \frac{a}{\sqrt{p}}\int \sqrt{\left(x + \frac{a}{2p}\right)^2 + \frac{4pr - q^2}{4p^2}}\,dx + \quad \text{(Standard Form IV)}$$

This form is reduced to form of the standard formula

$\int \sqrt{x^2 - a^2}\,dx$ if $4pr < q^2$ and

$\int \sqrt{x^2 + a^2}\,dx$ if $4pr > q^2$

For example, $\int \frac{x^2 + x + 1}{\sqrt{1 - x^2}}\,dx$

Let $I = \int \frac{x^2+x+1}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{-(1-x^2)+x+2}{\sqrt{1-x^2}}\,dx$$

$$= -\int \frac{(1-x^2)}{\sqrt{1-x^2}}\,dx + \int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{2}{\sqrt{1-x^2}}\,dx$$

$$= -\int \sqrt{1-x^2}\,dx + \int \frac{x\,dx}{\sqrt{1-x^2}} + 2\sin^{-1}x$$

$$= -x\frac{\sqrt{1-x^2}}{2} - \frac{1}{2}\sin^{-1}x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}} + 2\sin^{-1}x$$

$$= -\frac{x\sqrt{1-x^2}}{2} - \frac{1}{2}\sin^{-1}x + \frac{1}{2}\,.\,2\sqrt{1-x^2} + 2\sin^{-1}x$$

$$= -\frac{x\sqrt{1-x^2}}{2} - \frac{1}{2}\sin^{-1}x - \sqrt{1-x^2} + 2\sin^{-1}x$$

Thus $I = \frac{3}{2}\sin^{-1}x - \frac{1}{2}(x+2)\sqrt{1-x^2}$.

Standard Form VII : $\int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x},$

Here, $\int \frac{dx}{a+b\cos x}$

Put $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$$\int \frac{dx}{a+b\cos x} = \int \frac{1}{a+b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}\,dx$$

Multiplying numerator and denominator by $\sec^2\frac{x}{2}$ and put $\tan\frac{x}{2} = t$.

Next,

Let $I = \int \frac{dx}{a+b\sin x} = \int \frac{dx}{a+b\sin x}$

$$= \int \frac{1}{a+2b\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

Put $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

Multiplying numerator and denominator by $\sec^2\frac{x}{2}$ and put $\tan\frac{x}{2} = t$.

Standard Form VIII : $\int \frac{dx}{a\cos x + b\sin x}$, $a > b$

Let $I = \int \frac{dx}{a\cos x + b\sin x}$, $a > b$

Put $a = r\cos\theta$ and $b = r\sin\theta$, then $a^2+b^2 = r^2$ and $\tan\theta = \frac{b}{a}$

$$= \int \frac{dx}{r\cos x\cos\theta + r\sin\theta\sin x}$$

$$= \int \frac{dx}{\sqrt{a^2+b^2}(\cos\theta\cos x + \sin\theta\sin x)}$$

$$= \frac{1}{\sqrt{a^2+b^2}}\int \frac{dx}{\cos(x-\theta)} = \frac{1}{\sqrt{a^2+b^2}}\int \sec(x-\theta)\,dx$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log[\sec(x-\theta) + \tan(x-\theta)]$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left[\sec\left(x - \tan^{-1}\frac{b}{a}\right) + \tan\left(x - \tan^{-1}\frac{b}{a}\right)\right]$$

Standard Form IX : $\int \frac{p\sin x + q\cos x}{a\sin x + b\cos x}dx$

Here,

$$p\sin x + q\cos x = l(\text{Den}^r) + m(\text{Diff. coeff. of Den}^r)$$
$$= l(a\sin x + b\cos x) + m(a\cos x - b\sin x)$$

$$\int \frac{p\sin x + q\cos x}{a\sin x + b\cos x}\,dx = \int \frac{l(a\sin x + b\cos x) + m(a\cos x - b\sin x)}{a\sin x + b\cos x}\,dx$$

$$= l\int dx + m\int \frac{a\cos x - b\sin x}{a\sin x + b\cos x}dx$$

$$= l\,x + m\log(a\sin x + b\cos x)$$

Where l, m are determined by equating the coefficient of sinx and cosx from the two sides of the above identity.

Worked Out Examples

Ex. 1: Integrate $\int\sqrt{\frac{a+x}{x}}dx$

Solution:

Let $I = \int\sqrt{\frac{a+x}{x}}dx$

Put $x = t^2$, $\quad dx = 2t\,dt$

$$I = \int\frac{\sqrt{a+t^2}}{t}2t\,dt = 2\int\left[\sqrt{(\sqrt{a})^2 + t^2}\right]dt$$

$$= 2\left[\frac{t\sqrt{t^2+a}}{2} + \frac{a}{2}\log\left(\sqrt{t^2+a} + t\right)\right]$$

$$= \sqrt{x(x+a)} + a\log\left(\sqrt{x+a} + \sqrt{x}\right)$$

Ex. 2: Integrate $\int\frac{\sin 2x dx}{a\sin^2 x + b\cos^2 x}$

Solution:

$$\text{Let } I = \int\frac{\sin 2x dx}{a\sin^2 x + b\cos^2 x} = \int\frac{2\sin x\cos x\,dx}{a\sin^2 x + b\cos^2 x}$$

Put $a\sin^2 x + b\cos^2 x = t$

$(2a\sin x\cos x - 2b\cos x\sin x)\,dx = dt$

or $(a-b)\,2\sin x\cos x\,dx = dt$,

or $2\sin x\cos x\,dx = \frac{dt}{(a-b)}$

So, $I = \frac{1}{(a-b)}\int\frac{dt}{t}$

$$= \frac{1}{(a-b)}\log t = \frac{1}{(a-b)}\log(a\sin^2 x + b\cos^2 x)$$

$\therefore \quad I = \frac{1}{(a-b)}\log(a\sin^2 x + b\cos^2 x).$

Ex. 3: Integrate $\int\frac{(x+2)}{\sqrt{(4x-x^2)}}dx$

Solution:

$$\text{Let } I = \int\frac{(x+2)}{\sqrt{(4x-x^2)}}dx = \int\frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int\frac{(4x-2x)\,dx}{\sqrt{4x-x^2}} + 4\int\frac{dx}{\sqrt{4x-x^2}} = -\frac{1}{2}I_1 + 4I_2$$

Now, $I_1 = \int\frac{(4-2x)dx}{\sqrt{(4x-x^2)}}$

Put $4x - x^2 = t^2$

$(4-2x)\,dx = 2t\,dt$

Then, $I_1 = 2\int\frac{t\,dt}{t} = 2\int dt = 2t = 2\sqrt{4x-x^2}$

and $I_2 = \int\frac{dx}{\sqrt{4x-x^2}} = \int\frac{dx}{\sqrt{-(x^2-4x)}}$

$$= \int\frac{dx}{\sqrt{-[(x)^2 - 2.x.2 + (2)^2 - 4]}} = \int\frac{dx}{\sqrt{-[(x-2)^2 - (2)^2]}}$$

$$= \int\frac{dx}{\sqrt{(2)^2 - (x-2)^2}} = \sin^{-1}\left(\frac{x-2}{2}\right)$$

$$\therefore \quad I = -\frac{1}{2}I_1 + 4I_2 = -\sqrt{4x-x^2} + 4\sin^{-1}\frac{(x-2)}{2}$$

Ex. 4: Integrate $\int\frac{dx}{\sqrt{2+3x-2x^2}}$

Solution:

$$\text{Let } I = \int\frac{dx}{\sqrt{2+3x-2x^2}} = \int\frac{dx}{\sqrt{(1+2x)(2-x)}}$$

Put $2 - x = t^2$, $\quad -dx = 2t\,dt$

So, $I = -\int \frac{2t\,dt}{\sqrt{(5-2t^2)\,t^2}} = -\sqrt{2} \int \frac{dt}{\sqrt{\frac{5}{2} - t^2}}$

$$= \sqrt{2}\cos^{-1} t\sqrt{\frac{2}{5}} = \sqrt{2}\cos^{-1}\sqrt{\frac{2(2-x)}{5}}$$

$$\therefore \quad I = \sqrt{2}\cos^{-1}\sqrt{\frac{2(2-x)}{5}}$$

Ex. 5: Integrate $\int \frac{x\,dx}{x^4 + x^2 + 1}$

Solution:

Let $I = \int \frac{x\,dx}{x^4+x^2+1} = \int \frac{x\,dx}{(x^2)^2 + x^2 + 1}$

Put $x^2 = t$, $\quad 2x\,dx = dt$

So, $I = \frac{1}{2}\int \frac{dt}{t^2+t+1} = \frac{1}{2}\int \frac{dt}{(t)^2 + 2.t.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$

$$= \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(t+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x^2+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)$$

Ex. 6: Integrate $\int \frac{dx}{\sqrt{(a-x)(x-b)}}$

Solution:

Let $I = \int \frac{dx}{\sqrt{(a-x)(x-b)}}$

Put $x - b = t^2$,

$dx = 2t\,dt$

$$= \int \frac{2t\,dt}{\sqrt{(a - t^2 - b)\,.\,t^2}} = 2\int \frac{dt}{\sqrt{\left[\sqrt{a-b}\right]^2 - t^2}}$$

$$= 2\sin^{-1}\frac{t}{\sqrt{a-b}} = 2\sin^{-1}\sqrt{\frac{x-b}{a-b}}$$

$$\therefore \quad I = 2\sin^{-1}\sqrt{\frac{x-b}{a-b}}.$$

Ex. 7: Integrate $\int \frac{dx}{\sin x + \cos x}$

Solution:

Let $I = \int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\int \frac{1}{\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)}dx$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(x+\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(x+\frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$$

$$\therefore \quad I = \frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$$

Ex. 8: Integrate $\int \frac{\cos x\,dx}{\sin^2 x + 4\sin x + 5}$

Solution:

Let $\quad I = \int \frac{\cos x\,dx}{\sin^2 x + 4\sin x + 5}$

Put $\sin x = t$,

$\cos x\,dx = dt$

So, $I = \int \frac{dt}{t^2 + 4t + 5} = \int \frac{dt}{(t)^2 + 2.t.2 + (2)^2 - 4 + 5}$

$$= \int \frac{dt}{(t+2)^2 + (1)^2} = \tan^{-1}(t+2) = \tan^{-1}(\sin x + 2)$$

$$\therefore\ I = \tan^{-1}(\sin x + 2)$$

Ex. 9: Integrate $\int \frac{\sin x\, dx}{\sqrt{1+\sin x}}$

Solution:

$$\text{Let } I = \int \frac{\sin x}{\sqrt{1+\sin x}}\, dx = \int \frac{(1+\sin x - 1)}{\sqrt{1+\sin x}}\, dx$$

$$= \int \frac{1+\sin x}{\sqrt{1+\sin x}}\, dx - \int \frac{1}{\sqrt{1+\sin x}}\, dx$$

$$= \int \sqrt{1+\sin x}\, dx - \int \frac{1}{\sqrt{1+\sin x}}\, dx$$

$$= \int \sqrt{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\, dx - \int \frac{dx}{\sqrt{1+\sin x}}$$

$$= \int \left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right] dx - \int \frac{dx}{\sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}}$$

$$= -2\cos\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right) - \int \frac{dx}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}}$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \int \frac{dx}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}}\sin\frac{x}{2} + \frac{1}{\sqrt{2}}\cos\frac{x}{2}\right)}$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \int \frac{1}{\sin\frac{x}{2}\cos\frac{\pi}{4} + \cos\frac{x}{2}\sin\frac{\pi}{4}}\, dx$$

$$= 2\sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2} - \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)}$$

$$= 2\sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}} - \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)}$$

$$= 2\sqrt{1-\sin x} - \sqrt{2}\, \log\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)$$

$$\therefore\ I = 2\sqrt{1-\sin x} - \sqrt{2}\, \log\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)$$

Ex. 10: Integrate $\int \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$

Solution:

$$\text{Let } I = \int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x}$$

Put $b\tan x = at$,

$b\sec^2 x\, dx = a\, dt$

$$\text{So, } I = \frac{a}{b}\int \frac{dt}{a^2 + a^2t^2} = \frac{1}{ab}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{ab}\tan^{-1} t = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right)$$

$$\therefore\ I = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right)$$

Ex. 11: Integrate $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\, dx$

Solution:

$$\text{Let } I = \int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\, dx$$

We have,

$2\sin x + 3\cos x = l\,(\text{Den}^r) + m\,(\text{Diff. Coeff. of Den}^r)$

$= l\,(3\sin x + 4\cos x) + m(3\cos x - 4\sin x)$

or $2\sin x + 2\cos x = (3l - 4m)\sin x + (4l + 3m)\cos x$

Equating the coefficient of sinx and cosx, we get

$2 = 3l - 4m$

and $3 = 4l + 3m$

Solving these relations, we get

$l = \frac{18}{25}, m = \frac{1}{25}$

Thus, $I = \int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx$

$$= \int \frac{l(3\sin x + 4 \cos x) + m(3 \cos x - 4 \sin x)}{3 \sin x + 4 \cos x} dx$$

$$= \frac{18}{25} \int \frac{3 \sin x + 4 \cos x}{3 \sin x + 4 \cos x} dx + \frac{1}{25} \int \frac{3 \cos x - 4 \sin x}{3 \sin x + 4 \cos x} dx$$

$$= \frac{18}{25} \int dx + \frac{1}{25} \log (3 \sin x + 4 \cos x)$$

$$= \frac{18}{25} x + \frac{1}{25} \log (3 \sin x + 4 \cos x)$$

$$\therefore \quad I = \frac{18}{25} x + \frac{1}{25} \log (3 \sin x + 4 \cos x)$$

Ex. 12: Integrate $\int \frac{dx}{4 + 5 \sin x}$

Solution:

Let $I = \int \frac{dx}{4 + 5 \sin x} = \int \frac{dx}{4 + 10 \sin \frac{x}{2} \cos \frac{x}{2}}$

$$= \int \frac{\sec^2 \frac{x}{2} dx}{4 \sec^2 \frac{x}{2} + 10 \tan \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2} dx}{4 \tan^2 \frac{x}{2} + 10 \tan \frac{x}{2} + 4}$$

Put $\tan \frac{x}{2} = t$,

$\sec^2 \frac{x}{2} dx = 2dt$

So, $I = 2 \int \frac{dt}{4t^2 + 10t + 4} = \frac{1}{2} \int \frac{dt}{t^2 + \frac{5}{2} t + 1}$

$$= \frac{1}{2} \int \frac{dt}{(t)^2 + 2.t.\frac{5}{4} + \left(\frac{5}{4}\right)^2 + 1 - \frac{25}{16}} = \frac{1}{2} \int \frac{dt}{\left(t + \frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{2 \times \frac{3}{4}} \log \frac{t + \frac{5}{4} - \frac{3}{4}}{t + \frac{5}{4} + \frac{3}{4}} = \frac{1}{3} \log \left(\frac{2 \tan \frac{x}{2} + 1}{2 \tan \frac{x}{2} + 4} \right)$$

$$\therefore \quad I = \frac{1}{3} \log \left(\frac{2 \tan \frac{x}{2} + 1}{2 \tan \frac{x}{2} + 4} \right)$$

Ex. 13: Integrate $\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$

Solution:

Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$

Put $\sin^{-1} x = t$, $x = \sin t$,

$\therefore \quad \frac{1}{\sqrt{1 - x^2}} dx = dt$

So, $I = \int \sin t . t \, dt$

Integrating by parts

$$I = -t \cos t + \int \cos t \, dt = -t \cos t + \sin t$$

$$= -t\sqrt{1 - \sin^2 t} + \sin t = -\sin^{-1} x \sqrt{1 - x^2} + x$$

$$= x - \sin^{-1} x \sqrt{1 - x^2}$$

$$\therefore \quad I = x - \sin^{-1} x \sqrt{1 - x^2}$$

Ex. 14: Integrate $\int \frac{\log x}{(1 + \log x)^2} dx$ [2055 B.E.]

Solution:

Let $I = \int \frac{\log x}{(1 + \log x)^2} dx = \int \frac{(1 + \log x - 1)}{(1 + \log x)^2} dx$

$$= \int (1 + \log x)^{-1} .1 \, dx - \int \frac{dx}{(1 + \log x)^2}$$

Integrating by parts to the first integral,

$$= (1 + \log x)^{-1} x + \int \frac{1}{(1 + \log x)^2} \frac{1}{x} \, x \, dx - \int \frac{dx}{(1 + \log x)^2}$$

$$= \frac{x}{(1 + \log x)}$$

$$\therefore \quad I = \frac{x}{(1 + \log x)}$$

Ex.15: Integrate $\int \frac{dx}{\sin 2x - \sin x}$

Solution:

$$\text{Let } I = \int \frac{dx}{\sin 2x - \sin x} = \int \frac{\sin x \, dx}{\sin x(2 \sin x \cos x - \sin x)}$$

$$= \int \frac{\sin x}{\sin^2 x(2 \cos x - 1)} dx = \int \frac{\sin x \, dx}{(1 - \cos^2 x)(2 \cos x - 1)}$$

Put $\cos x = t$,

$-\sin x \, dx = dt$

$$\text{So, } I = -\int \frac{dt}{(1 - t^2)(2t - 1)} = -\int \frac{dt}{(1 + t)(1 - t)(2t - 1)}$$

Now,

$$\frac{1}{(1 + t)(1 - t)(2t - 1)} = \left[\frac{A}{(1 + t)} + \frac{B}{(1 - t)} + \frac{C}{(2t - 1)}\right]$$

$$1 = A(1 - t)(2t - 1) + B(1 + t)(2t - 1) + C(1 + t)(1 - t)$$

Put $t = 1$, $\quad 1 = 2B \quad \therefore B = \frac{1}{2}$.

Put $t = -1$, $\quad 1 = -6A \quad \therefore A = -\frac{1}{6}$

Put $t = 0$, $\quad 1 = -A - B + C \quad \therefore C = \frac{4}{3}$

$$\text{Thus } I = -\left[\int -\frac{1}{6} \frac{dt}{(1 + t)} + \int \frac{1}{2(1 - t)} dt + \int \frac{4}{3} \frac{dt}{(2t - 1)}\right]$$

$$= \frac{1}{6} \int \frac{dt}{(1 + t)} - \frac{1}{2} \int \frac{dt}{1 - t} + \frac{4}{3} \int \frac{dt}{(1 - 2t)}$$

$$= \frac{1}{6} \log (1 + t) + \frac{1}{2} \log (1 - t) - \frac{2}{3} \log (1 - 2t)$$

$$= \frac{1}{6} \log (1 + \cos x) + \frac{1}{2} \log (1 - \cos x) - \frac{2}{3} \log (1 - 2\cos x)$$

$$\therefore \quad I = \frac{1}{6} \log (1 + \cos x) + \frac{1}{2} \log (1 - \cos x) - \frac{2}{3} \log (1 - 2\cos x)$$

Ex. 16: Integrate $\int \frac{x^2 + 1}{x^4 + 1} dx$

Solution:

$$\text{Let } I = \int \frac{x^2 + 1}{x^4 + 1} dx = \int \frac{x^2\left(1 + \frac{1}{x^2}\right)}{x^2\left(x^2 + \frac{1}{x^2}\right)} dx = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2} dx$$

Put $x - \frac{1}{x} = t$,

$$\left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\text{So, } I = \int \frac{dt}{t^2 + 2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{\left(x - \frac{1}{x}\right)}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{(x^2 - 1)}{\sqrt{2}}$$

$$\therefore \quad I = \frac{1}{\sqrt{2}} \tan^{-1} \frac{(x^2 - 1)}{\sqrt{2}}$$

Ex. 17: Integrate $\int \frac{dx}{(x - 1)^2 (x - 2)^3}$

Solution:

$$\text{Let } I = \int \frac{dx}{(x - 1)^2 (x - 2)^3}$$

Put $x - 2 = \frac{1}{t}$, $\quad x = \frac{1}{t} + 2$,

$$dx = -\frac{1}{t^2}dt$$

$$= -\int \frac{dt}{t^2 \cdot \frac{1}{t^3}\left(\frac{1}{t}+2-1\right)^2} = -\int \frac{t^3}{(1+t)^2}dt$$

Put $1 + t = z$, $\quad t = z - 1$,

$dt = dz$

$$= -\int \frac{(z-1)^3}{z^2}dz = -\int \frac{z^3-1-3z^2+3z}{z^2}dz$$

$$= \int\left(-z+\frac{1}{z^2}+3-\frac{3}{z}\right)dz = -\frac{z^2}{2}-\frac{1}{z}+3z-3\log z$$

$$= -\frac{1}{2}(1+t)^2-\frac{1}{1+t}+3(1+t)-3\log(1+t)$$

$$= -\frac{1}{2}\left(1+\frac{1}{x-2}\right)^2-\frac{1}{1+\frac{1}{x-2}}+3\left(1+\frac{1}{x-2}\right)-3\log\left(1+\frac{1}{x-2}\right)$$

$$= -\frac{1}{2}\left(\frac{x-1}{x-2}\right)^2-\frac{x-2}{x-1}+3\left(\frac{x-1}{x-2}\right)-3\log\left(\frac{x-1}{x-2}\right)$$

Ex. 18: Integrate $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$

Solution:

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x}dx = \int \frac{\sqrt{\tan x}\sec^2 x}{\sin x \cos x.\sec^2 x}dx$

$$= \int \frac{\sqrt{\tan x}\sec^2 x}{\tan x}dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$$

Put $\tan x = t^2$,

$\sec^2 x\, dx = 2t\, dt$.

So, $I = \int \frac{2t\,dt}{t} = 2\int dt = 2t = 2\sqrt{\tan x}$

$\therefore \quad I = 2\sqrt{\tan x}$

Ex. 19: Integrate $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Solution:

Let $I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$

Put $x = \cos 2\theta$,

$dx = -2\sin 2\theta\, d\theta$

$$= \int \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\cdot(-2\sin 2\theta)\,d\theta$$

$$= \int \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right).(-2\sin 2\theta)\,d\theta = -2\int(\tan^{-1}\tan\theta)\sin 2\theta\, d\theta$$

$$= -2\int \theta \sin 2\theta\, d\theta$$

Integrating by parts,

$$= -2\theta.\frac{1}{2}(-\cos 2\theta)+\int 2.\frac{1}{2}(-\cos 2\theta)\,d\theta$$

$$= \theta\cos 2\theta-\frac{1}{2}.\sin 2\theta = \theta\cos 2\theta-\frac{1}{2}\sqrt{1-\cos^2 2\theta}$$

$$= \frac{x}{2}\cos^{-1}(x)-\frac{1}{2}\sqrt{1-x^2}$$

Exercise - 7

Integrate the following:

1. $\int \frac{3x^2}{1+x^6}dx$
2. $\int \frac{dx}{e^x+e^{-x}}$
3. $\int \frac{dx}{e^x+1}$
4. $\int \sqrt{\frac{a+x}{a-x}}\,dx$
5. $\int \frac{4x+3}{3x^2+3x+1}dx$
6. $\int \frac{dx}{x\sqrt{x^4-1}}$
7. $\int \frac{(x+1)\,dx}{\sqrt{4+8x-5x^2}}$
8. $\int \sqrt{\frac{1+x}{1-x}}\,dx$ [2060 B.E.]

9. $\int \sqrt{18x - 65 - x^2}\,dx$

10. $\int \frac{\cos x\,dx}{\sqrt{2\sin^2 x + 3\sin x + 4}}$

11. $\int \frac{dx}{\sqrt{1+\sin x}}$

12. $\int \frac{dx}{4 + 5\cos x}$

13. $\int \frac{dx}{1 + 3\sin^2 x}$

14. $\int \frac{(1+\cos x)}{\sin x \cos x}\,dx$

15. $\int \frac{x^2 \tan^{-1} x}{1+x^2}\,dx$ 2059 B.E.

16. $\int \frac{dx}{1+\tan x}$

17. $\int \frac{dx}{a^2 - b^2\cos^2 x}$ $(a > b)$

18. $\int \sqrt{\sec x - 1}\,dx$

19. $\int \frac{\cos x\,dx}{5 - 3\cos x}$

20. $\int \frac{dx}{1 - \cos x + \sin x}$

21. $\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}\,dx$

22. $\int \frac{(x + \sin x)}{1 + \cos x}\,dx$

23. $\int \frac{\cos x}{2\sin x + 3\cos x}\,dx$

24. $\int \frac{e^x(1+\sin x)}{1+\cos x}\,dx$

25. $\int \frac{dx}{\sin x(3 + 2\cos x)}$ 2057,062 B.E.

26. $\int \frac{dx}{5 + 4\cos x}$

27. $\int \frac{dx}{5 - 4\sin x}$

28. $\int \frac{\sin x\,dx}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}}$, $a \ne b$

29. $\int \frac{(1+\sin x)\,dx}{\sin x(1+\cos x)}$

30. $\int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\,dx$

31. $\int \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$

32. $\int \frac{dx}{x + \sqrt{x}}$

33. $\int \frac{x^2\,dx}{x^6 - 6x^3 + 5}$

34. $\int x^2 \sqrt{x^3 - 1}\,dx$

35. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

36. $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $\alpha < \beta$

37. $\int \frac{x e^x}{(x+1)^2}\,dx$ 2061 B.E.

38. $\int \frac{x^2\,dx}{(x+1)^2(x+2)}$

39. $\int \frac{x^3\,dx}{x^4 - x^2 - 12}$

40. $\int \frac{dx}{1+x^4}\,dx$

Answers

1. $\tan^{-1}(x^3)$ 2. $\tan^{-1}(e^x)$ 3. $-\log(e^{-x} + 1)$

4. $-a\cos^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$

5. $\frac{2}{3}\log(3x^2 + 3x + 1) + \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{3}(2x+1)$

6. $\frac{1}{2}\sec^{-1}(x^2)$

7. $\frac{9}{5\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{8}\right) - \frac{1}{5}\sqrt{4 + 8x - 5x^2}$

8. $\sin^{-1}x - \sqrt{1 - x^2}$ 9. $\frac{(x-9)}{2}\sqrt{18x - 65 - x^2} + 8\sin^{-1}\left(\frac{x-9}{4}\right)$

10. $\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{4\sin x + 3}{\sqrt{23}}\right)$ 11. $\sqrt{2}\log\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)$

12. $\frac{1}{3}\log\left(\frac{3 + \tan\frac{x}{2}}{3 - \tan\frac{x}{2}}\right)$ 13. $\frac{1}{2}\tan^{-1}(2\tan x)$

14. $\log\left(\tan x \tan\frac{x}{2}\right)$ 15. $x\tan^{-1}x - \frac{1}{2}(\tan^{-1}x)^2 - \frac{1}{2}\log(1+x^2)$

16. $\frac{x}{2} + \frac{1}{2}\log(\cos x + \sin x)$ 17. $\frac{1}{a\sqrt{a^2 - b^2}}\tan^{-1}\left(\frac{a\tan x}{\sqrt{a^2 - b^2}}\right)$

18. $-2\cosh^{-1}\left(\sqrt{2}\cos\frac{x}{2}\right)$ 19. $-\frac{x}{3} + \frac{5}{6}\tan^{-1}\left(2\tan\frac{x}{2}\right)$

20. $-\log\left(1 + \cot\frac{x}{2}\right)$ 21. $2x + \log(2\cos x + \sin x + 3)$

22. $x \tan\frac{x}{2}$

23. $\frac{3}{13}x + \frac{2}{13}\log(2\sin x + 3\cos x)$

24. $e^x \tan\frac{x}{2}$

25. $-\frac{1}{2}\log(1+\cos x) + \frac{1}{10}\log(1-\cos x) + \frac{2}{5}\log(3+\cos x)$

26. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)$

27. $\frac{2}{3}\tan^{-1}\left[\frac{1}{3}\left(5\tan\frac{x}{2} - 4\right)\right]$

28. $\frac{1}{\sqrt{b^2-a^2}}\cos^{-1}\left(\frac{\sqrt{b^2-a^2}}{b}\cos x\right)$

29. $\frac{1}{2}\log\tan\frac{x}{2} + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2}$

30. $\cos\alpha\cos^{-1}(\cos x\sec\alpha) - \sin\alpha\log\left(\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right)$

31. $\log\frac{1+\sin x}{2+\sin x}$

32. $2\log(1+\sqrt{x})$.

33. $\frac{1}{12}\log\frac{x^3-5}{x^3-1}$

34. $\frac{1}{6}\left[x^3\sqrt{x^6-1} - \log\left(x^3+\sqrt{x^6-1}\right)\right]$

35. $\frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

36. $2\log\left(\sqrt{x-\alpha} + \sqrt{x-\beta}\right)$

37. $\frac{e^x}{x+1}$

38. $-\frac{1}{x+1} - 3\log(x+1) + 4\log(x+2)$

39. $\frac{3}{14}\log(x^2+3) + \frac{2}{7}\log(x^2-4)$

40. $\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x} - \frac{1}{4\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}$

Chapter - 8

Definite Integral

8.1 Definition

The integral having the lower limit of x from x = a to x = b is defined by $\int_a^b f(x)\,dx$ and its integral is F(x) so that

$$\int_a^b f(x)\,dx = \left[F(x) + c\right]_a^b = [F(b) + c - F(a) - c]$$

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$

It shows that the difference of F(x) at x = b and x = a

i.e. F(b) - F(a) is the value of definite integral $\int_a^b f(x)\,dx$.

Definite integral is also defined as by the means of theorem known as *Fundamental Theorem of Integral Calculus*.

8.2 Fundamental Theorem of Integral Calculus

"If f(x) is integrable in (a, b) and the function F(x) defined on [a, b] is differentiable such that $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)"$$

Proof:

Let the interval [a, b] be divided into n parts by the set of points $\{x_0, x_1, x_2 \ldots\ldots x_n\}$ where $a = x_0 < x_1 < x_2 < \ldots\ldots\ldots < x_n = b$ with lengths

$$x_1 - x_0 = \delta, \quad x_2 - x_1 = \delta, \ldots\ldots, \quad x_n - x_{n-1} = \delta$$

Since $f(x) = F'(x)$ is integrable in (a, b)

So, by definition,

$$\int_a^b F'(x)\,dx = \lim_{\delta\to 0} \delta \sum_{r=1}^{n} F'(a+r\delta) \text{ provided the limit exists.}$$

By Mean Value theorem,

$$F(x_r) - F(x_{r-1}) = (x_r - x_{r-1})\,F'(a + r\delta)$$
$$= \delta F'(a + r\delta), \qquad x_{r-1} < a + r\delta < x_r$$

$$\therefore \int_a^b F'(x)\,dx = \lim_{\delta\to 0} \sum_{r=1}^{n} \delta F'(a + r\delta) = \sum_{r=1}^{n} \left[F(x_r) - F(x_{r-1})\right]$$

$$= \left[F(x_1) - F(x_0) + F(x_2) - F(x_1) + F(x_3) - F(x_2) + \ldots\right.$$
$$\left.\ldots\ldots + F(x_n) - F(x_{n-1})\right]$$

$$= F(x_n) - F(x_0) = F(b) - F(a)$$

Thus,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\therefore \int_a^b f(x)\,dx = F(b) - F(a)$$

8.3 Primitive and Integral

If f(x) is integrable in (a, b) and the function F(x) defined on [a, b] is differentiable such that $\frac{d}{dx}[F(x)] = f(x)$, then the function F(x) is called *Primitive* of f(x) and the *Integral* of f(x) is

$$\int_a^b f(x)\,dx = \lim_{\delta\to 0} \delta \sum_{r=1}^{n} f(a+r\delta)$$

The primitive of f(x) is not unique.

For, $\frac{d}{dx}[F(x)] = f(x)$ and

$$\frac{d}{dx}[F(x) + c] = \frac{d}{dx}[F(x)] + \frac{dc}{dx}$$

So [F (x)] and [F (x) + c] are primitive of f(x).

Thus the primitive of f(x) is not unique.

The distinction between *Primitive* and the *Integrals* is that while integrals can be calculated but primitive can not be calculated.

Worked Out Examples

Ex. 1. Integrate : $\int_0^{\pi/2} \sin^4 x\,dx$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \sin^4 x\,dx = \int_0^{\pi/2} (\sin^2 x)^2\,dx$$

$$= \int_0^{\pi/2} \left(\frac{1 - \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2} \left(1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\left[x - \frac{2\sin 2x}{2} + \frac{1}{2}x + \frac{\sin 4x}{8}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{\pi}{2} - 0 + \frac{\pi}{4} 0 - 0\right] = \frac{3\pi}{16}$$

$$\therefore \quad I = \frac{3\pi}{16}$$

Ex. 2. Integrate : $\int_0^1 x \tan^{-1} x\,dx$

Solution:

$$\text{Let } I = \int_0^1 x \tan^{-1} x\,dx$$

Integrating by parts

$$I = \left[\tan^{-1}x \cdot \frac{x^2}{2}\right]_0^1 - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\,dx$$

$$= \left(\frac{\pi}{8} - 0\right) - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{\pi}{8} - \left[\frac{x}{2}\right]_0^1 + \frac{1}{2}\left[\tan^{-1}x\right]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{4} = \frac{\pi-2}{4}$$

$$\therefore \quad I = \frac{\pi-2}{4}$$

Ex. 3. Integrate : $\displaystyle\int_0^{\pi/2} \frac{dx}{4\sin^2 x + 5\cos^2 x}$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \frac{dx}{4\sin^2 x + 5\cos^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{\sec^2 x\,(4\sin^2 x + 5\cos^2 x)}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{(4\tan^2 x + 5)}$$

Put $2\tan x = \sqrt{5}\,t$,

$2\sec^2 x\,dx = \sqrt{5}\,dt$

When $x = 0$, $t = 0$, when $x = \pi/2$, $t = \infty$

$$\text{So, } I = \frac{\sqrt{5}}{2}\int_0^{\infty} \frac{dt}{5(t^2+1)} = \frac{1}{2\sqrt{5}}\left[\tan^{-1} t\right]_0^{\infty}$$

$$= \frac{1}{2\sqrt{5}} \cdot \frac{\pi}{2} = \frac{\pi}{4\sqrt{5}}$$

$$\therefore \quad I = \frac{\pi}{4\sqrt{5}}$$

Ex. 4. Integrate : $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x}$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \frac{dx}{2+\cos x} = \int_0^{\pi/2} \frac{dx}{2 + \left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)}$$

$$= \int_0^{\pi/2} \frac{\sec^2\left(\frac{x}{2}\right)dx}{2 + \sec^2\left(\frac{x}{2}\right)} = \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2} + 3}$$

Put $\tan\frac{x}{2} = t$,

$\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$

When $x = 0$, $t = 0$, when $x = \frac{\pi}{2}$, $t = 1$

$$I = \int_0^1 \frac{2dt}{t^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - 0\right) = \frac{\pi}{3\sqrt{3}}$$

$$\therefore \quad I = \frac{\pi}{3\sqrt{3}}$$

Ex. 5. Integrate : $\displaystyle\int_0^{\pi/2} \frac{dx}{5+4\sin x}$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \frac{dx}{5+4\sin x} = \int_0^{\pi/2} \frac{dx}{5 + 8\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{5\sec^2\frac{x}{2} + 8\tan\frac{x}{2}} = \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{5 + 5\tan^2\frac{x}{2} + 8\tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = 1$, $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

When $x = 0, t = 0$, when $x = \pi/2, t = 1$

$$\therefore \quad I = \int_0^1 \frac{2dt}{5t^2 + 8t + 5} = \frac{2}{5}\int_0^1 \frac{dt}{t^2 + \frac{8t}{5} + 1}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{(t)^2 + 2.t\frac{4}{5} + \left(\frac{4}{5}\right)^2 + 1 - \frac{16}{25}}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(t + \frac{4}{5}\right)^2 + \left(\frac{9}{5}\right)^2}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(t + \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{2}{5}\,\frac{5}{3}\left[\tan^{-1}\frac{t + 4/5}{3/5}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{1 + (4/5)}{(3/5)} - \tan^{-1}\frac{(4/5)}{(3/5)}\right] = \frac{2}{3}\left\{\tan^{-1} 3 - \tan^{-1}\frac{4}{3}\right\}$$

$$= \frac{2}{3}\tan^{-1}\frac{3 - \frac{4}{3}}{1 + 3 \times \frac{4}{3}} = \frac{2}{3}\tan^{-1}\frac{1}{3}$$

$$\therefore \quad I = \frac{2}{3}\tan^{-1}\frac{1}{3}$$

Ex. 6. Integrate : $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \cos\alpha \cos x}$

Solution:

$$\text{Let} \quad I = \int_0^{\pi/2} \frac{dx}{1 + \cos\alpha \cos x}$$

$$= \int_0^{\pi/2} \frac{dx}{1 + \cos\alpha \left(2\cos^2\frac{x}{2} - 1\right)}$$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{\sec^2\frac{x}{2} + 2\cos\alpha - \cos\alpha \sec^2\frac{x}{2}}$$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{\left(1 + \tan^2\frac{x}{2} + 2\cos\alpha - \cos\alpha - \cos\alpha \tan^2\frac{x}{2}\right)}$$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{1 + \cos\alpha + (1 - \cos\alpha)\tan^2\frac{x}{2}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\tan^2\frac{x}{2}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sec^2\frac{\alpha}{2}\sec^2\frac{x}{2}\,dx}{1 + \tan^2\frac{\alpha}{2}\tan^2\frac{x}{2}}$$

Put $\tan\frac{\alpha}{2}\ \tan\frac{x}{2} = t$, $\tan\frac{\alpha}{2}.\frac{1}{2}\ \sec^2\frac{x}{2}\ dx = dt$,

When $x = 0, t = 0$, when $x = \pi/2, t = \tan\frac{\alpha}{2}$

$$\text{So,} \quad I = \frac{1}{2}.\frac{1}{2}\int_0^{\tan(\alpha/2)} \frac{\sec^2\left(\frac{\alpha}{2}\right)}{\tan\left(\frac{\alpha}{2}\right)}\,\frac{2dt}{(1 + t^2)}$$

$$= \frac{2}{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}\int_0^{\tan(\alpha/2)} \frac{dt}{1 + t^2}$$

$$= \frac{2}{\sin\alpha}\left[\tan^{-1} t\right]_0^{\tan(\alpha/2)} = \frac{2}{\sin\alpha}\left(\frac{\alpha}{2} - 0\right) = \frac{\alpha}{\sin\alpha}$$

$\therefore\ I = \frac{\alpha}{\sin\alpha}$

Ex. 7. Integrate : $\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$

Solution:

Let, $I = \int_2^3 \frac{dx}{(x-1)\sqrt{(x-1)^2-1}}$

Put $x - 1 = \sec\theta$, $dx = \sec\theta \tan\theta\, d\theta$

When $x = 2$, $\theta = 0$, when $x = 3$, $\theta = \frac{\pi}{3}$

$$= \int_0^{\pi/3} \frac{\sec\theta \tan\theta\, d\theta}{\sec\theta\sqrt{\sec^2\theta - 1}} = \int_0^{\pi/3} \frac{\sec\theta \tan\theta\, d\theta}{\sec\theta \tan\theta}$$

$$= \int_0^{\pi/3} d\theta = [\theta]_0^{\pi/3} = \frac{\pi}{3}$$

$\therefore\ I = \frac{\pi}{3}$

Ex. 8. Integrate : $\int_0^{\pi/2} \frac{x \sin x \cos x\, dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

Solution:

Let, $I = \int_0^{\pi/2} \frac{x \sin x \cos x\, dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$

$$= \int_0^{\pi/2} x\left[\frac{\sin x \cos x}{(a^2\cos^2 x + b^2\sin^2 x)}\right] dx$$

Integrating by parts,

$$= \left[x\int \frac{\sin x \cos x\, dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}\right]_0^{\pi/2} - \int_0^{\pi/2} \left[1. \int \frac{\sin x \cos x\, dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}\right] dx$$

$$= [x I_1]_0^{\pi/2} - \int_0^{\pi/2} I_1\, dx$$

Where, $I_1 = \int \frac{\sin x \cos x\, dx}{(a^2 \cos x + b^2 \sin^2 x)^2}$

Put, $a^2\cos^2 x + b^2\sin^2 x = t$,

$(-2a^2 \cos x \sin x + 2b^2 \sin x \cos x)\, dx = dt$

$2(b^2 - a^2) \sin x \cos x\, dx = dt$

So, $I_1 = \frac{1}{2(b^2-a^2)} \int \frac{dt}{t^2} = \frac{1}{2(b^2-a^2)}\left(-\frac{1}{t}\right)$

$$= -\frac{1}{2(b^2-a^2)(a^2\cos^2 x + b^2\sin^2 x)}$$

Thus, $I = \left[-\frac{x}{2(b^2-a^2)(a^2\cos^2 x + b^2\sin^2 x)}\right]_0^{\pi/2}$

$$+ \frac{1}{2(b^2-a^2)}\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)}$$

$$= -\frac{\pi}{4(b^2-a^2)\, b^2} + \frac{1}{2(b^2-a^2)}\int_0^{\pi/2} \frac{\sec^2 dx}{(a^2 + b^2\tan^2 x)}$$

Put, $b \tan x = at$, $\quad b\sec^2 x\, dx = a\, dt$,

When $x = 0$, $t = 0$, $\quad$ when $x = \pi/2$, $t = \infty$

$$= -\frac{\pi}{4b^2(b^2-a^2)} + \frac{1}{2(b^2-a^2)}\frac{a}{b}\int_0^\infty \frac{dt}{a^2(1+t^2)}$$

$$= -\frac{\pi}{4b^2(b^2-a^2)} + \frac{1}{2ab(b^2-a^2)}\left[\tan^{-1} t\right]_0^\infty$$

$$= -\frac{\pi}{4b(b^2-a^2)} + \frac{1}{2ab(b^2-a^2)}\frac{\pi}{2}$$

$$= -\frac{\pi}{4b^2(b^2-a^2)}\left(\frac{1}{ab} - \frac{1}{b^2}\right)$$

$$= -\frac{\pi}{4(b^2-a^2)}\left[\frac{(b-a)}{ab^2}\right] = \frac{\pi}{4ab^2(b+a)}$$

Ex. 9. Integrate : $\int_0^{\pi/2} \sin^6\theta \cos^3\theta \, d\theta$

Solution:

Let, $I = \int_0^{\pi/2} \sin^6\theta \cos^3\theta \, d\theta = \int_0^{\pi/2} \sin^6\theta \, (1 - \sin^2\theta) \cos\theta d\theta$

Put $\sin\theta = t$, $\cos\theta \, d\theta = dt$

When $\theta = 0$, $t = 0$, when $\theta = \pi/2$, $t = 1$.

$$= \int_0^1 t^6 (1 - t^2) \, dt = \int_0^1 t^6 \, dt - \int_0^1 t^8 \, dt$$

$$= \left[\frac{t^7}{7}\right]_0^1 - \left[\frac{t^9}{9}\right]_0^1 = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}$$

$\therefore \quad I = \frac{2}{63}$

Ex. 10. Integrate : $\int_0^{\pi/4} \sqrt{\tan\theta} \, d\theta$

Solution:

Let, $I = \int_0^{\pi/4} \sqrt{\tan\theta} \, d\theta$

Put $\tan\theta = t^2$, $\sec^2\theta \, d\theta = 2t \, dt$,

$d\theta = \frac{2t}{1 + t^4} dt$

When $\theta = 0$, $t = 0$, when $\theta = \pi/4$, $t = 1$

$$= \int_0^1 \frac{t.\,2t\,dt}{1 + t^4} = 2\int_0^1 \frac{t^2}{t^2\left(t^2 + \frac{1}{t^2}\right)} dt = 2\int_0^1 \frac{dt}{\left(t^2 + \frac{1}{t^2}\right)}$$

Now, $2\int \frac{dt}{\left(t^2 + \frac{1}{t^2}\right)} = \int \left[\frac{\left(1 + \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} + \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t^2 + \frac{1}{t^2}\right)}\right] dt$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2} + \int \frac{\left(1 - \frac{1}{t^2}\right) dt}{\left(t + \frac{1}{t}\right)^2 - 2}$$

Put $t - \frac{1}{t} = x$ in the first integral,

$\left(1 + \frac{1}{t^2}\right) dt = dx$

Put $t + \frac{1}{t} = y$, in the second integral,

$\left(1 - \frac{1}{t^2}\right) dt = dy$

Then, $2\int \frac{dt}{t^2 + \frac{1}{t^2}} = \int \frac{dx}{x^2 + 2} + \int \frac{dy}{y^2 - 2}$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + \frac{1}{2.\sqrt{2}} \log \frac{y - \sqrt{2}}{y + \sqrt{2}}$$

$$2\int \frac{dt}{t^2 + \frac{1}{t^2}} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t^2 - 1}{\sqrt{2}\,t} + \frac{1}{2\sqrt{2}} \log \frac{t^2 + 1 - \sqrt{2}\,t}{t^2 + 1 + \sqrt{2}\,t}$$

Thus, $2\int_0^1 \frac{dt}{t^2 + \frac{1}{t^2}} = \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{t^2-1}{\sqrt{2}t} + \frac{1}{2\sqrt{2}} \log \frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right]_0^1$

$$= \frac{1}{\sqrt{2}}.\,0 + \frac{1}{2\sqrt{2}} \log \frac{2 - \sqrt{2}}{2 + \sqrt{2}} + \frac{1}{\sqrt{2}} \frac{\pi}{2} - \frac{1}{2\sqrt{2}} \log 1$$

$$= \frac{1}{2\sqrt{2}} \log \frac{2 - \sqrt{2}}{2 + \sqrt{2}} + \frac{\pi}{2\sqrt{2}}$$

$$\therefore \int_0^{\pi/4} \sqrt{\tan\theta} \, d\theta = \frac{1}{2\sqrt{2}} \log \frac{2 - \sqrt{2}}{2 + \sqrt{2}} + \frac{\pi}{2\sqrt{2}}$$

Exercise -8

Integrate the following

1. $\int_0^a \frac{x+a}{a^2+x^2}\,dx$
2. $\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$
3. $\int_0^{\pi/2} \cos^4x\,dx$
4. $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx$
5. $\int_0^a x\sqrt{a^2-x^2}\,dx$
6. $\int_0^1 \frac{dx}{(1+x)\sqrt{x^2+2x}}$ [2059 B.E.]
7. $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$
8. $\int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$
9. $\int_1^e \frac{dx}{x\sqrt{1-(\log x)^2}}$
10. $\int_0^1 \frac{1-x}{1+x}\,dx$ [2061 B.E.]
11. $\int_1^2 \frac{dx}{x(1+x^4)}$
12. $\int_0^1 \frac{1-4x+2x^2}{\sqrt{2x-x^2}}\,dx$
13. $\int_0^1 \frac{1-x^2}{1+x^2}\,dx$
14. $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\,dx$
15. $\int_1^2 \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$
16. $\int_0^1 \tan^{-1}x\,dx$
17. $\int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$
18. $\int_0^{\pi/2} e^x(\sin x+\cos x)\,dx$
19. $\int_0^{\pi/2} \sin^3\theta\cos^3\theta\,d\theta$
20. $\int_0^{\pi/2} \frac{dx}{1+2\cos x}$
21. $\int_0^{\pi/2} \frac{dx}{4+5\sin x}$
22. $\int_0^{\pi} \frac{dx}{3+2\sin x+\cos x}$

23. $\int_0^{\pi/4} \frac{(\sin x+\cos x)\,dx}{(9+16\sin 2x)}$ [2061 B.E.]
24. $\int_0^{\pi/2} (\sqrt{\tan x}+\sqrt{\cot x})\,dx$
25. $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2}$
26. $\int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$

Answers

1. $\frac{1}{2}\log 2+\frac{\pi}{4}$
2. $\frac{\pi^2}{8}$
3. $\frac{3\pi}{16}$
4. $\frac{\pi}{8}(\beta-\alpha)^2$
5. $\frac{a^3}{3}$
6. $\frac{\pi}{3}$
7. $\frac{\pi}{2\sqrt{2}}$
8. $\frac{1}{2}\log(2+\sqrt{3})$
9. $\frac{\pi}{2}$
10. $-1+2\log 2$
11. $\frac{1}{4}\log\frac{32}{17}$
12. 0
13. $\frac{\pi}{2}-1$
14. $\frac{\pi}{4}$
15. $\frac{1}{\sqrt{3}}$
16. $\frac{\pi}{4}-\frac{1}{2}\log 2$
17. $\frac{8}{21}$
18. $e^{\pi/2}$
19. $\frac{1}{24}$
20. $\frac{1}{\sqrt{3}}\log(2+\sqrt{3})$
21. $\frac{1}{3}\log 2$
22. $\frac{\pi}{4}$
23. $\frac{1}{20}\log 3$
24. $\pi\sqrt{2}$
25. $\frac{\pi(a^2+b^2)}{4a^3b^3}$
26. $\log\frac{4}{3}$

8.4 Properties of Definite Integrals

Property 1:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

For $\int_a^b f(x)\,dx = [F(x)+c]_a^b$

$$= F(b)+c-F(a)-c$$

$= F(b) - F(a)$

and $\int_a^b f(t)\,dt = [F(t) + c]_a^b$

$= F(b) + c - F(a) - c$

$= F(b) - F(a)$

$\therefore \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Property 2:

$$\int_a^b f(x)\,dt = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b$$

For $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= [F(x)]_a^c + [F(x)]_c^b$

$= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a) = \int_a^b f(x)\,dx$

$\therefore \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Property 3:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

For $-\int_b^a f(x)\,dx = -[F(x)]_b^a$

$= -[F(a) - F(b)]$

$= F(b) - F(a)$

$= \int_b^a f(x)\,dx$

$\therefore \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property 4:

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

For

$\int_0^a f(a-x)\,dx = \int_0^a f(a-x)\,dx$

Put $a - x = t$, $dx = -dt$

$= -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt$

$= \int_0^a f(x)\,dx$

$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Property 5:

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f(2a-x) = f(x)$$

$$= 0 \quad \text{if } f(2a-x) = -f(x)$$

For $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

Put $x = 2a - t$ in second integral

$dx = -dt$

So, $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^0 f(2a-t)(-dt)$

$= \int_0^a f(x)\,dx - \int_a^0 f(2a-t)\,dt$

$= \int_0^a f(x)\,dx + \int_0^a f(2a-t)\,dt$

$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$$\therefore \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \qquad \text{if } f(2a-x) = f(x)$$

$$\text{and} \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx \qquad \text{if } f(2a-x) = -f(x)$$

$$= 0, \text{ if } f(2a-x) = -f(x)$$

Even and Odd Functions

A function f(x) is said to be *even* if $f(-x) = -f(x)$ and *odd* if $f(-x) = -f(x)$.

For example, $f(x) = x^3$ and $\qquad f(x) = \cos x$

$f(-x) = (-x)^3, \qquad f(-x) = \cos(-x)$

$= -x^3, \qquad = \cos x$

$f(-x) = -f(x), \qquad f(-x) = f(x)$

x^3 is *odd* function, and cosx is *even* function.

Property 6:

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \qquad \textbf{if f(x) is even}$$

$$= 0 \qquad \textbf{if f(x) is odd}$$

For $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

Put $x = -t$ in the first integral,

$dx = -dt$,

$$\int_{-a}^{a} f(x)\,dx = \int_a^0 f(-t)(-dt) + \int_0^a f(x)\,dx$$

$$= -\int_a^0 f(-t)\,dt + \int_0^a f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx \quad \text{if f(x) is even.}$$

i.e. $f(-x) = f(x)$

So, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if f(x) is even.

and $\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx \quad$ if $f(-x) = -f(x)$

$= 0$, if f(x) is odd

Worked Out Examples

Ex. 1: $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} = \frac{\pi}{4}$

Solution:

Let $I = \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$

Put $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$, when $x = a$, $\theta = \frac{\pi}{2}$

So, $I = \int_0^{\pi/2} \frac{a\cos\theta}{a\sin\theta + a\sqrt{1 - \sin^2\theta}}\,d\theta$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta}\,d\theta \qquad \text{.....(1)}$$

By using properties of definite integral

$$I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)}\,d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta + \sin\theta}\,d\theta \qquad \text{.....(2)}$$

Adding (1)and(2), we get

$$2I = \int_0^{\pi/2} \frac{(\sin\theta + \cos\theta)}{(\cos\theta + \sin\theta)}\,d\theta$$

$$= \int_0^{\pi/2} d\theta = [\theta]_0^{\pi/2}$$

or $\quad 2I = \frac{\pi}{2}$.

Hence $I = \frac{\pi}{4}$.

Ex. 2: $\int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} x\, f(\sin x)\, dx$

Solution:

Let $I = \int_0^{\pi} x\, f(\sin x)\, dx$ ....(1)

By using properties of definite integral

$$I = \int_0^{\pi} (\pi - x)\, f\{\sin(\pi - x)\}\, dx$$

$$= \int_0^{\pi} (\pi - x)\, f\{\sin x\}\, dx$$

$$= \pi \int_0^{\pi} f(\sin x)\, dx - \int_0^{\pi} x\, f(\sin x)\, dx$$

or $\quad I = \pi \int_0^{\pi} f(\sin x)\, dx - I$

or $\quad 2I = \pi \int_0^{\pi} f(\sin x)\, dx.$

$\therefore \quad I = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx$

Ex. 3: $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx = 0$

Solution:

Let $I = \int_0^{\pi/2} \frac{(\sin x - \cos x)\, dx}{1 + \sin x \cos x}$

By using properties of definite integral

$$I = \int_0^{\pi/2} \frac{\left\{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)\right\}}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}\, dx$$

$$= \int_0^{\pi/2} \frac{(\cos x - \sin x)\, dx}{1 + \cos x \sin x} = -\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx$$

$I = -I,$

or $\quad 2I = 0,$

$\therefore \quad I = 0$

Ex. 4: $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}}\, dx = \frac{\pi}{4}$

Solution:

$$\text{Let, } I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{1 + \sqrt{\cot x}}\, dx = \int_0^{\pi/2} \frac{\sqrt{\frac{\cos x}{\sin x}}}{1 + \sqrt{\frac{\cos x}{\sin x}}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad \text{..........(1)}$$

By using properties of definite integral

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\, dx$$

or $\quad I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$ ....(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$\therefore \quad I = \frac{\pi}{4}.$

Ex. 5: $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$

Solution:

Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx$

By using properties of definite integral,

$$I = \int_0^{\pi} \frac{(\pi - x) \tan (\pi - x)\, dx}{\sec (\pi - x) + \cos (\pi - x)}$$

$$= \int_0^{\pi} \frac{-(\pi - x) \tan x\, dx}{-\sec x - \cos x}$$

$$= \int_0^{\pi} \frac{-(\pi - x) \tan x\, dx}{-(\sec x + \cos x)}$$

$$= \int_0^{\pi} \frac{\pi \tan x\, dx}{\sec x + \cos x} - \int_0^{\pi} \frac{x \tan x\, dx}{\sec x + \cos x}$$

$$= \pi\int_0^{\pi} \frac{\tan x\, dx}{\sec x + \cos x} - I$$

or $2I = \int_0^{2\times(\pi/2)} \frac{\tan x\, dx}{\sec x + \cos x} = \int_0^{2\times(\pi/2)} \frac{\sin x}{1 + \cos^2 x} dx$

or $2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx \quad \left(\square\, f\left(2\frac{\pi}{2} - x\right) = f(x)\right)$

Put $\cos x = t$, $\quad -\sin x\, dx = dt$,

When $x = 0, t = 1$, $\quad$ when, $x = \frac{\pi}{2}, t = 0$

$$I = -\pi \int_1^0 \frac{dt}{1 + t^2}$$

$$= -\pi \left[\tan^{-1} t\right]_1^0 = \frac{\pi^2}{4}.$$

$\therefore \quad I = \frac{\pi^2}{4}.$

Ex. 6: $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx = \frac{\pi^2}{16}$ 2060 B.E.

Solution:

Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$

By using properties of definite integral

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\left[\cos\left(\frac{\pi}{2} - x\right)\right]^4 + \left[\sin\left(\frac{\pi}{2} - x\right)\right]^4} dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x \sin x}{\sin^4 x + \cos^4 x} dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sec^4 x \,.\, \cos x \sin x}{\tan^4 x + 1} dx - I$$

or $2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx$

Put $\tan^2 x = t$, $2 \tan x \sec^2 x\, dx = dt$

When $x = 0, t = 0$, $\quad$ when $x = \frac{\pi}{2}, t = \infty$

So, $2I = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{2(t^2 + 1)}$

or $I = \frac{\pi}{8} \left[\tan^{-1} t\right]_0^{\infty}$,

or $I = \frac{\pi}{8}\left(\frac{\pi}{2} - 0\right)$.

$\therefore \quad I = \frac{\pi^2}{16}$.

Ex. 7: Show that $\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = \frac{\pi}{2}\log\frac{1}{2}$

Solution:

Let $I = \int_0^{\pi/2} \log \sin x\, dx$

By using properties of definite integral,

$$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \log \cos x\, dx$$

Adding (1) and (2), we get,

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log (\sin x \cos x)\, dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

Put $2x = t$,

$$dx = \frac{dt}{2}$$

So, $2I = \int_0^{\pi} \log \sin t \frac{dt}{2} - \log 2\, [x]_0^{\pi/2}$

$$= \frac{1}{2}\int_0^{\pi} \log \sin t\, dt - \log 2\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{1}{2}\int_0^{2(\pi/2)} \log \sin t\, dt - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2} \times 2\int_0^{\pi/2} \log \sin t\, dt - \frac{\pi}{2}\log 2 \qquad \left[\because \sin\left(\frac{2\pi}{2} - t\right) = \sin t\right]$$

$$= \int_0^{\pi/2} \log \sin x\, dx - \frac{\pi}{2}\log 2$$

or $2I = I - \frac{\pi}{2}\log 2.$

or $I = -\frac{\pi}{2}\log 2$

$\therefore \quad I = \frac{\pi}{2}\log\frac{1}{2}.$

Ex. 8: Show that $\int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta = \frac{\pi}{8}\log 2$

Solution:

Let, $I = \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$

By using properties of definite integral,

$$I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{1 + \tan\theta + 1 - \tan\theta}{1 + \tan\theta}\right] d\theta = \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$$

or $I = \log 2\, [\theta]_0^{\pi/4} - I$

or $2I = \log 2\left(\frac{\pi}{4} - 0\right)$

$\therefore \quad I = \frac{\pi}{8}\log 2.$

Exercise -9

Show that

1. $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx =$

3. $\int_0^{\pi} x \sin^2 x \, dx = \frac{\pi^2}{4}$

4. $\int_0^{\pi/2} \log (\tan x) \, dx = 0$

5. $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx = \frac{a}{2}$

6. $\int_0^{\pi/2} \sin 2x \log (\tan x) \, dx =$

7. $\int_0^{1} \frac{\log x}{\sqrt{1 - x^2}} dx = \frac{\pi}{2} \log \frac{1}{2}$ [2059/060 …]

8. $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\tan x}} = \frac{\pi}{4}$

9. $\int_0^{1} \frac{\log (1 + x)}{1 + x^2} dx = \frac{\pi}{8} \log 2$ [2059/061/062 …]

10. $\int_0^{\pi} \frac{x \tan dx}{\sec x + \tan x} = \frac{\pi}{2} (\pi - 2)$

11. $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)$

12. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$

13. $\int_0^{\pi} x \log (\sin x) \, dx = \frac{\pi^2}{2} \log \frac{1}{2}$

14. $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log (\sqrt{2} + 1)$

15. $\int_0^{1} \cot^{-1} (1 - x + x^2) \, dx = \frac{\pi}{2} - \log 2$

16. $\int_0^{\pi} \frac{x}{a^2 \sin^2 x + b^2 \cos^2 x} dx \; (a, b > 0) = \frac{\pi^2}{2ab}$

17. $\int_0^{\pi} \frac{x \, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi^2 (a^2 + b^2)}{4a^3 b^3}$

8.5 Definite Integral as the Limit of a Sum

If f(x) is a continuous and single- valued function defined in the interval [a, b] where a and b are finite with a < b and if the interval [a, b] be divided into n equal parts, each of length h, by the points, a, a + h, a + 2h, a + 3h ...., a + nh, where nh = b – a, then the definite integral of f(x) is defined as the *Limit of a Sum* by

$$\int_a^b f(x) \, dx = \lim_{h \to 0} h \, [f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1) h)]$$

or $$\int_a^b f(x) \, dx = \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh), \text{ where } nh = b - a$$

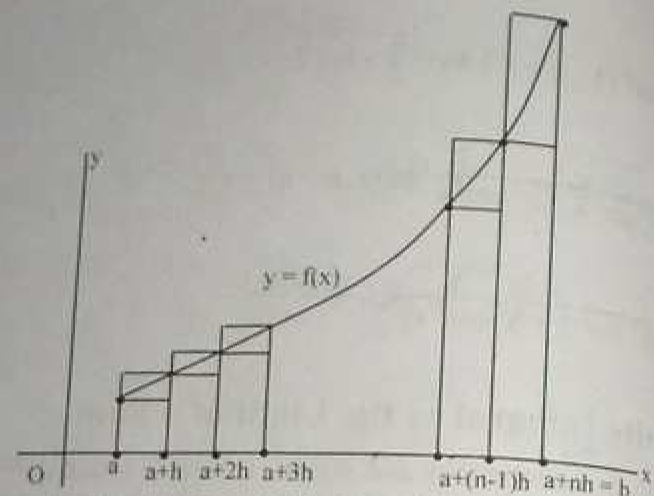
or $$\int_a^b f(x) \, dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a + rh), \text{ where } nh = b - a$$

$$\therefore \int_a^b f(x) \, dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a + rh), \text{ or } \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh)$$

8.6 Geometrical Interpretation of Definite Integral as the Limit of a Sum

If f(x) is continuous on the interval [a, b] then the area bounded by the curve y = f(x), x-axis and the ordinate x = a and x = b is represented by the definite integral

$$\int_a^b f(x) \, dx.$$



Let $y = f(x)$ be continuous function defined on $[a,b]$. The inte[rval] $[a, b]$ is divided into n equal parts by the po[ints] $a, a + h, a + 2h, \ldots\ldots, a + nh$, each of the length h so tha[t]

$b - a = nh.$

Thus, the value of y at these points

$a, a + h, a + 2h, a + 3h, \ldots\ldots, a + nh$ are

$f(a), f(a + h), f(a + 2h), f(a + 3h) \ldots\ldots, f(a + nh)$

Completing the rectangles as shown in the figure such that t[he] inner rectangles and outer rectangles are formed.

The area of all inner rectangles are

$hf(a), hf(a + h), h\, f(a + 2h), h\, f(a + 3h) \ldots\ldots, h\, f(a + (n-1)h)$

and area of all outer rectangles are

$h\, f(a + h), h\, f(a + 2h), h\, f(a + 3h), \ldots\ldots, h\, f(a + nh)$

Let S_1 and S_2 denote the sum of the areas of all *Inner* and *Out[er]* *rectangles*,

$$S_1 = hf(a) + h\, f(a + h) + h\, f(a + 2h) + \ldots\ldots + h\, f(a + (n-1)h)$$

$$= h[f(a) + f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n-1)h)]$$

$$= h \sum_{r=0}^{n-1} f(a + rh) \qquad \ldots\ldots(1)$$

and $S_2 = h\, f(a + h) + h\, f(a + 2h) + h\, f(a + 3h) + \ldots\ldots + h\, f(a + nh)$

$$= h[f(a + h) + f(a + 2h) + f(a + 3h) + \ldots\ldots + f(a + nh)]$$

$$S_2 = h \sum_{r=1}^{n} f(a + rh) \qquad \ldots\ldots(2)$$

Let S denote the sum of the areas between two areas of (1) and (2) and we see from the figure that

$S_1 < S < S_2$

$$\Rightarrow \quad h \sum_{r=0}^{n-1} f(a + rh) < S < h \sum_{r=1}^{n} f(a + rh)$$

$$\Rightarrow \quad \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh) = S = \lim_{h \to 0} h \sum_{r=1}^{n} f(a + rh)$$

$$\therefore \int_a^b f(x)\, dx = \lim_{h \to 0} \sum_{r=0}^{n-1} f(a + rh) = \lim_{h \to 0} h \sum_{r=1}^{n} f(a + rh)$$

8.7 Summation of Series

We have,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} \Sigma f(a + rh), \qquad nh = b - a$$

In special case, when $a = 0$, and $b = 1$,

$$\text{So, } \int_0^1 f(x)\, dx = \lim_{h \to 0} h \Sigma f(rh) \qquad \text{where } nh = 1,\ h = \frac{1}{n}$$

$$= \lim_{n \to \infty} \frac{1}{n} \Sigma f\left(\frac{r}{n}\right)$$

$$= \lim_{n \to \infty} \Sigma \frac{1}{n} f\left(\frac{r}{n}\right)$$

$$\text{Hence, } \lim_{n \to \infty} \Sigma \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx$$

It shows that the series represented by *Definite Integral* as we write, x for $\frac{r}{n}$, dx for $\frac{1}{n}$ and the symbol $\lim_{n \to \infty} \Sigma$ for $\int_0^1$.

8.8 Some Important Formulae

1. $$\int_a^b f(x)\, dx = \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh),$$

$$= \lim_{h \to 0} h \sum_{r=1}^{n} f(a + rh), \text{ where } nh = b - a$$

2. $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

3. $\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(z_i)\,\delta_i$

4. $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

5. $1 + 2 + 3 + 4 + \ldots\ldots + n = \frac{n(n+1)}{2}$

6. $1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

7. $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

8. $\cos\alpha + \cos(\alpha + h) + \cos(\alpha + 2h) + \ldots\ldots + \cos(\alpha + nh - h)$

$= \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \cos\frac{1}{2}$ [Angle of first term + Angle of last term]

9. $\sin\alpha + \sin(\alpha + h) + \sin(\alpha + 2h) + \ldots\ldots + \sin(\alpha + nh - h)$

$= \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \sin\frac{1}{2}$ [Angle of first term + Angle of last term]

Worked Out Examples

Evaluate the following series by using Integral.

Ex. 1: $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}\right)$

Solution:

Here, $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}\right)$

The r^{th} term $= \frac{1}{n+r}$, where $r = 1, 2, \ldots\ldots n$

So,

The limit $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \frac{1}{\left(1 + \frac{r}{n}\right)}$

Replacing $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx, $\sum_{r=1}^{n}$ by $\int_0^1$, we get

$= \int_0^1 \frac{1}{1+x}\,dx = \log(1+x)\Big]_0^1 = \log 2$

$\therefore \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}\right) = \log 2$

Ex. 2: $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots + \frac{1}{8n}\right)$

Solution:

Here, $\lim_{n\to\infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots + \frac{1}{8n}\right]$

The r^{th} term $= \frac{n^2}{(n+r)^3}$, where $r = 0, 1, 2, \ldots\ldots n$

So,

Given series $= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{n^2}{(n+r)^3}$

$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} \frac{n^2}{n^2\left(1 + \frac{r}{n}\right)^3} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} \frac{1}{\left(1 + \frac{r}{n}\right)^3}$

Replacing $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx, $\sum_{r=0}^{n-1}$ by $\int_0^1$, we get

$= \int_0^1 \frac{1}{(1+x)^3}\,dx = \left[-\frac{1}{2(x+1)^2}\right]_0^1 = -\frac{1}{2}\left(\frac{1}{4} - 1\right) = \frac{3}{8}$

$\therefore \lim_{n\to\infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots\ldots\ldots + \frac{1}{8n}\right] = \frac{3}{8}$

Ex. 3: $\lim_{n\to\infty} \left(\frac{1}{1+n^3} + \frac{4}{8+n^3} + \frac{9}{27+n^3} \ldots\ldots\ldots + \frac{1}{2n}\right)$

Solution:

Here $\lim_{n\to\infty} \left(\frac{1}{1+n^3} + \frac{4}{8+n^3} + \frac{9}{27+n^3} \ldots + \frac{1}{2n}\right)$

The r^{th} term $= \frac{r^2}{r^3 + n^3}$, where r = 1, 2, ... n

So,

$$\text{Given series} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{r^3 + n^3} = \sum_{r=1}^{n} \frac{r^2}{n^3\left(1 + \frac{r^3}{n^3}\right)}$$

$$= \sum_{r=1}^{n} \frac{1}{n}\left(\frac{r}{n}\right)^2 \frac{1}{1 + \frac{r^3}{n^3}} = \int_0^1 \frac{x^2}{(1 + x^3)} dx$$

$$= \frac{1}{3}\left[\log(1 + x^3)\right]_0^1 = \frac{1}{3}\log 2$$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{1+n^3} + \frac{4}{8+n^3} + \frac{9}{27+n^3} \ldots + \frac{1}{2n}\right) = \log 2.$$

excluded from course

Exercise-10

Evaluate the following:

1. $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots\ldots + \frac{n}{2n^2}\right)$
2. $\lim_{n\to\infty}\left(\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \ldots\ldots + \frac{1}{n}\right)$
3. $\lim_{n\to\infty} \frac{r^3}{r^4+n^4}$
4. $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{2}\right)\right]^{1/n}$
5. $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$

Answers

1. $\frac{\pi}{4}$ 2. $\frac{\pi}{4} + \frac{1}{2}\log 2$
3. $\frac{1}{4}\log 2$ 4. $\frac{4}{e}$ 5. e^{-1}

8.9 Improper Integrals

The integral $\int_a^b f(x)\,dx$ is called an *Infinite Integral* or *Improper Integral* if the integrand f(x) becomes infinite within the interval $a \le x \le b$ i.e. the limit of integration a or b (or both) become infinite.

For example, $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$, $\int_a^{\infty} \frac{dx}{x^2}$

$\int_{-\infty}^{b} \frac{dx}{x^2+1}$, $\int_a^b \frac{dx}{x(a-x)}$

To evaluate the value of infinite integrals, the integral can be defined as the following.

1. $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$,

   provided the limit exists.

2. $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to\infty}\int_{-a}^{b} f(x)\,dx$,

   provided the limit exists.

3. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$

   $= \lim_{a\to\infty}\int_{-a}^{0} f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx$

   provided the limit exists.

4. If $f(x) \to \infty$ as $x \to a$, then

   $\int_a^b f(x)\,dx = \lim_{h\to 0}\int_{a+h}^{b} f(x)\,dx,\ h > 0.$

5. If $f(x) \to \infty$ as $x \to b$,then

   $\int_a^b f(x)\,dx = \lim_{h\to 0}\int_a^{b-h} f(x)\,dx,\ h > 0.$

6. If $f(x) \to \infty$ as $x \to c$, $a < c < b$, then

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\int_a^{c-h} f(x)\,dx + \lim_{h'\to 0}\int_{c+h}^{b} f(x)\,dx,$$
$h > 0,\ h' > 0.$

Worked Out Examples

Evaluate, if possible, the following improper integrals:

Ex. 1: $\int_0^\infty x^2 e^{-x}\,dx$

Solution:

Here,

$$\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[x^2\frac{e^{-x}}{-1}\right]_0^b - \lim_{b\to\infty}\int_0^b 2x\frac{e^{-x}}{-1}\,dx$$

(Integrating by parts)

$$= \lim_{b\to\infty}\left[\frac{-b^2}{e^b}\right] + \lim_{b\to\infty}\int_0^b 2x\,e^{-x}\,dx$$

$$= \lim_{b\to\infty}\frac{-2b}{e^b} + \lim_{b\to\infty}\left[2x\frac{e^{-x}}{-1}\right]_0^b - \lim_{b\to\infty}\int_0^b 2\times\frac{e^{-x}}{-1}\,dx$$

$\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{b\to\infty}\left[\frac{2b}{e^b}\right] + \lim_{b\to\infty}\left[\frac{-2b}{e^b}\right] + \lim_{b\to\infty}\left[\frac{2e^{-x}}{-1}\right]_0^b \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{b\to\infty}\left[-\frac{2}{e^b}\right] + \lim_{b\to\infty}\left[-\frac{2}{e^b}\right] + \lim_{b\to\infty}\left[-2e^{-b} + 2\right]$$

$$= 0+0-0+2 = 2.$$

$$\therefore \int_0^\infty x^2 e^{-x}\,dx = 2.$$

Ex. 2: $\int_{-\infty}^\infty \frac{dx}{x^3}$

Solution:

The number 0 is the interior point of $(-\infty, \infty)$ and $\frac{1}{x^3}\to\infty$ as $x\to 0$, then the integral is defined as

$$\int_{-\infty}^\infty \frac{dx}{x^3} = \lim_{h\to 0}\lim_{a\to\infty}\int_{-a}^{0-h}\frac{dx}{x^3} + \lim_{h'\to 0}\lim_{b\to\infty}\int_{0+h'}^{b}\frac{dx}{x^3}$$

$$= \lim_{h\to 0}\lim_{a\to\infty}\int_{-a}^{-h}\frac{dx}{x^3} + \lim_{h'\to 0}\lim_{b\to\infty}\int_{+h'}^{b}\frac{dx}{x^3}$$

$$= \lim_{h\to 0}\lim_{a\to\infty}\left[\frac{-1}{2x^2}\right]_{-a}^{-h} + \lim_{h'\to 0}\lim_{b\to\infty}\left[\frac{-1}{2x^2}\right]_{h'}^{b}$$

$$= \lim_{h\to 0}\lim_{a\to\infty}\left[\frac{-1}{2h^2} + \frac{1}{2a^2}\right] + \lim_{h'\to 0}\lim_{b\to\infty}\left[\frac{-1}{2b^2} + \frac{1}{2h'^2}\right]$$

$$= \lim_{h\to 0}\left[\frac{-1}{2h^2}\right] + \lim_{h'\to 0}\left[\frac{1}{2h'^2}\right]$$

The integral does not exist if $h = h'$ the principal value

$$= \lim_{h\to 0}\left[\frac{-1}{2h^2} + \frac{1}{2h^2}\right] = 0$$

Ex. 3: $\int_2^\infty \frac{dx}{x^2-1}$

Solution:

$$\text{Here} \int_2^\infty \frac{dx}{x^2-1} = \lim_{b\to\infty}\int_2^b \frac{dx}{x^2-1} = \lim_{b\to\infty}\frac{1}{2}\left[\log\frac{x-1}{x+1}\right]_2^b$$

$$= \frac{1}{2}\lim_{b\to\infty}\log\frac{b-1}{b+1} - \frac{1}{2}\log\frac{1}{3}$$

$$= \lim_{b\to\infty}\log\frac{1-\frac{1}{b}}{1+\frac{1}{b}} - \frac{1}{2}\log\frac{1}{3} = \log 1 - \frac{1}{2}\log 3^{-1}$$

$$= -\frac{1}{2}\log 3^{-1} = \frac{1}{2}\log 3$$

$$\therefore \int_2^\infty \frac{dx}{x^2-1} = \frac{1}{2}\log 3.$$

Ex. 4: $\int_{-\infty}^\infty \frac{x\,dx}{x^4+1}$

Solution:

$$\text{Here} \int_{-\infty}^\infty \frac{x\,dx}{x^4+1} = \int_{-\infty}^0 \frac{x\,dx}{x^4+1} + \int_0^\infty \frac{x\,dx}{x^4+1}$$

Put $x^2 = t$, $2x\,dx = dt$

So, $\displaystyle\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int\frac{dt}{t^2+1} = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1} x^2$

Thus, $$\int_{-\infty}^{\infty}\frac{x\,dx}{x^4+1} = \lim_{a\to\infty}\int_{-a}^{0}\frac{x\,dx}{x^4+1} + \lim_{b\to\infty}\int_{0}^{b}\frac{x\,dx}{x^4+1}$$

$$= \lim_{a\to\infty}\frac{1}{2}\left[\tan^{-1}x^2\right]_{-a}^{0} + \lim_{b\to\infty}\frac{1}{2}\left[\tan^{-1}x^2\right]_{0}^{b}$$

$$= \lim_{a\to\infty}\frac{1}{2}\left[0 - \tan^{-1}a^2\right] + \lim_{b\to\infty}\frac{1}{2}\left[\tan^{-1}b^2 - 0\right]$$

$$= -\frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2} = -\frac{\pi}{4} + \frac{\pi}{4} = 0$$

$$\therefore \int_{-\infty}^{\infty}\frac{x\,dx}{x^4+1} = 0.$$

Ex. 5: $\displaystyle\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ [2058 B.E.]

Solution:

Here $\displaystyle\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Since $f(x) = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}} \to \infty$ as $x \to 1$.

Thus, $$\int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \lim_{h\to 0}\int_0^{1-h}\frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

Put $\sin^{-1}x = t$, $\dfrac{1}{\sqrt{1-x^2}}\,dx = dt$

Now $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int t\,dt = \frac{t^2}{2} = \frac{1}{2}(\sin^{-1}x)^2$

$$= \lim_{h\to 0}\left[\frac{1}{2}(\sin^{-1}x)^2\right]_0^{1-h} = \frac{1}{2}\lim_{h\to 0}\left[\sin^{-1}(1-h)\right]^2$$

$$= \frac{1}{2}\left[\sin^{-1}1\right]^2 = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{1}{2}\cdot\frac{\pi^2}{4} = \frac{\pi^2}{8}$$

$$\therefore \int_0^1 \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \frac{\pi^2}{8}.$$

Ex. 6: $\displaystyle\int_0^1 \log x\,dx$

Solution:

Here $\displaystyle\int_0^1 \log x\,dx$

Integrating by parts

$$\int \log x\,.1\,dx = \log x.\,x - \int\frac{1}{x}.\,x\,dx = x\log - x$$

So, $$\int_0^1 \log x\,dx = \lim_{h\to 0}\int_{0+h}^{1}\log x\,dx = \lim_{h\to 0}\left[x\log x - x\right]_h^1$$

$$= \lim_{h\to 0}\left[0 - 1 - h\log h + h\right]$$

$$= -1 + \lim_{h\to 0} h(1 - \log h)$$

$$= -1 + \lim_{h\to 0}\frac{(1-\log h)}{\frac{1}{h}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= -1 + \lim_{h\to 0}\frac{-1/h}{-\frac{1}{h^2}} = -1 + \lim_{h\to 0}\frac{1}{h}\times\frac{h^2}{1}$$

$$= -1 + 0 = -1$$

$$\therefore \int_0^1 \log x\,dx = -1.$$

8.10 Some Standard Improper Integrals

Ex. 7: The integral $\displaystyle\int_0^{\infty} e^{-ax}\sin bx\,dx\ (a > 0)$

Solution:

Here $\displaystyle\int_0^{\infty} e^{-ax}\sin bx\,dx\ (a > 0)$

The infinite integral is defined as

$$\int_0^\infty e^{-ax}\sin bx\,dx = \lim_{t\to\infty}\int_0^t e^{-ax}\sin bx\,dx$$

$$= \lim_{t\to\infty}\left[e^{-ax}\frac{(-a\sin bx - b\cos bx)}{a^2+b^2}\right]_0^t \quad \text{(Using formula)}$$

$$= -\lim_{t\to\infty}\left[e^{-at}\frac{(a\sin bt + b\cos bt)}{a^2+b^2}\right] + \frac{b}{a^2+b^2}$$

$$= -\lim_{t\to\infty} e^{-at}\frac{a\sin b\,t + b\cos bt}{a^2+b^2} + \frac{b}{a^2+b^2}$$

Since $\lim_{t\to\infty} e^{-at} = 0$ and a sin bt + b cos bt is bounded

$$\lim_{t\to\infty} e^{-at}\frac{a\sin b\,t + b\cos bt}{a^2+b^2} = 0$$

$$= 0 + \frac{b}{a^2+b^2} = \frac{b}{a^2+b^2}$$

$$\therefore \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

Ex. 8: The integral $\int_0^\infty e^{-ax}\cos bx\,dx\ (a>0)$ 2059 B.E.

Solution:

Here $\int_0^\infty e^{-ax}\cos bx\,dx\ (a>0)$

The infinite integral is defined as

$$\int_0^\infty e^{-ax}\cos bx\,dx = \lim_{t\to\infty}\int_0^t e^{-ax}\cos bx\,dx$$

$$= \lim_{t\to\infty}\int_0^t e^{-ax}\left(\frac{e^{ibx}+e^{-ibx}}{2}\right) \quad \text{(Using formula)}$$

$$= \lim_{t\to\infty}\frac{1}{2}\int_0^t \left[e^{-(a-ib)x} + e^{-(a+ib)x}\right]dx$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[\frac{e^{-(a-ib)x}}{-(a-ib)} - \frac{e^{-(a+ib)x}}{a+ib}\right]_0^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[\frac{e^{-(a-ib)t}}{-(a-ib)} - \frac{e^{-(a+ib)t}}{a+ib}\right] + \frac{1}{2}\left[\frac{1}{a-ib} + \frac{1}{a+ib}\right]$$

$$= 0 + 0 + \frac{1}{2}\frac{(a+ib+a-ib)}{(a-ib)(a+ib)} = \frac{2a}{a^2-i^2b^2}$$

$$\therefore \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

Ex. 9: The integral $\int_0^\infty e^{-x^2}dx$

Solution:

Let, $I = \int_0^\infty e^{-x^2}dx$

Replacing x by ax, dx = adx

$$I = a\int_0^\infty e^{-a^2x^2}dx$$

Multiplying both sides by e^{-a^2}

$$Ie^{-a^2} = a\int_0^\infty e^{-a^2(1+x^2)}dx$$

Integrating with respect to 'a' on both sides as a = 0 to ∞.

$$I\int_0^\infty e^{-a^2}da = \int_0^\infty a\left\{\int_0^\infty e^{-a^2(1+x^2)}dx\right\}da$$

$$= \int_0^\infty\left\{\int_0^\infty e^{-a^2(1+x^2)}\,a\,da\right\}dx$$

$$= \int_0^\infty \lim_{\varepsilon\to\infty}\left[-\frac{e^{-a^2(1+x^2)}}{2(1+x^2)}\right]_0^\varepsilon dx$$

$$= \int_0^\infty \lim_{\varepsilon\to\infty}\left[\frac{e^{-\varepsilon^2(1+x^2)}}{2(1+x^2)} + \frac{1}{2(1+x^2)}\right]dx$$

$$= \int_0^\infty\left[0 + \frac{1}{2(1+x^2)}\right]dx$$

or $\quad I.I = \int_0^\infty \frac{1}{2(1+x^2)}dx = \frac{1}{2}\left[\tan^{-1}x\right]_0^\infty = \frac{1}{2}\frac{\pi}{2}$

or $\quad I^2 = \frac{\pi}{4}$

$\therefore \quad I = \frac{\sqrt{\pi}}{2}$.

Ex. 10: The integral $\int_0^\infty \frac{\sin bx}{x}dx$

Solution:

Let $u = \int_0^\infty \frac{e^{-ax}\sin bx}{x}dx$

Differentiating with respect to b under the integral sign, we get

$$\frac{du}{db} = \int_0^\infty \frac{e^{-ax}\, x\cos bx}{x}dx = \int_0^\infty e^{-ax}\cos bx\, dx$$

$$\frac{du}{db} = \frac{a}{a^2+b^2},\ a>0.$$

Integrating,

$$u = a\int \frac{db}{a^2+b^2} = a.\frac{1}{a}\tan^{-1} b + c$$

$u = \tan^{-1}\frac{b}{a} + c$ where c is constant of integration,

When $b = 0$, $u = \int_0^\infty \frac{e^{ax}\sin bx}{x}dx = 0,\qquad \therefore\ c = 0$

Thus $u = \tan^{-1}\frac{b}{a}$

or $\int_0^\infty \frac{e^{ax}\sin bx}{x}dx = \tan^{-1}\frac{b}{a}$

Letting a = 0 on both sides, we get

$$\int_0^\infty \frac{\sin bx}{x}dx = \frac{\pi}{2}$$

Ex. 11: The integral $\int_0^\infty e^{-x}x^n dx$

Solution:

Let $I_n = \int_0^\infty e^{-x}x^n dx = \lim_{b\to\infty}\int_0^b e^{-x}x^n dx$

Integrating by parts

$$I_n = \lim_{b\to\infty}\left[e^{-x}x^n\right]_0^b + \lim_{b\to\infty} n\int_0^b e^{-x}x^{n-1}dx$$

or $I_n = \lim_{b\to\infty}\left[-\frac{b^n}{e^b}\right] + n\lim_{b\to\infty}\int_0^b e^{-x}x^{n-1}dx$

Now,

$$\lim_{b\to\infty}\left[\frac{b^n}{e^b}\right] = \lim_{b\to\infty}\frac{b^n}{e^b} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{b\to\infty}\frac{nb^{n-1}}{e^b} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{b\to\infty}\frac{n(n-1)b^{n-2}}{e^b} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

By continuing this process, we get

$$\lim_{b\to\infty}\frac{n(n-1)(n-2)\ldots\ldots 3.2.1}{e^b} = 0$$

So, $I_n = 0 + n\lim_{b\to\infty}\int_0^b e^{-x}x^{n-1}dx = \int_0^\infty e^{-x}x^{n-1}dx = n\,I_{n-1}$

$\therefore\ I_n = n\,I_{n-1}$

Exercise -11

Evaluate, if possible, the following improper integrals

1. $\int_{-\infty}^\infty \frac{dx}{1+x^2}$
2. $\int_0^\infty \frac{dx}{9+x^2}$
3. $\int_1^\infty \frac{\log x}{x^2}dx$
4. $\int_1^\infty \frac{x dx}{(1+x^2)^2}$
5. $\int_1^\infty \frac{1}{x^3}dx$
6. $\int_1^\infty \frac{1}{\sqrt{x}}dx$
7. $\int_{-1}^1 \frac{1}{x^2}dx$
8. $\int_0^\infty \frac{x\,dx}{x^2+4}$

Ex. 10: The integral $\int_0^\infty \frac{\sin bx}{x}dx$

Solution:

Let $u=\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx$

Differentiating with respect to b under the integral sign, we get

$$\frac{du}{db}=\int_0^\infty \frac{e^{-ax}\,x\cos bx}{x}dx=\int_0^\infty e^{-ax}\cos bx\,dx$$

$$\frac{du}{db}=\frac{a}{a^2+b^2},\ a>0,$$

Integrating,

$$u=a\int\frac{db}{a^2+b^2}=a.\frac{1}{a}\tan^{-1}b+c$$

$u=\tan^{-1}\frac{b}{a}+c$ where c is constant of integration,

When $b=0$, $u=\int_0^\infty \frac{e^{ax}\sin bx}{x}dx=0$, $\therefore\ c=0$

Thus $u=\tan^{-1}\frac{b}{a}$

or $\int_0^\infty \frac{e^{ax}\sin bx}{x}dx=\tan^{-1}\frac{b}{a}$

Letting $a=0$ on both sides, we get

$$\int_0^\infty \frac{\sin bx}{x}dx=\frac{\pi}{2}$$

Ex. 11: The integral $\int_0^\infty e^{-x}x^n dx$

Solution:

Let $I_n=\int_0^\infty e^{-x}x^n dx=\lim_{b\to\infty}\int_0^b e^{-x}x^n dx$

Integrating by parts

$$I_n=\lim_{b\to\infty}\left[e^{-x}x^n\right]_0^b+\lim_{b\to\infty}n\int_0^b e^{-x}x^{n-1}dx$$

or $$I_n=\lim_{b\to\infty}\left[-\frac{b^n}{e^b}\right]+n\lim_{b\to\infty}\int_0^b e^{-x}x^{n-1}dx$$

Now,

$$\lim_{b\to\infty}\left[\frac{b^n}{e^b}\right]=\lim_{b\to\infty}\frac{b^n}{e^b}\qquad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{b\to\infty}\frac{nb^{n-1}}{e^b}\qquad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{b\to\infty}\frac{n(n-1)b^{n-2}}{e^b}\qquad\left(\frac{\infty}{\infty}\text{ form}\right)$$

By continuing this process, we get

$$\lim_{b\to\infty}\frac{n(n-1)(n-2)\ldots\ldots3.2.1}{e^b}=0$$

So, $I_n=0+n\lim_{b\to\infty}\int_0^b e^{-x}x^{n-1}dx=\int_0^\infty e^{-x}x^{n-1}dx=n\,I_{n-1}$

$\therefore\ I_n=n\,I_{n-1}.$

Exercise -11

Evaluate, if possible, the following improper integrals

1. $\int_{-\infty}^\infty \frac{dx}{1+x^2}$
2. $\int_0^\infty \frac{dx}{9+x^2}$
3. $\int_1^\infty \frac{\log x}{x^2}dx$
4. $\int_1^\infty \frac{x\,dx}{(1+x^2)^2}$
5. $\int_1^\infty \frac{1}{x^3}dx$
6. $\int_1^\infty \frac{1}{\sqrt{x}}dx$
7. $\int_{-1}^1 \frac{1}{x^2}dx$
8. $\int_0^\infty \frac{x\,dx}{x^2+4}$

9. $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$ [2062 B.E.]

10. $\int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$ [2055/061 B.E.] 11. $\int_0^a \sqrt{\frac{a-x}{x}} dx$

12. $\int_0^2 \frac{dx}{(1-x)^2}$ 13. $\int_{-1}^{2} \frac{dx}{x^3} dx$ [2058 B.E.]

14. $\int_0^1 \sqrt{\frac{1+x}{1-x}} dx$ [2059 B.E.]

15. Show that $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2$

Answers

1. π 2. $\frac{\pi}{6}$ 3. 1

4. $\frac{1}{4}$ 5. $\frac{1}{2}$ 6. Does not exist

7. Does not exist, principal value = - 2 8. Does not exist,

9. $\frac{\pi}{2}$ 10. Does not exist. 11. $\frac{a\pi}{2}$

12. Does not exists. 13. does not exist, principal value = $\frac{3}{8}$

14. $\frac{\pi}{2} + 1$

✧✧✧

8.11 Differentiation under Integral sign

This is one of the strongest methods, which helps us to integrate the functions, which are not easily integrable. Firstly, the integrand of definite integral is differentiated with rerspect a quantity of which the limits of integration are independent .It can be calculated in the following ways:

Case I:

If we know acertain integral (or the value can be easily obtained), then on differentiating both sides with respect to a certain quantity, we get a new integral in left hand side and its value on the right hand side.

Case II:

If on differentiating the given integral with respect to a certain gauantity it takes the form which can be easily integrated ,then we integrate with respect to the quantity with which the original was differentiated.

If the value of the a definite integral $\int_a^b f(x,\alpha)\,dx$ is a function of α

i.e. $F(\alpha) = \int_a^b f(x,\alpha)\,dx$,then the differentiation under integral sign is the differentiation of $F(\alpha)$ i.e $F'(\alpha)$.

8.11.1 Leibnitz' rule

If $f(x,\alpha)$ and $\frac{\partial f(x,\alpha)}{\partial \alpha}$ be continuous function of x and α ,then

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha} dx$$

Proof:

Let $F(\alpha) = \int_a^b f(x,\alpha)\,dx$,then

$$F(\alpha + F(\alpha)\delta\alpha) = \int_a^b f(x, \alpha+\delta\alpha)\,dx$$

Hence $F(\alpha+\delta\alpha) - F(\alpha) = \int_a^b f(x, \alpha+\delta\alpha)dx - \int_a^b f(x,\alpha)\,dx$

$$= \int_a^b [f(x, \alpha+\delta\alpha) - f(x,\alpha)]\, dx$$

$$\frac{F(\alpha+\delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \frac{[f(x,\alpha+\delta\alpha) - f(x,\alpha)]}{\delta\alpha}\, dx$$

Taking the limit $\delta\alpha \to 0$ on both sides, we get

$$\lim_{\delta\alpha \to 0} \frac{F(\alpha+\delta\alpha) - F(\alpha)}{\delta\alpha} = \lim_{\delta\alpha \to 0} \int_a^b \frac{[f(x,\alpha+\delta\alpha) - f(x,\alpha)]}{\delta\alpha}\, dx$$

$$\therefore \frac{dF}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\, dx \text{ . Proved.}$$

8.11.2 Rule of differentiation under the integral sign when the limits of integration are functions of the parameter

The limits of the definite integral are any parameter, then the rule of differentitation under integral sign is stated below without proof.

If $f(x, \alpha)$ and $\frac{\partial f(x,\alpha)}{\partial \alpha}$ be continuous function of x and α, then

$$\frac{d}{d\alpha}\left[\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\, dx\right]$$

$$= \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\, dx - \frac{d\phi}{d\alpha} f[(\phi(\alpha),\alpha] + \frac{d\psi}{d\alpha} f[(\psi(\alpha),\alpha]$$

Worked Out Examples

Ex.1: Prove that $\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$

Solution:

We know $\int_0^\infty e^{-ax}\, dx = \left[\frac{e^{-ax}}{-a}\right]_0^\infty = -\frac{1}{a}(0-1) = \frac{1}{a}$

Differentiating both sides n times with respect to a under the sign of integration, we get

$$\int_0^\infty (-x)^n e^{-ax}\, dx = \frac{(-1)^n n!}{a^{n+1}}$$

$$\therefore \int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

Ex.2: **Find the value of** $\int_0^\pi \frac{dx}{a + b\cos x}$ **(where a> 0, |b| <a) and deduce that** $\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$

Solution:

Let $I = \int_0^\pi \frac{dx}{a + b\cos x}$

$$= \int_0^\pi \frac{dx}{a\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int_0^\pi \frac{dx}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}}$$

$$= \frac{1}{a-b}\int_0^\pi \frac{\sec^2\frac{x}{2}\, dx}{\left(\frac{a+b}{a-b}\right) + \tan^2\frac{x}{2}}$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\left[\tan^{-1}\left\{\tan\frac{x}{2}\sqrt{\frac{a-b}{a+b}}\right\}\right]_0^\pi$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\,(\tan^{-1}\infty - \tan^{-1}0)$$

$$= \frac{2}{\sqrt{a^2-b^2}}\,\frac{\pi}{2} = \frac{\pi}{\sqrt{a^2-b^2}}$$

$$I = \frac{\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

Now differentiating both sides with respect to "a", we get

$$\int_0^{\pi} (-1)(a + b\cos x)^{-2}\, dx = \frac{1}{2}\, \frac{2\pi a}{(a^2 - b^2)^{3/2}} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

Hence $\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = -\frac{\pi a}{(a^2 - b^2)^{3/2}}$.

Ex.3: Prfove that $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi(a^2 + b^2)}{4a^3b^3}$

Solution:

Here we have $I = \int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2} = \frac{1}{b}\cdot\frac{1}{a}\left[\tan^{-1}\frac{a\tan x}{b}\right]_0^{\pi/2}$$

$$= \frac{1}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2ab}$$

$$I = \frac{\pi}{2ab}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab} \quad ...(1)$$

Now differentiating both sides with respect to "a", we get

$$\int_0^{\pi/2} \frac{-2a\sin^2 x\, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{-\pi}{2a^2 b}$$

$$\int_0^{\pi/2} \frac{\sin^2 x\, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^2 b} \quad ...(2)$$

Again differentiating both sides of (1) with respect to "b", we get

$$\int_0^{\pi/2} \frac{\cos^2 x\, dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4ab^2} \quad ...(3)$$

Adding (2) and (3) and putting $\sin^2 x + \cos^2 x = 1$, we get

$$\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4a^2 b} + \frac{\pi}{4ab^2} = \frac{\pi(a^2 + b^2)}{4a^3b^3}.$$

Ex.4: Evaluate $\int_0^1 \frac{x^{\alpha} - 1}{\log x}\, dx$

Solution:

Let $I = \int_0^1 \frac{x^{\alpha} - 1}{\log x}\, dx \quad ...(1)$

Differentiating it with respect to α under the sign of integration, we get

$$\frac{dI}{d\alpha} = \int_0^1 \frac{x^{\alpha}\log x - 0}{\log x}\, dx = \int_0^1 x^{\alpha}\, dx$$

$$= \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$$

$$\therefore \frac{dI}{d\alpha} = \frac{1}{\alpha+1}$$

Integrating it both sides with respect to α, we get

$$I = \log(\alpha + 1) + c \quad ...(2)$$

From (1), when $\alpha = 0$, $I = 0$

Using $I = 0$ when $\alpha = 0$ in (2)

$$0 = \log(1 + 0) + c \quad \therefore c = 0$$

Hence (2) gives

$$I = \log(\alpha + 1).$$

Ex5: Evaluate $\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1 + x^2)}\, dx$

Solution:

Let $I = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1 + x^2)}\, dx \quad ...(1)$

Differentiating it with respect to a under the sign of integration, we get

$$\frac{dI}{da} = \int_0^{\infty} \frac{1}{x(1 + x^2)}\, \frac{x}{1 + a^2x^2}\, dx$$

$$=\int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\,dx$$

$$=\frac{1}{1-a^2}\int_0^\infty \left[\frac{1}{1+x^2}-\frac{a^2}{1+a^2x^2}\right]dx$$

$$=\frac{1}{1-a^2}\left[\tan^{-1}x - a\tan^{-1}(ax)\right]_0^\infty$$

$$=\frac{1}{1-a^2}\left[\frac{\pi}{2}-a\frac{\pi}{2}\right]=\frac{\pi}{2}\,\frac{1-a}{1-a^2}=\frac{\pi}{2}\,\frac{1}{1+a}$$

$$\therefore \frac{dI}{da}=\frac{\pi}{2}\,\frac{1}{1+a}$$

Integrating it both sides with respect to a , we get

$$I=\frac{\pi}{2}\log(a+1)+c \qquad ...(2)$$

From (1) ,when a =0 , I =0

Using I = 0 when a =0 in (2)

$$0=\frac{\pi}{2}\log(1+0)+c \qquad \therefore\ c=0$$

Hence (2) gives

$$I=\frac{\pi}{2}\log(a+1).$$

Ex6: Evaluate $\int_0^\infty \frac{e^{-x}\sin bx}{x}\,dx$

Solution:

$$\text{Let } I=\int_0^\infty \frac{e^{-x}\sin bx}{x}\,dx \qquad ..(1)$$

Differentiating it with respect to b under the sign of integration, we get

$$\frac{dI}{db}=\int_0^\infty \frac{e^{-x}}{x}\,x\cos bx\,dx=\int_0^\infty e^{-x}\cos bx\,dx$$

$$=\left[\frac{e^{-x}(-\cos bx+b\sin bx)}{1+b^2}\right]_0^\infty$$

$$=0+\frac{1}{1+b^2}=\frac{1}{1+b^2}$$

$$\therefore \frac{dI}{db}=\frac{1}{1+b^2}$$

Integrating it both sides with respect to b, we get

$$I=\tan^{-1}b+c \qquad ...(2)$$

From (1) ,when b =0 , I =0

Using I = 0 when b =0 in (2)

$$0=\frac{\pi}{2}\log(1+0)+c \qquad \therefore\ c=0$$

Hence (2) gives

$$I=\tan^{-1}b.$$

Ex.7: Evaluate $\int_0^\alpha \frac{\log(1+\alpha x)}{1+x^2}\,dx$ **and hence show that**

$$\int_0^1 \frac{\log(1+x)}{1+x^2}=\frac{\pi}{8}\log 2.$$

Solution:

$$\text{Let } I=\int_0^\alpha \frac{\log(1+\alpha x)}{1+x^2}\,dx \qquad ..(1)$$

Differentiating it with respect to b under the sign of integration, By using the formula

$$\frac{d}{d\alpha}\left(\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right)=\int_{\phi(\alpha)}^{\psi(\alpha)}\frac{\partial f(x,\alpha)}{\partial\alpha}dx-\frac{d\phi}{d\alpha}f[\phi(\alpha),\alpha]+\frac{d\psi}{d\alpha}f[\psi(\alpha),\alpha], \text{ we get}$$

$$\frac{dI}{d\alpha}=\int_0^\alpha \frac{\partial}{\partial\alpha}\left[\frac{\log(1+\alpha x)}{1+x^2}\right]dx-0+\frac{d\alpha}{d\alpha}f(\alpha,\alpha)$$

$$\frac{dI}{d\alpha}=\int_0^\alpha \frac{x}{(1+x^2)(1+\alpha x)}\,dx+\frac{\log(1+\alpha^2)}{1+\alpha^2}$$

Converting into partial fractions, we get

$$\frac{dI}{d\alpha} = -\frac{\alpha}{1+\alpha^2}\int_0^\alpha \frac{dx}{1+\alpha x} + \frac{1}{2(1+\alpha^2)}\int_0^\alpha \frac{2x\,dx}{1+x^2}$$

$$+\frac{\alpha}{1+\alpha^2}\int_0^\alpha \frac{dx}{1+x^2} + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

or $$I = -\frac{\alpha}{1+\alpha^2}\,\frac{1}{\alpha}\left[\log(1+\alpha x)\right]_0^\alpha + \frac{1}{2(1+\alpha^2)}\left[\log(1+x^2)\right]_0^\alpha$$

$$+\frac{\alpha}{1+\alpha^2}\left[\tan^{-1}x\right]_0^\alpha + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$= -\frac{\log(1+\alpha^2)}{1+\alpha^2} + \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$= \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha$$

$$\therefore \frac{dI}{d\alpha} = \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha$$

Integrating it both sides with respect to α, we get

$$I = \frac{1}{2}\log(1+\alpha^2).\tan^{-1}\alpha - \frac{1}{2}\int \frac{2\alpha}{1+\alpha^2}.\tan^{-1}\alpha\,d\alpha + \int \frac{\alpha}{1+\alpha^2}\tan^{-1}\alpha \ldots$$

$$I = \frac{1}{2}\log(1+\alpha^2).\tan^{-1}\alpha + c \quad \ldots(2)$$

From (1), when $\alpha = 0$, $I = 0$

Using $I = 0$ when $\alpha = 0$ in (2), we get

$$0 = 0 + c \qquad \therefore c = 0$$

Hence (2) gives

$$I = \frac{1}{2}\log(1+\alpha^2).\tan^{-1}\alpha$$

or $$\int_0^\alpha \frac{\log(1+\alpha x)}{1+x^2}\,dx = \frac{1}{2}\log(1+\alpha^2).\tan^{-1}\alpha$$

On putting $\alpha = 1$ in this equation, we have

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{1}{2}\log(1+1).\tan^{-1}(1) = \frac{1}{2}\log 2\cdot\frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$$

Exercise -12

Evaluate, by using the rule of differentiation under the sign of integration

1. $\int_0^\infty \frac{1-e^{-ax}}{x}\,e^{-x}\,dx$
2. $\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx$ also deduce that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$
3. $\int_0^\infty \frac{\cos\lambda x}{x}\,(e^{-ax} - e^{-bx})\,dx$
4. $\int_0^1 \frac{x^a - x^b}{\log x}\,dx$
5. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$
6. $\int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta$ ($a>0, b>0$)
7. $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx$
8. $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx$
9. $\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\,dx$
10. $\int_0^{\pi/2} \log\left(\frac{a+b\sin x}{a-b\sin x}\right).\frac{dx}{\sin x}$

Answers

1. $\log(1+a)$
2. $\cot^{-1}\alpha$
3. $\frac{1}{2}\log\frac{b^2+\lambda^2}{a^2+\lambda^2}$
4. $\log\frac{a+1}{b+1}$
5. $\log\frac{b}{a}$
6. $\pi\log(a+b)$
7. $\frac{\pi}{b}\log(a+b)$.
8. $-\pi\alpha$
9. $\pi\sin^{-1}a$
10. $\pi\sin^{-1}\frac{b}{a}$

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Chapter - 9

Reduction Formula and Beta and Gamma Functions

9.1 Reduction formula

The formula of the given integral which is connected with some integrals of lower order is called a *Reduction Formula*. It is the simplest form of the given integral. In most of the cases, the reduction formula is obtained by the method of integration by parts.

9.2 Some Important Formula

1. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

Let $I_n = \int x^n e^{ax}\,dx$

Integrating by parts

$$I_n = x^n\frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

$$= x^n\frac{e^{ax}}{a} - \frac{n}{a}I_{n-1}$$

$$\therefore\ I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a}I_{n-1}$$

2. $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n} I_{n-2}$ 2059 B.E.

Let $I_n = \int \sin^n x\,dx$ = sin^{n-1} x . sin x

$$I_n = \sin^{n-1}x(-\cos x) - (n-1)\int \sin^{n-2}x\cos x(-\cos x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$I_n = \sin^{n-1}x\cos x + (n-1)I_{n-1} - (n-1)I_n$$

$$(1+n-1)I_n = -\sin^{n-1}x\cos x + (n-1)I_n$$

$$\therefore\ I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{(n-1)}{n}I_{n-2}$$

For example, $I_6 = \int \sin^6 x\,dx$ [2059 B.E.]

By using formula

$$I_6 = -\frac{\sin^5 x\cos x}{6} + \frac{5}{6}I_4$$

$$= -\frac{\sin^5 x\cos x}{6} + \frac{5}{6}\left[-\frac{\sin^3 x\cos x}{4} + \frac{3}{4}I_2\right]$$

$$= -\frac{\sin^5 x\cos x}{6} - \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}I_2$$

$$= -\frac{\sin^5 x\cos x}{6} - \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}\left[-\frac{\sin x\cos x}{2} + \frac{1}{2}I_0\right]$$

But $I_0 = \int (\sin x)^0\,dx = \int dx = x$

$$\therefore \int \sin^6 x\,dx = -\frac{\sin^5 x\cos x}{6} - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x$$

3. $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}$

Let $I_n = \int \cos^n x\,dx$

$$= \int \cos^{n-1}x\cos x\,dx$$

Integrating by parts

$$I_n = \cos^{n-1}x(\sin x) - (n-1)\int \cos^{n-2}x\sin x(\sin x)\,dx$$

$$= \cos^{n-1}x\sin x - (n-1)\int \cos^{n-2}x\sin^2 x\,dx$$

$$= \cos^{n-1}x\sin x - (n-1)\int \cos^{n-2}x\,(1-\cos^2 x)\,dx$$

$$= \cos^{n-1}x\sin x - (n-1)\int \cos^{n-2}x\,dx + (n-1)\int \cos^n x\,dx$$

$$I_n = \cos^{n-1}x\sin x - (n-1)I_{n-1} + (n-1)I_n$$

$$(1+n-1)I_n = -\cos^{n-1}x\sin x + (n-1)I_n$$

$$\therefore\ I_n = -\frac{\cos^{n-1}\sin x}{n} + \frac{(n-1)}{n}I_{n-2}$$

4. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$

Let $I_n = \int \tan^n x\,dx$

$$= \int \tan^{n-2}x\tan^2 x\,dx$$

$$= \int \tan^{n-2}x\,(\sec^2 x - 1)\,dx$$

$$I_n = \int \tan^{n-2}x\sec^2 x\,dx - \int \tan^{n-2}x\,dx$$

$$= \frac{1}{n-1}\int d(\tan^{n-1}x) - I_{n-2}$$

$$\therefore\ I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

5. $\displaystyle\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - I_{n-2}$ [2061 B.E.]

Let $I_n = \int \cot^n x\,dx$

$$= \int \cot^{n-2}x\cot^2 x\,dx$$

$$= \int \cot^{n-2}x\,(\mathrm{cosec}^2 x - 1)\,dx$$

$$I_n = \int \cot^{n-2}x\,\mathrm{cosec}^2 x\,dx - \int \cot^{n-2}x\,dx$$

$$= -\frac{1}{n-1}\int d(\cot^{n-1}x) - I_{n-2}$$

$$\therefore\ I_n = -\frac{\cot^{n-1}x}{n-1} - I_{n-2}$$

For example,

$I_7 = \int \cot^7 x\,dx$ [2061 B.E.]

$$= -\frac{\cot^6 x}{6} - I_5 = -\frac{\cot^6 x}{6} - \left[-\frac{\cot^4 x}{4} - I_3\right]$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} + \left[-\frac{\cot^2 x}{2} - I_1\right]$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} + \int -\cot x \, dx$$

$$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \log \sin x$$

6. $\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

Let $I_n = \int \sec^n x \, dx$

$= \int \sec^{n-2} x \sec^2 x \, dx$

Integrating by parts

$I_n = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \tan x \, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$

or, $I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$

or, $I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

or, $(1 + n - 2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\therefore \quad I_n = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} I_{n-2}$

For example,

Let $I_5 = \int \sec^5 x \, dx$

Applying reduction formula,

$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} I_3$

$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4}\left[\frac{\sec x \tan x}{2} + \frac{1}{2} I_1\right]$

$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \int \sec x \, dx$

$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \int \sec x \, dx$

$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \log (\sec x + \tan x)$

7. $\int \operatorname{cosec}^n x \, dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

Let $I_n = \int \operatorname{cosec}^n x \, dx$

$= \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$

Integrating by parts

$I_n = -\operatorname{cosec}^{n-2} x \cot x - \int (n-2) \operatorname{cosec}^{n-3} x \operatorname{cosec} x (-\cot x) \cot x \, dx$

$= -\operatorname{cosec}^{n-2} x \cot x + (n-2) \int \operatorname{cosec}^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$

or, $I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) \int \operatorname{cosec}^n x \, dx - (n-2) \int \operatorname{cosec}^{n-2} x \, dx$

or, $I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_n - (n-2) I_{n-2}$

or, $(1 + n - 2) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

$\therefore \quad I_n = -\frac{\operatorname{cosec}^{n-2} x \cot x}{(n-1)} + \frac{(n-2)}{(n-1)} I_{n-2}$

For example,

Let $I_5 = \int \operatorname{cosec}^5 x \, dx$

Applying reduction formula

$I_5 = -\frac{1}{4} \operatorname{cosec}^3 x \cot x + \frac{3}{4} I_3$

$= -\frac{1}{4} \operatorname{cosec}^3 x \cot x + \frac{3}{4}\left[\frac{-\operatorname{cosec} x \cot x}{2} + \frac{1}{2} I_1\right]$

$= -\frac{1}{4} \operatorname{cosec}^3 x \cot x - \frac{3}{8} \operatorname{cosec} x \cot x + \frac{3}{8} \int \operatorname{cosec} x \, dx$

$= -\frac{1}{4} \operatorname{cosec}^3 x \cot x - \frac{3}{8} \operatorname{cosec} x \cot x + \frac{3}{8} \int \operatorname{cosec} x \, dx$

$= -\frac{1}{4} \operatorname{cosec}^3 x \cot x - \frac{3}{8} \operatorname{cosec} x \cot x + \frac{3}{8} \log (\operatorname{cosec} x - \cot x)$

8. $\int \cos^m x \cos nx \, dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1, n-1}$

Le $I_{m,n} = \int \cos^m x \cos nx\, dx$

Integrating by parts

$$I_{m,n} = \cos^m x \frac{\sin nx}{n} - \int m \cos^{m-1} x(-\sin x) \frac{\sin nx\, dx}{n} dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \ \sin x \sin nx\, dx$$

But, $\cos (nx - x) = \cos nx \cos x + \sin nx \sin x$

$\cos (n-1) x - \cos nx \cos x = \sin nx \sin x$

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\, [\cos(n-1)\, x - \cos nx \cos x]\, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos (n-1)\, x\, dx$$

$$- \frac{m}{n}\int \cos^m x \text{ con } nx\, dx$$

or, $I_{m,n} = \dfrac{\cos^m x \sin nx}{n} + \dfrac{m}{n} I_{m-1,n-1} - \dfrac{m}{n} I_{m,n}$

or, $\left(1 + \dfrac{m}{n}\right) I_{m,n} = \dfrac{\cos^m x \sin nx}{n} + \dfrac{m}{n} I_{m-1,n-1}$

$\therefore \quad I_{m,n} = \dfrac{\cos^m x \sin nx}{m+n} + \dfrac{m}{m+n} I_{m-1,n-1}$

For example,

Let $I_{3,5} = \int \cos^3 x \cos 5x\, dx$

$$= \frac{\cos^3 x \sin 5x}{3+5} + \frac{3}{3+5} I_{2,4}$$

$$= \frac{1}{8} \cos^3 x \sin 5x + \frac{3}{8}\left[\frac{\cos^2 x \sin 4x}{2+4} + \frac{2}{2+4} I_{1,3}\right]$$

$$= \frac{1}{8} \cos^3 x \sin 5x + \frac{3}{48} \cos^2 x \sin 4x + \frac{1}{8} I_{1,3}$$

$$= \frac{1}{8} \cos^3 x \sin 5x + \frac{3}{48} \cos^2 x \sin 4x$$

$$+ \frac{1}{8}\left[\frac{\cos x \sin 3x}{1+3} + \frac{1}{1+3} I_{0,2}\right]$$

$$= \frac{1}{8} \cos^3 x \sin 5x + \frac{1}{16} \cos^2 x \sin x\, 4x + \frac{1}{32} \cos x \sin 3x + \frac{1}{32}\int \cos 2x\, dx$$

$$= \frac{1}{8} \cos^3 x \sin 5x + \frac{1}{16} \cos^2 x \sin 4x + \frac{1}{32} \cos x \sin 3x + \frac{1}{64} \sin 2x$$

9. $\displaystyle\int \cos^m x \sin nx\, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$

Let $I_{m,n} = \displaystyle\int_0^{\pi/2} \cos^m x \sin nx\, dx$

Integrating by parts

$$I_{m,n} = \cos^m x \frac{(-\cos nx)}{n} - \int m \cos^{m-1} x\, (-\sin x) \frac{(-\cos nx)}{n} dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x (\cos nx \sin x)\, dx \quad \text{....(1)}$$

But, $\sin (nx - x) = \sin nx \cos x - \cos nx \sin x$

$\cos nx \sin x = \sin nx \cos x - \sin (nx - x)$

Substituting in (1)

$$I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x\, [\sin nx \cos x - \sin (nx - x)]\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx\, dx + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

or, $\left(1 + \dfrac{m}{n}\right) I_{m,n} = -\dfrac{\cos^m x \cos nx}{n} + \dfrac{m}{n} I_{m-1,n-1}$

or, $\dfrac{(n+m)}{n} I_{m,n} = -\dfrac{\cos^m x \cos nx}{n} + \dfrac{m}{n} I_{m-1,n-1}$

$\therefore \quad I_{m,n} = -\dfrac{\cos^m x \cos nx}{m+n} + \dfrac{m}{m+n} I_{m-1,n-1}$

10. $\displaystyle\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} J_{n-2}$

or,

$\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} J_{n-2}$

Let $J_n = \int_0^{\pi/2} \sin^n x \, dx$

$= \int_0^{\pi/2} \sin^{n-1} x . \sin x \, dx$

Integrating by parts .

$J_n = \left[\sin^{n-1} x \, (-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} (n-1) \sin^{n-2} x \cos x \, (-\cos x) \, dx$

$= 0 + (n-1) \int_0^{\pi/2} \sin^{n-2} x \, (1 - \sin^2 x) \, dx$

$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \, dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx$

$J_n = (n-1) J_{n-2} - (n-1) J_n$

$(1 + n - 1) J_n = (n-1) J_{n-2}$

$\therefore J_n = \frac{n-1}{n} J_{n-2}$

For example,

Let $J_6 = \int_0^{\pi/2} \sin^6 x \, dx$

$= \frac{6-1}{6} J_4 = \frac{5}{6} . \frac{(4-1)}{4} J_2$

$= \frac{5}{6} . \frac{3}{4} . \frac{(2-1)}{2} J_0 = \frac{5}{6} . \frac{3}{4} . \frac{1}{2} \int_0^{\pi/2} dx$

$= \frac{5}{16} [x]_0^{\pi/2} = \frac{5\pi}{32}.$

11. $I_{m,n} = \int_0^{\pi/2} \cos^m x \, \cos nx \, dx = \left[\frac{m(m-1)}{m^2 - n^2}\right] I_{m-2,n}$

Let $I_{m,n} = \int_0^{\pi/2} \cos^m x \, \cos nx \, dx$

Integrating by parts,

$I_{m,n} = \left[\frac{\cos^m x \sin nx}{n}\right]_0^{\pi/2} + \frac{m}{n} \int_0^{\pi/2} \cos^{m-1} x \, \sin x \, \sin nx \, dx$

$= 0 + \frac{m}{n} \int_0^{\pi/2} (\cos^{m-1} x \, \sin x) \sin nx \, dx$

Again integrating by parts,

$I_{m,n} = \frac{m}{n} \left[\cos^{m-1} x \sin x \left(-\frac{\cos nx}{n}\right)\right]_0^{\pi/2}$

$- \frac{m}{n} \int_0^{\pi/2} \left[(m-1) \cos^{m-2} x \, (-\sin x) \sin nx + \cos^{m-1} x \cos x\right] \left(-\frac{\cos nx}{n}\right) dx$

$= 0 - \frac{m(m-1)}{n^2} \int_0^{\pi/2} \left[\cos^{m-2} x \, (\sin^2 x) \cos nx\right] dx$

$+ \frac{m}{n^2} \int_0^{\pi/2} \cos^m x \cos nx \, dx$

$= 0 - \frac{m(m-1)}{n^2} \int_0^{\pi/2} \left[\cos^{m-2} x \, (1 - \cos^2 x) \cos nx\right] dx + \frac{m}{n^2} I_{m,n}$

or, $\left(1 - \frac{m}{n^2}\right) I_{m,n} = -\frac{m(m-1)}{n^2} \int_0^{\pi/2} \cos^{m-2} x \cos nx \, dx$

$+ \frac{m(m-1)}{n^2} \int_0^{\pi/2} \cos^m x \cos nx \, dx$

or, $\left(\frac{n^2 - m}{n^2}\right) I_{m,n} = -\frac{m(m-1)}{n^2} I_{m-2,n} + \frac{m(m-1)}{n^2} I_{m,n}$

or, $\left[\frac{n^2 - m}{n^2} - \frac{m(m-1)}{n^2}\right] I_{m,n} = -\frac{m(m-1)}{n^2} I_{m-2,n}$

or, $\left(\frac{n^2 - m - m^2 + m}{n^2}\right) I_{m,n} = -\frac{m(m-1)}{n^2} I_{m-2,n}$

or, $\left(\frac{n^2 - m^2}{n^2}\right) I_{m,n} = -\frac{m(m-1)}{n^2} I_{m-2,n}$

or, $I_{m,n} = -\frac{m(m-1)}{n^2 - m^2} I_{m-2,n}$

$\therefore I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$

Exercise-13

1. Obtain a reduction formula for $\int_0^{\pi/2} \cos^n x \, dx$ and hence evaluate $\int_0^{\pi/2} \cos^{10} x \, dx$ [2060/062 BE]

2. Obtain a reduction formula for $\int \sec^n x \, dx$ and hence find $\int \sec^6 x \, dx$

3. If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, show that $I_n + I_{n-2} = \frac{1}{n-1}$ and hence deduce the values of I_5.

4. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx \, dx$, prove that

$$I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

5. Prove that $\int_0^{\pi/2} \cos^5 x \sin 3x \, dx = \frac{1}{3}$

6. Prove that $\int_0^{\pi/2} \cos^m x \sin mx \, dx = \frac{1}{2^{m+1}}\left(2 + \frac{2^2}{2} + \frac{2^3}{3} + \dots + \frac{2^m}{m}\right)$

7. If m, n are positive integers, then show that

$$I_{m,n} = \int_a^b (x-a)^m (b-x)^n \, dx = \frac{n(b-a)}{m+n+1} I_{m,n-1}$$

8. Obtain a reduction formula for $\int \cos^m x \cos nx \, dx$ and hence show that $\int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}$

Answers

1. $\frac{63\pi}{512}$

2. $\frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$, $\frac{\sec^4 x \tan x}{5} + \frac{4 \sec^2 x \tan x}{15} + \frac{8}{15} \tan x$

3. $\frac{1}{2} \log 2 - \frac{1}{4}$

8. $\frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1, n-1}$

9.3 Beta and Gamma Functions

The integrals

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \; m > 0, n > 0 \text{ and}$$

$$\int_0^\infty e^{-x} x^{n-1} \, dx, \; n > 0$$

are called *First Eulerian Integral* (or Beta function) and *Second Eulerian Integral* (or Gamma function) respectively and denoted by $\beta(m, n)$ and $\Gamma(n)$, we write,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \; m > 0, n > 0$$

and $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$, $n > 0$ respectively.

9.4 Some Important Formulae

1. $\beta(m, n) = \beta(n, m)$

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

By using properties of definite integral

$$\beta(m,n) = \int_0^1 (1-x)^{m-1} [1-(1-x)]^{n-1} \, dx$$

$$\text{or, } \beta(m,n) = \int_0^1 x^{n-1} (1-x)^{m-1} \, dx = \beta(n, m)$$

$\therefore \beta(m, n) = \beta(n, m)$

2. $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\,dx = \int_0^\infty e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^\infty = -0 + 1 = 1$$

$\therefore \Gamma(1) = 1$

3. $\Gamma(n+1) = n\,\Gamma(n)$ **for n is any positive number.**

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1}\,dx = \int_0^\infty e^{-x} x^n\,dx$$

$$= -\left[x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\,dx$$

$$= 0 + n\Gamma(n) = n\Gamma(n) \qquad \left(\because \lim_{x\to\infty} \frac{x^n}{e^x} = 0\right)$$

$\therefore \Gamma(n+1) = n\,\Gamma(n)$

4. $\Gamma(n+1) = n!$ **for n is any positive integer.**

We have,

$\Gamma(n+1) = n\Gamma(n)$ for any positive number.

$= n(n-1)\,\Gamma(n-1)$

$= n(n-1)(n-2)\,\Gamma(n-2)$

By this process

$\Gamma(n+1) = n(n-1)(n-2) \ldots\ldots\ldots 2.1\Gamma(1)$

$= n(n-1)(n-2) \ldots\ldots\ldots\ldots 2.1.1 \qquad (\because \Gamma(1) = 1)$

$\therefore \Gamma(n+1) = n!$ for n is any positive integer.

5. $\displaystyle\int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$

Put $kx = t$, $kdx = dt$

When $x = 0$, $t = 0$, when $x = \infty$, $t = \infty$

So,

$$\int_0^\infty e^{-kx} x^{n-1}\,dx = \int_0^\infty e^{-t}\left(\frac{t}{k}\right)^{n-1}\frac{dt}{k}$$

$$= \frac{1}{k^n}\int_0^\infty e^{-t} t^{n-1}\,dt = \frac{1}{k^n}\Gamma(n)$$

$$\therefore \int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$$

6. $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Integrating by parts

$$\beta(m, n) = \left[(1-x)^{n-1}\frac{x^m}{m}\right]_0^1 + (n-1)\int_0^1 (1-x)^{n-2}\frac{x^m}{m}\,dx$$

$$= 0 + \frac{n-1}{m}\int_0^1 (1-x)^{n-2} x^m\,dx$$

Again integrating by parts, we get

$$\beta(m, n) = \frac{(n-1)}{m}\,\frac{(n-2)}{m+1}\int_0^1 (1-x)^{n-3} x^{m+1}\,dx$$

By this process, we get

$$\beta(m, n) = \frac{(n-1)(n-2)(n-3)\ldots\ldots 2.1}{m(m+1)(m+2)\ldots(m+n-2)}\int_0^1 (1-x)^{n-n} x^{m+n-2}\,dx$$

$$= \frac{(n-1)!}{m(m+1)(m+2)\ldots\ldots\ldots\ldots(m+n-2)}\left[\frac{x^{m+n-1}}{(m+n-1)}\right]_0^1$$

$$= \frac{(n-1)!\,1.2.3\ldots\ldots\ldots\ldots(m-2)(m-1)}{1.2.3\ldots\ldots\ldots(m-2)(m-1)\,m(m+1)(m+2)\ldots\ldots(m+n-2)(m+n-1)}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)!} = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(m+n)}$$

$$\therefore \beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

7. $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

We know $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Put $x = \frac{1}{1+y}$, $dx = -\frac{1}{(1+y)^2} dy$

When $x = 0, y = \infty$, when $x = 1, y = 0$

$$\therefore \beta(m, n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1 - \frac{1}{1+y}\right)^{n-1} \cdot \frac{-1}{(1+y)^2} dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m+1}} \frac{(y)^{n-1}}{(1+y)^{n-1}} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\therefore \beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Similarly if we write $\beta(m, n) = \beta(n, m)$,then

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\therefore \beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

8. $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}, 0 < m < 1$

$$\Gamma(m)\Gamma(1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(m+1-m)} = \beta(m, 1-m)$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+1-m}} dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)} dx = \frac{\pi}{\sin m\pi} \quad \left(\square \int_0^\infty \frac{x^{n-1}}{(1+x)} = \frac{\pi}{\sin n\pi}\right)$$

$$\therefore \Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

9. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

We know $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m, n)$.

Let $m = n = \frac{1}{2}$

So

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1} dx$$

$$= \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx$$

Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$, when $x = 1$, $\theta = \pi/2$

$$= \int_0^{\pi/2} \frac{1}{\sin\theta\cos\theta} . 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\,[\theta]_0^{\pi/2} = 2 . \frac{\pi}{2} = \pi$$

or $\Gamma\left(\frac{1}{2}\right) . \Gamma\left(\frac{1}{2}\right) = \pi$,

$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

10. $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}, p, q > -1$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \int_0^{\pi/2} (\sin^2\theta)^{p/2}(1 - \sin^2\theta)^{q/2} d\theta$$

Put $\sin^2\theta = x$, $2\sin\theta\cos\theta\, d\theta = dx$, $d\theta = \frac{dx}{x^{1/2}(1-x)^{1/2}}$

When $\theta = 0$, $x = 0$, when $\theta = \pi/2$, $x = 1$

So,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \int_0^1 x^{\frac{p}{2}} (1-x)^{\frac{q}{2}} \frac{dx}{2x^{1/2}(1-x)^{1/2}}$$

$$= \frac{1}{2}\int_0^1 x^{\frac{p}{2}-\frac{1}{2}} (1-x)^{\frac{q}{2}-\frac{1}{2}} dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{p+1}{2}-1} (1-x)^{\frac{q+1}{2}-1} dx$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

11. $$\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

If we put $q = 0$ in the above result, then we get

$$\int_0^{\pi/2} \sin^p \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{p+1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

12. $$\int_0^{\pi/2} \cos^q\theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)}$$

For,

If we put $p = 0$ in the above result then we get,

$$\int_0^{\pi/2} \sin^q d\theta = \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{q+2}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{q+2}{2}\right)}$$

$$\therefore \int_0^{\pi/2} \sin^q\theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)}$$

13. $$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

For,

Here, $\int_0^\infty e^{-x^2} dx$

Put $x^2 = t$, $2x dx = dt$, $dx = \frac{dt}{2t^{1/2}}$

When $x = 0$, $t = 0$, when $x = \infty$, $t = \infty$

So, $\int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{dt}{2t^{1/2}}$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt = \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$$

$$\therefore \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

Worked Out Examples

Ex. 1: Show that i. $\Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$ ii. $\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(6)} = \frac{\pi}{128}$

i. $\Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$

Solution:

Here, $\Gamma\left(\frac{9}{2}\right) = \frac{9}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$

$= \frac{105}{16}\sqrt{\pi}$

$\therefore \Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$

ii. $\dfrac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(6)} = \dfrac{\pi}{128}$

Solution:

Here $\dfrac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(6)} = \dfrac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(5+1)}$

$= \dfrac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2} \times \sqrt{\pi}}{5 \times 4 \times 3 \times 2 \times 1} = \dfrac{\pi}{128}$

$\therefore \dfrac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma 6} = \dfrac{\pi}{128}$

Ex. 2: Show that $\int_0^{\pi/2} \sin^3 x \cos^5 x \, dx = \frac{1}{24}$

Solution:

Here, $\int_0^{\pi/2} \sin^3 x \cos^5 x \, dx = \dfrac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{3+5+2}{2}\right)} = \dfrac{\Gamma(2)\,\Gamma(3)}{2\Gamma(5)}$

$= \dfrac{1 \times 2 \times 1}{2 \times 4 \times 3 \times 2 \times 1} = \dfrac{1}{24}$

$\therefore \int_0^{\pi/2} \sin^3 x \cos^5 x \, dx = \frac{1}{24}$

Ex. 3: Use Gamma function to evaluate $\int_0^1 x^6 \sqrt{1 - x^2} \, dx$

Solution:

Given integral is

$\int_0^1 x^6 \sqrt{1 - x^2} \, dx$

Put $x^2 = t$, $\quad 2x\,dx = dt$, $\quad dx = \frac{dt}{2\sqrt{t}}$

When, $x = 0$, $t = 0$, when $x = 1$, $t = 1$

$\therefore \int_0^1 x^6 \sqrt{1 - x^2} \, dx = \int_0^1 \dfrac{t^3 (1 - t)^{1/2} dt}{2\, t^{1/2}} = \frac{1}{2}\int_0^1 t^{5/2} (1 - t)^{1/2} dt$

$= \frac{1}{2}\int_0^1 t^{\frac{7}{2}-1} (1 - t)^{\frac{3}{2}-1} dt = \frac{1}{2}\beta\left(\frac{7}{2}, \frac{3}{2}\right)$

$= \frac{1}{2}\dfrac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}+\frac{3}{2}\right)} = \frac{1}{2}\dfrac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{10}{2}\right)}$

$= \dfrac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{2 \times 4 \times 3 \times 2 \times 1} = \frac{5}{256}\sqrt{\pi}.\sqrt{\pi} = \frac{5}{256}\pi$

$\therefore \int_0^1 x^6 \sqrt{1 - x^2} \, dx = \frac{5}{256}\pi.$

Ex. 4: Use Gamma function to prove: $\int_0^{\pi/8} \cos^3 4x \, dx = \frac{1}{6}$

Solution:

Given integral is

$\int_0^{\pi/8} \cos^3 4x \, dx$

Put, $4x = \theta$, $4\,dx = d\theta$,

When $x = 0$, $\theta = 0$, when $x = \pi/8$, $\theta = \pi/2$

$\therefore \int_0^{\pi/4} \cos^3 4x \, dx = \frac{1}{4}\int_0^{\pi/4} \cos^3 \theta \, d\theta = \frac{1}{4}\dfrac{\sqrt{\pi}\,\Gamma\left(\frac{3+1}{2}\right)}{2\Gamma\left(\frac{3+2}{2}\right)}$

$$= \frac{\sqrt{\pi}}{4} \frac{\Gamma(2)}{2\Gamma\left(\frac{5}{2}\right)} = \frac{\sqrt{\pi}}{4} \frac{1}{2 \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi}} = \frac{1}{6}$$

$$\therefore \int_0^{\pi/4} \cos^3 4x \, dx = \frac{1}{6}$$

Ex. 5: Prove that $\frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) = 1.3.5 \ldots\ldots (2n-3)(2n-1)$

Solution:

Given gamma function is

$$\frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \left(n + \frac{1}{2} - 1\right) \Gamma\left(n + \frac{1}{2} - 1\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \Gamma\left(n - \frac{3}{2}\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right) \Gamma\left(n - \frac{5}{2}\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right) \ldots\ldots \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \frac{(2n-1)}{2} \frac{(2n-3)}{2} \frac{(2n-5)}{2} \ldots\ldots\ldots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \frac{1.3.5 \ldots\ldots\ldots (2n-3)(2n-1)}{2^n} \sqrt{\pi}$$

$$= 1.3.5 \ldots\ldots\ldots\ldots (2n-3)(2n-1)$$

$$\therefore \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) = 1.3.5 \ldots\ldots\ldots\ldots (2n-3)(2n-1)$$

Ex. 6: Prove that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \ldots\ldots\ldots \Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$

Solution:

Here, $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \ldots\ldots\ldots \Gamma\left(\frac{8}{9}\right)$

$$= \Gamma\left(\frac{1}{9}\right).\Gamma\left(\frac{2}{9}\right).\Gamma\left(\frac{3}{9}\right).\Gamma\left(\frac{4}{9}\right).\Gamma\left(\frac{5}{9}\right).\Gamma\left(\frac{6}{9}\right).\Gamma\left(\frac{7}{9}\right).\Gamma\left(\frac{8}{9}\right)$$

$$= \Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right).\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{7}{9}\right).\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{6}{9}\right).\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)$$

$$= \Gamma\left(\frac{1}{9}\right)\Gamma\left(1 - \frac{1}{9}\right).\Gamma\left(\frac{2}{9}\right)\Gamma\left(1 - \frac{2}{9}\right).\Gamma\left(\frac{3}{9}\right)\Gamma\left(1 - \frac{3}{9}\right)\Gamma\left(\frac{4}{9}\right)\Gamma\left(1 - \frac{4}{9}\right)$$

$$= \frac{\pi}{\sin\frac{\pi}{9}} \cdot \frac{\pi}{\sin\frac{2\pi}{9}} \cdot \frac{\pi}{\sin\frac{3\pi}{9}} \cdot \frac{\pi}{\sin\frac{4\pi}{9}}$$

$$= \frac{\pi^4}{\sin 20° \sin 40° \sin 60° \sin 80°}$$

$$= \frac{4\pi^4}{\sqrt{3} \sin 20° \ (2\sin 40° \sin 80°)}$$

$$= \frac{4\pi^4}{\sqrt{3} \sin 20° \ [\cos(40° - 80°) - \cos(40° + 80°)]}$$

$$= \frac{1}{\sqrt{3} \sin 20°} \frac{4\pi^4}{(\cos 40° - \cos 120°)}$$

$$= \frac{1}{\sqrt{3} \sin 20°} \frac{4\pi^4}{\left(\cos 40° + \frac{1}{2}\right)} = \frac{4\pi^4}{\left(\sqrt{3} \sin 20° \cos 40° + \frac{\sqrt{3}}{2} \sin 20°\right)}$$

$$= \frac{8\pi^4}{\sqrt{3}\,(2\sin 20° \cos 40°) + \sqrt{3} \sin 20°}$$

$$= \frac{8\pi^4}{\sqrt{3}\,[\sin(20° + 40°) + \sin(20° - 40°)] + \sqrt{3} \sin 20°}$$

$$= \frac{8\pi^4}{\sqrt{3} \sin 60° - \sqrt{3} \sin 20° + \sqrt{3} \sin 20°} = \frac{8\pi^4}{\sqrt{3} \cdot \frac{\sqrt{3}}{2}} = \frac{16}{3}\pi^4$$

$$\therefore \Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \ldots\ldots\ldots \Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$$

Ex. 7: Use Gamma Function to prove that $\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{\pi}{3}$ [2057/068 B.E.]

Solution:

Given integral is

$$\int_0^1 \frac{dx}{(1-x^6)^{1/6}}$$

Put $x^6 = t$, $\quad 6x^5\,dx = dt$, $\quad dx = \dfrac{dt}{6t^{5/6}}$

When $x = 0$, $t = 0$, when, $x = 1$, $t = 1$

$$\text{So, } \int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{1}{6}\int_0^1 \frac{dt}{(1-t)^{1/6}\,t^{5/6}} = \frac{1}{6}\int_0^1 t^{-5/6}(1-t)^{-1/6}\,dt$$

$$= \frac{1}{6}\int_0^1 t^{\frac{1}{6}-1}(1-t)^{\frac{5}{6}-1}\,dt = \frac{1}{6}\,\beta\left(\frac{1}{6}, \frac{5}{6}\right)$$

$$= \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}+\frac{5}{6}\right)} = \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\,\frac{\pi}{\sin\frac{\pi}{6}} = \frac{1}{6}\cdot\frac{\pi}{\frac{1}{2}} = \frac{\pi}{3}$$

$$\therefore \int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{\pi}{3}.$$

Ex. 8: Prove $\displaystyle\int_0^\infty \sqrt{y}\, e^{-y^2}\,dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy = \frac{\pi}{2\sqrt{2}}$ [2061 B.S.]

Solution:

$$\text{Here, } \int_0^\infty \sqrt{y}\, e^{-y^2}\,dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy$$

Put $y^2 = t$, $\quad 2y\,dy = dt$, $\quad dy = \dfrac{dt}{2\,t^{1/2}}$

When $y = 0$, $t = 0$, when $y = \infty$, $t = \infty$

$$\text{So, } \int_0^\infty \sqrt{y}\, e^{-y^2}\,dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy$$

$$= \frac{1}{2}\int_0^\infty \frac{t^{1/4}e^{-t}}{t^{1/2}}\,dt \times \frac{1}{2}\int_0^\infty \frac{e^{-t}\,dt}{t^{1/4}\,.\,t^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{1}{4}-\frac{1}{2}}\,e^{-t}\,dt \times \frac{1}{2}\int_0^\infty e^{-t}\,t^{-\frac{1}{4}-\frac{1}{2}}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/4}\,dt \times \frac{1}{2}\int_0^\infty e^{-t}\,t^{-3/4}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{\frac{3}{4}-1}\,dt \times \frac{1}{2}\int_0^\infty e^{-t}\,t^{\frac{1}{4}-1}\,dt$$

$$= \frac{1}{2}\,\Gamma\left(\frac{3}{4}\right) \times \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right) = \frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{4}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{4.\frac{1}{\sqrt{2}}} = \frac{\pi}{2\sqrt{2}}$$

$$\therefore \int_0^\infty \sqrt{y}\, e^{-y^2}\,dy \,.\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy = \frac{\pi}{2\sqrt{2}}.$$

Ex. 9: Prove that $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)$

Solution:

We have, $\quad \beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ ....(1)

Put $m = n$, we get

$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \beta(n, n) = \int_0^1 (1-x)^{n-1}\,x^{n-1}\,dx$$

Put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$

So,

$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \int_0^{\pi/2} (1-\sin^2\theta)^{n-1}(\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2n-2}\theta\,\sin^{2n-2}\theta\,\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2n-1}\theta\,\sin^{2n-1}\theta\,d\theta$$

$$= \frac{2}{2^{2n-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1}\,d\theta$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} (\sin 2\theta)^{2n-1}\, d\theta$$

Put $2\theta = t$, $d\theta = \frac{dt}{2}$

or, $\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \frac{2}{2^{2n-1}} \int_0^{\pi} (\sin t)^{2n-1} \frac{dt}{2} = \frac{1}{2^{2n-1}} \int_0^{\pi} (\sin t)^{2n-1} dt$

$\therefore \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{2n-1}} . 2\int_0^{\pi/2} \sin^{2n-1} t\, dt$ ....(2)

Again, put $m = \frac{1}{2}$ in (1), we get

$$\beta\left(\frac{1}{2}, n\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)}$$

or, $\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} = \beta\left(\frac{1}{2}, n\right) = \int_0^1 (1-x)^{\frac{1}{2}-1} x^{n-1} dx$

$$= \int_0^1 (1-x)^{\frac{-1}{2}} x^{n-1} dx$$

Put $x = \sin^2\theta$, $dx = 2\sin\theta \cos\theta\, d\theta$

So, $\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} = \int_0^{\pi/2} (1 - \sin^2\theta)^{-1/2} (\sin^2\theta)^{n-1}\, 2\sin\theta \cos\theta\, d\theta$

or, $\frac{\sqrt{\pi}\,\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} = 2\int_0^{\pi/2} \sin^{2n-1}\theta\, d\theta$

$\therefore \frac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma\left(n+\frac{1}{2}\right)} = \int_0^{\pi/2} \sin^{2n-1} t\, dt$ ....(3)

From (2) and (3),

$$\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{2n-1}} . 2\, \frac{\sqrt{\pi}\,\Gamma(n)}{2\Gamma\left(n+\frac{1}{2}\right)}$$

$$\therefore \sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)$$

This formula is known as *Duplication Formula.*

Exercise-14

1. Find the value of i. $\Gamma\left(\frac{7}{2}\right)$ ii. $\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)}$
2. Evaluate i. $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$
   ii. $\int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta\, d\theta$
3. Use Beta Gamma function to evaluate
   i. $\int_0^a x^2 (a^2 - x^2)^{3/2} dx$ ii. $\int_0^1 x^{3/2} (1-x)^{1/2} dx$
   iii. $\int_0^{2a} x^5 \sqrt{2ax - x^2}\, dx$ iv. $\int_0^{2a} x^{9/2} (2a - x)^{-1/2} dx$
   v. $\int_0^a x^4 \sqrt{a^2 - x^2}\, dx$ vi. $\int_0^1 \frac{x^6 . dx}{\sqrt{1-x^2}}$
   vii. $\int_0^a \frac{x^4 . dx}{\sqrt{a^2 - x^2}}$ viii. $\int_0^a x^3 (a^2 - x^2)^{5/2} dx$
5. Using Beta Gamma Function, show that
   i. $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta\, d\theta = \frac{5\pi}{192}$ [2058/059 B.E.]
   ii. $\int_0^{\pi/6} \cos^2 6\theta \sin^4 3\theta\, d\theta = \frac{7\pi}{192}$
   iii. $\int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2}\, dx = \frac{5\pi}{2^{11}}$

iv. $\int_0^{\pi/4} \sin^4 x \cos^2 x \, dx = \frac{3\pi - 4}{192}$

6. Prove that $\int_0^{\infty} x^2 e^{-x^4} dx \times \int_0^{\infty} e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$

7. Prove that $\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$

8. Show that $\beta(m, n)\, \beta(m + n, l) = \beta(n, l)\, \beta(n + l, m)$

9. Show that $\int_0^{\infty} e^{-x^2} x^{\alpha} dx = \frac{1}{2} \Gamma\left(\frac{\alpha + 1}{2}\right)$

10. Prove that

$$\int_a^b (x - a)^m (b - x)^n dx = (b - a)^{m+n+1} \frac{\Gamma(m + 1)\, \Gamma(n + 1)}{\Gamma(m + n + 2)}, \ m > -1, n > -1$$

[Hint: Put $x - a = (b - a)t$]

Answers

1. i. $\frac{15}{8}\sqrt{\pi}$ ii. $\frac{\pi}{64}$

2. i. $\frac{\pi}{32}$ ii. $\frac{3\pi}{16\sqrt{2}}$

3. i. $\frac{\pi a^6}{32}$ ii. $\frac{3\pi}{128}$ iii. $\frac{33\pi a^7}{16}$ iv. $\frac{63\pi a^5}{8}$

v. $\frac{\pi a^6}{32}$ (vi) $\frac{5\pi}{32}$ vii. $\frac{3\pi a^4}{16}$ viii. $\frac{2a^9}{63}$

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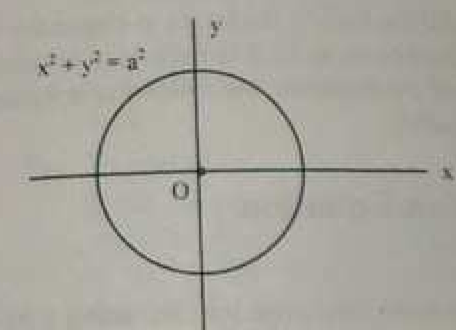
Chapter - 10

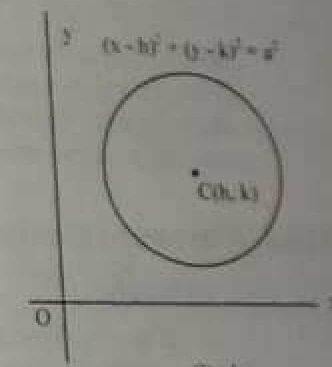
Tracing of Curves

10.1 Introduction

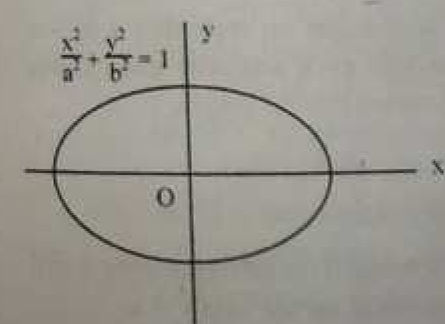
We are familiar with curves of the equations such as circle, parabola, ellipse and hyperbola which are shown in the following figures.



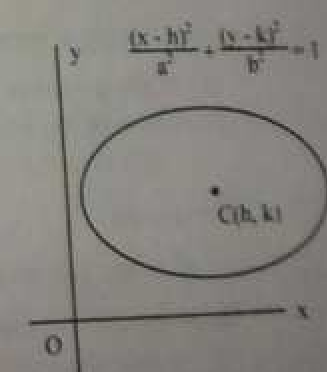
Circle



Circle



Ellipse



Ellipse

Exercise

...ace the following curves

$-x^2y^2=a^2$

$6y+9=4x-4$

$...+\frac{y^{2/3}}{b^{2/3}}=1$

$...^2+y^2)=y^2(a^2-y^2)$

...cosθ

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Chapter - 11

Quadrature

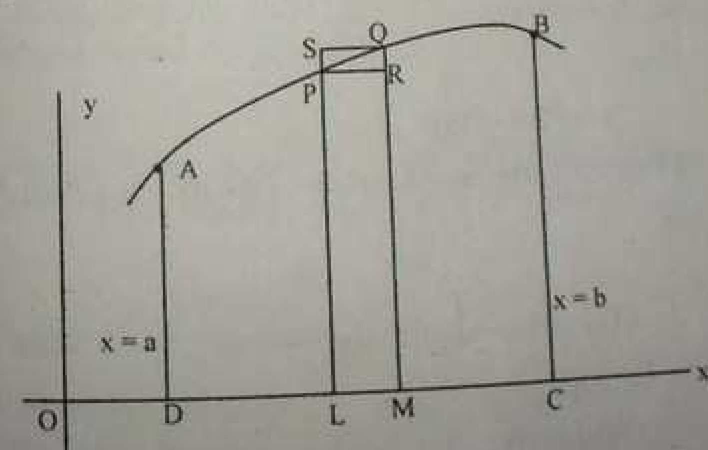
11.1 Quadrature

Quadrature is the method for finding out the area bounded by any portion of a plane curve. The definite integral $\int_a^b f(x)\,dx = \int_a^b y\,dx$ represents the area bounded by the curve $y = f(x)$, the x-axis and the two fixed ordinates $x = a$ and $x = b$.

11.2 Areas in Cartesian Coordinates

If $y = f(x)$ be single valued continuous functions in the interval (a,b), then the area bounded by the curve $y = f(x)$ the x-axis and the ordinates $x = a$ an $dx = b$ is defined by $\int_a^b f(x)\,dx$

Let the curve AB represents the equation $y = f(x)$ and ABCD represents the area bounded by the curve, the x-axis and ordinates $x = a$ and $x = b$.



Take $P(x, y)$ be any point on the curve such that the variable area ADLP = A, which is bounded by the curve $y = f(x)$, the x-axis, the fixed ordinates AD where OD = a and variable ordinates PL where OL = x.

Take a point $Q(x + \delta x, y + \delta y)$ near to $P(x, y)$ so that δx and δy are increments of x and y, δA be the increment area such that

δA = the area LMQP

From the figure, we see that

Area of rect. LMRP < Area LMQP < Area of rect. LMQS

or, $y\delta x < \delta A < (y + \delta y)\,\delta x$

or, $y < \dfrac{\delta A}{\delta x} < (y + \delta y)$

When $Q \to P$, $\delta x \to 0$, $\delta y \to 0$, $\dfrac{\delta A}{\delta x} \to \dfrac{dA}{dx}$

So, $y = \lim\limits_{\delta x \to 0} \dfrac{\delta A}{\delta x} = y$

or, $\dfrac{dA}{dx} = y = f(x)$

or, $A = \int dA = \int f(x)\,dx$

$A = F(x) + c$

But A = 0 when x = a

Therefore,

$0 = F(a) + c,$

or, $c = -F(a).$

Thus,

$A = F(x) - F(a)$

When x = b, the area A becomes the required Area ADCB

So,

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

$$\therefore\ A = \int_a^b y\,dx$$

Note :

In the some manner, it can be shown that the area bounded by the curve x = f(y), the y-axis and the lines y = c, y = d is

$$A = \int_c^d f(y)\,dy = \int_c^d x\,dy.$$

11.3 Area Enclosed by Two Curves

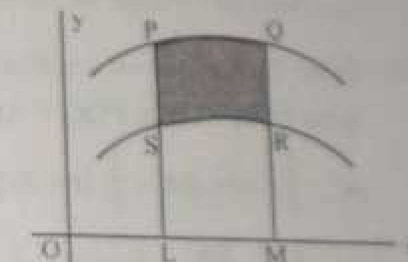
Let the area bounded by two curves $y = f_1(x)$ and $y = f_2(x)$ and the two given ordinates x = a and x = b is PQRS where

OL = a, OM = b, then the required area PQRS is given by

Area PQRS = Area PLMQ - Area SLMR

$$= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

$$= \int_a^b [f_2(x) - f_1(x)]\,dx$$



Note 1

If an area comes out to be negative then we ignore the negative sign and take into consideration only the numerical value.

Note 2

If the curve is symmetrical then we find the area of one symmetrical portion and multiply it by the number of symmetrical parts.

Note 3

If an area is divided into one or more parts then we find their area separately and add their numerical values.

Note 4

While choosing the values of x for limits of integration, we should take from left to right whereas the values of y from depth to height.

11.4 Area in Polar Coordinates

Let $r = f(\theta)$ be single valued continuous function defined in the interval $\alpha \le \theta \le \beta$ then the area bounded by the curve and the radii $\theta = \alpha$ and $\theta = \beta$ is given by $\displaystyle\int_\alpha^\beta \frac{1}{2} r^2\,d\theta$

Let $r = f(\theta)$ be a curve APB where $\angle AOX = \alpha$, $\angle BOX = \beta$.

A denote the area AOP, if we take a point $Q(r + \delta r, \theta + \delta\theta)$ be new point to $P(r, \theta)$ then

Area AOQ = A + δA.

Area POQ = A + δA - A = δA

Draw the arcs PM and QM of circles with centre O

So, Area of circular sector PON $= \frac{1}{2} r^2 \delta\theta$

Area of circular sector MOQ $= \frac{1}{2} (r + \delta r)^2 \delta\theta$

From the figure, we see that

Area PON < Area POQ < Area MOQ

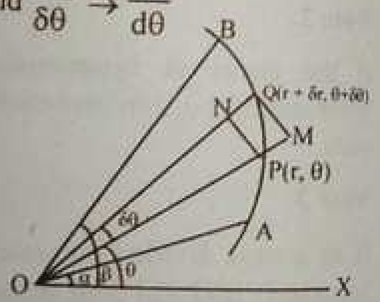
or, $\frac{1}{2} r^2 \delta\theta < \delta A < \frac{1}{2} (r + \delta r)^2 \delta\theta$

or, $\frac{1}{2} r^2 < \frac{\delta A}{\delta\theta} < \frac{1}{2} (r + \delta r)^2$

When $Q \to P$, $\delta\theta \to 0$, $\delta r \to 0$ and $\frac{\delta A}{\delta\theta} \to \frac{dA}{d\theta}$

So, $\frac{1}{2} r^2 = \lim_{\delta\theta \to 0} \frac{\delta A}{\delta\theta} = \frac{1}{2} r^2$

$\therefore \frac{dA}{d\theta} = \frac{1}{2} r^2$



$$\int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_\alpha^\beta \frac{dA}{d\theta} d\theta = [A]_{\theta=\alpha}^{\theta=\beta}$$

= (Value of A when θ = β) - (Value of A when θ = α)

= Area of sector AOB - 0 = Area of sector AOB

$$\therefore \text{ Area of sector AOB} = \frac{1}{2}\int_\alpha^\beta r^2 d\theta = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta$$

Note

The bounded by two curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ and two given radii vectors $\theta = \alpha$ and $\theta = \beta$ is $\frac{1}{2}\int_\alpha^\beta (r_2^2 - r_1^2) d\theta$

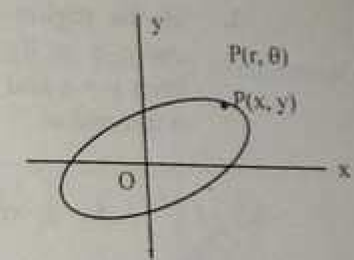
11.5 Area of Closed Curves

Let $x = f(t)$ and $y = g(t)$ be the equation of the closed curves in parametric form., then the area bounded by the curve and $t = t_1$ to $t = t_2$ is defined by

$$A = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt.$$

Let P (r, θ) be the polar coordinates of a point P(x, y) on the curve, then we have

$x = 4\cos\theta$, $y = r\sin\theta$



Differentiating it with respect to t,

$$\frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt}$$

and $$\frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dr}{dt}$$

Now, $x\frac{dy}{dt} - y\frac{dx}{dt}$

$$= r\cos\theta\left(r\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{dr}{dt}\right) - r\sin\theta\left(-r\sin\theta\frac{d\theta}{dt} + \cos\theta\frac{dr}{dt}\right)$$

$$= r^2\cos^2\theta\frac{d\theta}{dt} + r\sin\theta\cos\theta\frac{dr}{dt} + r^2\sin^2\theta\frac{d\theta}{dt} - r\sin\theta\cos\theta\frac{dr}{dt}$$

$$= r^2(\cos^2\theta + \sin^2\theta)\frac{d\theta}{dt} = r^2\frac{d\theta}{dt}$$

$$r^2\frac{d\theta}{dt} = x\frac{dy}{dt} - y\frac{dx}{dt}$$

We know that the area of closed curve

$$= \frac{1}{2}\int r^2 d\theta$$

$$= \frac{1}{2}\int\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt = \frac{1}{2}\int r^2\frac{d\theta}{dt} dt$$

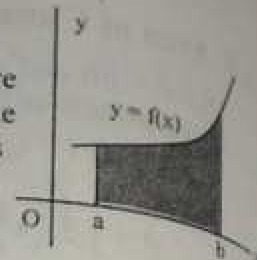
Taking the limits of t from $t = t_1$ to $t = t_2$.

$$\therefore A = \frac{1}{2}\int_{t_1}^{t_2}(x\,dy - y\,dx)$$

11.6 Some Important Formulae

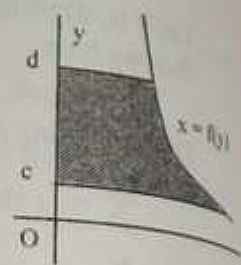
1. If the region bounded by the curve $y = f(x)$, x-axis with two ordinate $x=a$ and $x = b$, then area is defined as

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$



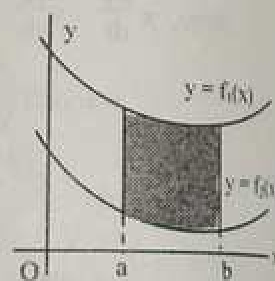
2. If the region is bounded by the curve $x = f(y)$, y-axis, with two lines $y = c$ and $y = d$, then the area is defined as

$$A = \int_c^d x\,dy = \int_c^d f(y)\,dy$$



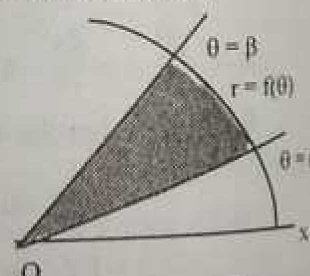
3. If the region is bounded by the curves $y = f_1(x)$ and $y = f_2(x)$, x-axis with two ordinates $x = a$ to $x = b$, then the area is defined as

$$A = \int_a^b [f_1(x) - f_2(x)]\,dx$$

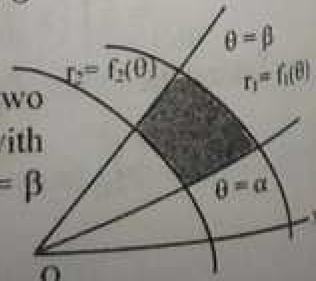


4. If the region is bounded by the curve $r = f(\theta)$, the two radii vectors $\theta = \alpha$ to $\theta = \beta$, then the area is defined as

$$A = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta$$



5. If the region is bounded by the two curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ with the two radii vectors $\theta = \alpha$ to $\theta = \beta$ then the area is defined as



$$A = \frac{1}{2}\int_\alpha^\beta (r_2^2 - r_1^2)\,d\theta$$

Worked Out Examples

Ex. 1: Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum

Solution:

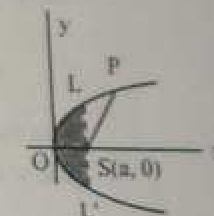
Equation of parabola $y^2 = 4ax$

The parabola bounded by its latus rectum into two symmetrical parts by the x-axis.

Required area $= 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$

$$= 2\int_0^a \sqrt{4a}\,x^{1/2}\,dx$$

$$= 2\sqrt{4a}\left[\frac{2x^{3/2}}{3}\right]_0^a = \frac{8a^2}{3} \text{ square unit.}$$



Ex. 2: Show that the area bounded by semi-cubical parabola and a double ordinate $y^2 = ax^3$ of the is 2/5 of the area of the rectangle formed by the ordinate and its distance from the vertex.

Solution:

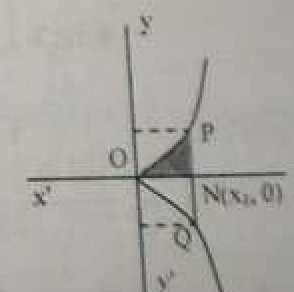
Here, $y^2 = ax^3$

The area OPN is bounded by the curve $y^2 = ax^3$, the x-axis and two ordinates $x = 0$ and $x = x_1$

So, the area

$$OPN = \int_0^{x_1} y\,dx = \int_0^{x_1} \sqrt{ax^3}\,dx$$

$$= \sqrt{a}\left[\frac{2x^{5/2}}{5}\right]_0^{x_1} = \frac{2\sqrt{a}}{5}\,x_1^{5/2}$$



$$\text{Area, } OPQ = \frac{2.2\sqrt{a}}{5}\,x_1^{5/2} = \frac{4\sqrt{a}}{5}\,x_1^{5/2} \quad \text{..........(1)}$$

Area of rectangle formed by this ordinate and its distance from the vertex is

$$PQ.\,ON. = 2y_1.x_1$$

$= 2\sqrt{ax_1^3}\; x_1 = 2\sqrt{a}\; x_1^{5/2}$ ......(2)

From (1) and (2),

Area OPQ $= \frac{2}{5}\left(\frac{2\sqrt{a}}{5}x_1^{5/2}\right) = \frac{2}{5}$ (Area of the rectangle)

Ex. 3: Show that area of the hypocycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ is $\frac{3}{8}\pi ab$

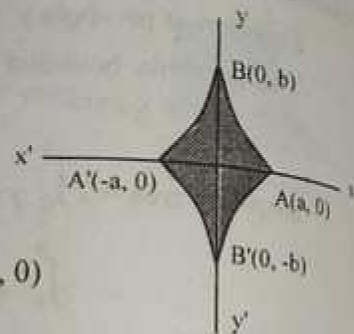
Solution:

The equation of hypo-cycloid is

$$\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}} = 1$$

or, $y = b\left(1 - \frac{x^{2/3}}{a^{2/3}}\right)^{3/2}$,

$$y = \frac{b}{a}(a^{2/3} - x^{2/3})^{3/2}$$



The curve meets x-axis at $(\pm a, 0)$ and y - axis at $(0, \pm b)$

Required area = 4 area of OAB $= 4\int_0^a y\,dx$

$$= 4\int_0^a b\left(1 - \frac{x^{2/3}}{a^{2/3}}\right)^{3/2} dx = \frac{4b}{a}\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx$$

Put $x = a\sin^3\theta$, $dx = 3a\sin^2\theta\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$, when $x = a$, $\theta = \frac{\pi}{2}$

$$= 4.3a.\,b\int_0^{\pi/2} \cos^4\theta \sin^2\theta\, d\theta = 4.3a.b\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\,\Gamma\left(\frac{4+2+2}{2}\right)}$$

$$= \frac{4.3a\,b.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2.3.2.1} = \frac{3\pi}{8}\,ab$$

$\therefore$ The required area $= \frac{3\pi}{8}$ ab square unit.

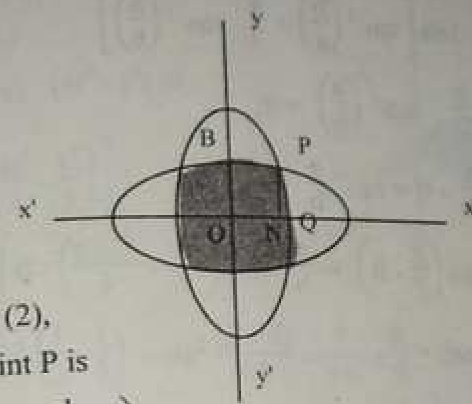
Ex. 4: Show that the area common to the two ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, a>b, is $2ab\tan^{-1}\frac{2ab}{a^2 - b^2}$.

Solution:

Two curves are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ....(1)$$

and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ ....(2)



Solving (1) and (2), the common point P is

$$\left(\frac{ab}{\sqrt{a^2+b^2}}, \frac{ab}{\sqrt{a^2+b^2}}\right)$$

Also the point Q is (b, 0)

Required area = 4 area OQPB

= 4[area ONPB + area NQPN]

$$= 4\left[\int_0^{ab/\sqrt{a^2+b^2}} \{y \text{ from } (1)\}dx + \int_{ab/\sqrt{a^2+b^2}}^{b} \{y \text{ from } (2)\}dx\right]$$

$$= 4\int_0^{ab/\sqrt{a^2+b^2}} \frac{b}{a}\sqrt{a^2 - x^2}\,dx + 4\int_{ab/\sqrt{a^2+b^2}}^{b} \left(\frac{a}{b}\sqrt{b^2 - x^2}\right)dx$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{ab/\sqrt{a^2+b^2}} + \frac{4a}{b}\left[\frac{x\sqrt{b^2 - x^2}}{2} + \frac{b^2}{2}\sin^{-1}\frac{x}{b}\right]_{ab/\sqrt{a^2+b^2}}^{b}$$

$$= \frac{4b}{a}\left[\frac{ab}{2\sqrt{a^2+b^2}}\sqrt{a^2 - \frac{a^2b^2}{a^2+b^2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right]$$

$$+ \frac{4a}{b}\left[0 + \frac{b^2}{2}.\frac{\pi}{2} - \frac{ab}{2\sqrt{a^2+b^2}}\sqrt{b^2 - \frac{a^2b^2}{a^2+b^2}} - \frac{b^2}{2}\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right]$$

$$= \frac{2a^2b^2}{a^2+b^2} + 2ab\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) + \pi ab - \frac{2a^2b^2}{a^2+b^2} - 2ab\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right)$$

area of a loop

$(a^2 - x^2)$ is

of the curve is

$= x^2 (a^2 - x^2)$

for $x = 0$,

symmetrical

area of a loop

$$= -2a^2\int_{\pi/4}^{0}\frac{\sin\theta}{\cos\theta}\,2\sin\theta\cos\theta\,d\theta = 4a^2\int_0^{\pi/4}\sin^2\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/4}\left(\frac{1-\cos2\theta}{2}\right)d\theta = 2a^2\left[\theta-\frac{\sin2\theta}{2}\right]_0^{\pi/4}$$

$$= 2a^2\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{a^2}{2}(\pi-2)$$

∴ The required area = $\frac{a^2}{2}(\pi-2)$ square unit.

Ex. 7: Find the area above the x-axis included between the parabola $y^2 = ax$ and the circle $x^2+y^2=2ax$

Solution:

The equation of circle $x^2+y^2=2ax$ ....(1)

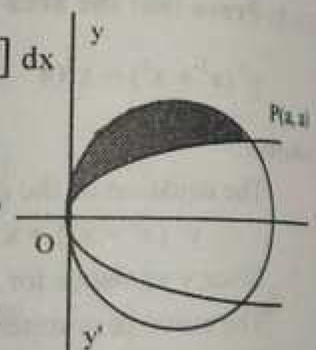
and the parabola $y^2 = ax$ . ...(2)

Common point of the curves is $x = 0$ and $x = a$.

The required area

$$= \int_0^a [y \text{ from (1)}]\,dx - \int_0^a [y \text{ from (2)}]\,dx$$

$$= \int_0^a \sqrt{2ax-x^2}\,dx - \int_0^a \sqrt{ax}\,dx$$



Put $x = 2a\sin^2\theta$ in the first integral,

$dx = 4a\sin\cos\theta d\theta$

When $x = 0$, $\theta = 0$, when $x = a$, $\theta = \frac{\pi}{4}$

$$=\int_0^{\pi/4}\sqrt{2a.2a\sin^2\theta-4a^2\sin^2\theta}\;.4a\sin\theta\cos\theta\,d\theta-\sqrt{a}\left[\frac{2x^{3/2}}{3}\right]_0^a$$

$$=\int_0^{\pi/4}2a\sin\theta\cos\theta.\;4a\sin\theta\cos\theta\,d\theta-\sqrt{a}\left[\frac{2x^{3/2}}{3}\right]_0^a$$

$$=8a^2\int_0^{\pi/4}\sin^2\theta\cos^2\theta\,d\theta-\sqrt{a}\left[\frac{2a^{3/2}}{3}-0\right]$$

$$=2a^2\int_0^{\pi/4}4\sin^2\theta\cos^2\theta\,d\theta-\frac{2a^2}{3}$$

$$=2a^2\int_0^{\pi/4}(2\sin\theta\cos\theta)^2\,d\theta-\frac{2a^2}{3}$$

$$=a^2\int_0^{\pi/4}2\sin^2 2\theta\,d\theta-\frac{2a^2}{3}=a^2\int_0^{\pi/4}(1-\cos4\theta)\,d\theta-\frac{2a^2}{3}$$

$$=a^2\left[\theta-\frac{1}{4}\sin4\theta\right]_0^{\pi/4}-\frac{2a^2}{3}=a^2\left[\frac{\pi}{4}-0\right]-\frac{2a^2}{3}=a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$$

∴ The required area = $a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$ square unit.

Ex. 8: Find the area of two loops of the curve $a^2y^2 = a^2x^2 - x^4$ [2061 B.E.]

Solution:

Here, the equation of the curve is

$a^2y^2 = a^2x^2 - x^4$

Since y vanishes to $x = 0$- and $x = \pm a$, the curve is symmetrical on both axes, so two loops are formed.

So the required area,

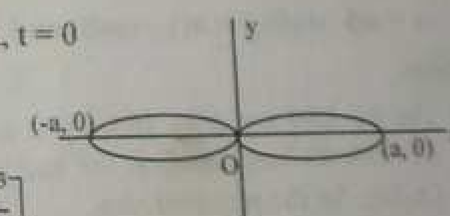
$$A = 4\int_0^a y\,dx = 4\int_0^a \frac{x}{a}\sqrt{a^2-x^2}\,dx$$

Put $a^2 - x^2 = t^2$

$-x\,dx = t\,dt$

When $x = 0$, $t = a$, when $x = a$, $t = 0$

$$=\frac{4}{a}\int_a^0 t(-t\,dt)$$



$$=-\frac{4}{a}\left[\frac{t^3}{3}\right]_a^0=-\left[0-\frac{a^3}{3}\right]$$

$$=\frac{4}{a}\cdot\frac{a^3}{3}=\frac{4a^2}{3}$$

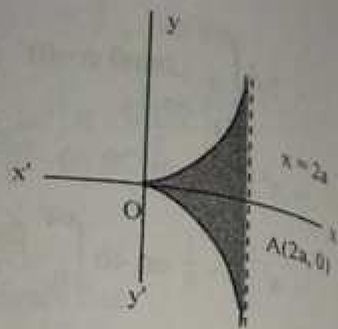
∴ Area = $\frac{4a^2}{3}$ square unit.

Ex. 9: Find the area of the curve $y^2(2a-x) = x^3$ and its asymptotes

Solution:

The curve $y^2(2a-x) = x^3$

It is symmetrical on x-axis and passes through the origin. The equation of it asymptote $x = 2a$. So the required area.



$$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} x\sqrt{\frac{x}{2a-x}}\,dx$$

Put $x = 2a\sin^2\theta$,

$dx = 4a\sin\theta\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$, when $x = 2a$, $\theta = \pi/2$

So,

$$A = 2\int_0^{\pi/2} 2a\sin^2\theta\sqrt{\frac{2a\sin^2\theta}{2a(1-\sin^2\theta)}}\;4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta = 16a^2\frac{\sqrt{\pi}\,\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+2}{2}\right)}$$

$$= \frac{8a^2\sqrt{\pi}\,\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2.1} = 3\pi a^2$$

$\therefore$ The required area = $3\pi a^2$ square unit.

Ex. 10: Find the area included between an arc of cycloid

$x = a(\theta - \sin\theta)$ $y = a(1 - \cos\theta)$ and its base

Solution:

The cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ passes through origin. Let OA its base and B is the maximum point whose coordinates is $(\pi, 2a)$. So the required area,

$$A = 2\int_0^{a\pi} y\,dx$$

$$= 2\int_0^{\pi} a(1-\cos\theta)\,d\{a(\theta - \sin\theta)\}$$

$$= 2\int_0^{\pi} a(1-\cos\theta)\,a(1-\cos\theta)\,d\theta$$

$$= 2a^2\int_0^{a\pi}(1 - \cos\theta + \cos^2\theta)\,d\theta$$

$$= 2a^2\int_0^{a\pi}\left(1 - \cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2a^2\left[\theta - 2\sin\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= 2a^2\left[\pi - 0 + \frac{\pi}{2} + 0 - 0\right] = 3\pi a^2$$

$\therefore$ The required area = $3\pi a^2$ square unit.

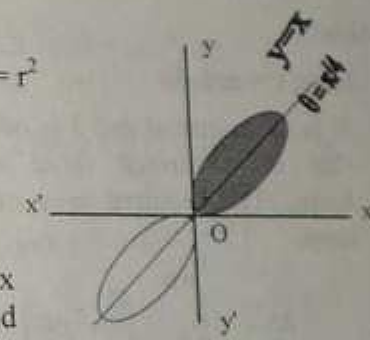
Ex. 11: Find the area of one loop of the curve $r^2 = a^2\sin 2\theta$

Solution:

The equation of the curve is

$r^2 = a^2\sin 2\theta$

or, $r^2 = a^2.2\sin\theta\cos\theta$



Put $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

So that,

$r^4 = a^2.2r\sin\theta.\,r\cos\theta$

or, $(x^2 + y^2)^2 = a^2xy$

It is symmetrical on the line $y = x$ and tangents at origin is x axis and y-axis. So, θ varies from 0 to $\pi/4$.

The required area of a loop,

$$A = 2\int_0^{\pi/4}\frac{1}{2}r^2\,d\theta = 2\int_0^{\pi/4}\frac{1}{2}a^2\sin 2\theta\,d\theta$$

$$= a^2\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/4} = a^2\left[0 + \frac{1}{2}\right] = \frac{a^2}{2}$$

$\therefore$ The required area $= \frac{a^2}{2}$ square unit.

Ex. 12: Find the area of the loop of the curve $r = a\sin 2\theta$

Solution:

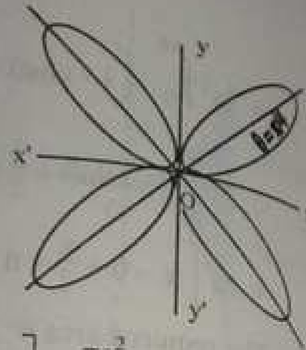
Here, the curve, $r = a\sin 2\theta$. Since 2 is even so it is formed four equal loops. The required area of the loop,

$$A = 4\int_0^{\pi/4} 2\left(\frac{1}{2}r^2\, d\theta\right)$$

$$= 4\int_0^{\pi/4} a^2 \sin^2 2\theta\, d\theta$$

$$= 4a^2\int_0^{\pi/4}\left(\frac{1-\cos 4\theta}{2}\, d\theta\right)$$

$$= 2a^2\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = 2a^2\left[\frac{\pi}{4} - 0\right] = \frac{\pi a^2}{2}$$



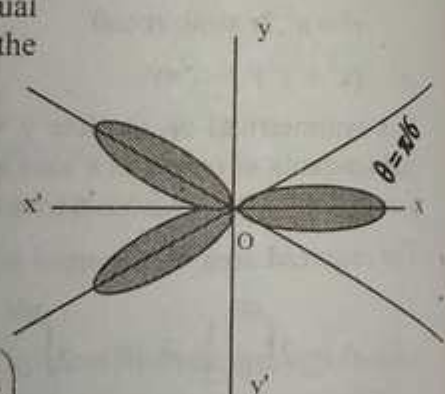
$\therefore$ The required area = = $\frac{\pi a^2}{2}$ square unit.

Ex. 13: Find the area of the loops of the curve $r = a\cos 3\theta$

Solution:

Here, $r = a\cos 3\theta$

It is symmetrical and 3 is odd, so that it is formed three equal loops. The required area of the loop,



$$A = 3\int_0^{\pi/6} 2\left(\frac{1}{2}r^2\, d\theta\right)$$

$$= 3\int_0^{\pi/6} a^2\cos^2 3\theta\, d\theta$$

$$= 3\int_0^{\pi/6}\left(\frac{1+\cos 6\theta}{2}\, d\theta\right)$$

$$= \frac{3a^2}{2}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/6} = \frac{3a^2}{2}\left[\frac{\pi}{6} - 0\right] = \frac{\pi a^2}{4}$$

$\therefore$ The required area = $\frac{\pi a^2}{4}$ square unit.

Ex. 14: Find the area between the Cardioid $r = a(1 + \cos\theta)$ a nd the circle $r = \frac{3a}{2}$

Solution:

Here, the Cardiod is

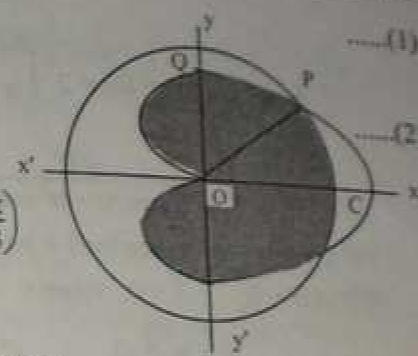
$r = a(1 + \cos\theta)$ .....(1)

and circle is

$r = \frac{3a}{2}$ .....(2)

Solving (1) and (2),

The common point P is $\left(\frac{3a}{2}, \frac{\pi}{3}\right)$



Area

$= 2\{\text{Area OCP} + \text{Area PQO}\}$

$$= 2\left\{\frac{1}{2}\int_0^{\pi/3}\left(\frac{3a}{2}\right)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} a^2(1+\cos\theta)^2\, d\theta\right\}$$

$$= \int_0^{\pi/3}\frac{9a^2}{4}\, d\theta + a^2\int_{\pi/3}^{\pi}(1 + 2\cos\theta + \cos^2)\, d\theta$$

$$= \frac{9a^2}{4}[\theta]_{\pi/3}^{\pi} + a^2\int_{\pi/3}^{\pi}\left[1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \frac{9a^2}{4}\left(\frac{\pi}{3} - 0\right) + a^2\left[\theta + 2\,\text{cisn}\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_{\pi/3}^{\pi}$$

$$= \frac{3\pi a^2}{4} + a^2\left[\pi + 0 + \frac{\pi}{2} + 0 - \frac{\pi}{3} - 2\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\pi}{3} - \frac{1}{4}\frac{\sqrt{3}}{2}\right]$$

$$= \frac{3\pi a^2}{4} + a^2\left(\pi - \frac{9\sqrt{3}}{8}\right) = \frac{7\pi a^2}{4} - \frac{9\sqrt{3}\,a^2}{8}$$

$$= a^2\left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8}\right)$$

$\therefore$ The required area = $a^2\left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8}\right)$ square unit.

Ex. 15: Find the area of the curve $r^2(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2$

Solution:

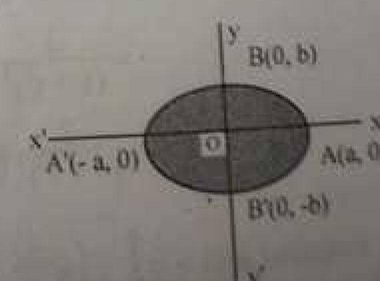
Here, $r^2(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2$

or, $r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$

Since $r = \pm a$ for $\theta = 0$

And $r = \pm b$ for $\theta = \pi/2$

Thus, the required area is given by



$$A = 4\int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = 2\int_0^{\pi/2} \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta} \, d\theta = 2\int_0^{\pi/2} \frac{a^2b^2\sec^2\theta}{a^2\tan^2\theta + b^2} \, d\theta$$

Put $a\tan\theta = bt$, $a\sec^2\theta d\theta = bdt$

When $\theta = 0$, $t = 0$, when $\theta = \pi/2$, $t = \infty$

$$= 2\int_0^{\infty} \frac{a^2b^2 \frac{b}{a} dt}{b^2t^2 b^2} = 2ab\int_0^{\infty} \frac{dt}{t^2+1} = 2ab\left[\tan^{-1} t\right]_0^{\infty}$$

$$= 2ab\frac{\pi}{2} = \pi ab$$

$\therefore$ The required area $= \pi ab$ square unit.

Ex. 16: Find the area enclosed by the curves $x(1+t^2) = 1-t^2$, $y(1+t^2) = 2t$

Solution:

Here, the equation of curve is

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)\times(-2t) - (1-t^2)\times 2t}{(1+t^2)^2}$$

$$= \frac{2t(-1-t^2-1+t^2)}{(1+t^2)^2} = \frac{-4t}{(1+t^2)^2}$$

and $\dfrac{dy}{dt} = \dfrac{(1+t^2)\times 2 - 2t\times 2t}{(1+t^2)^2}$

$$= \frac{2(1+t^2-2t^2)}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

Now, $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = \dfrac{1-t^2}{1+t^2}\dfrac{2(1-t^2)}{(1+t^2)^2} - \dfrac{2t}{1+t^2}\times\dfrac{-4t}{(1+t^2)^2}$

$$= \frac{2}{(1+t^2)^3}[1 - 2t^2 + t^4 + 4t^2]$$

$$= \frac{2}{(1+t^2)^3}[1 + 2t^2 + t^4] = \frac{2(1+t^2)^2}{(1+t^2)^3} = \frac{2}{1+t^2}$$

Required area $= \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt = \frac{1}{2}\int_{-\infty}^{\infty}\frac{2}{1+t^2}\,dt$

$$= \int_{-\infty}^{\infty}\frac{1}{1+t^2}dt = \int_0^{\infty}\frac{1}{1+t^2}dt = \left[2\tan^{-1}t\right]_0^{\infty} = 2\frac{\pi}{2} = \pi$$

$\therefore$ The required area $= \pi$ square unit.

Exercise-15

1. Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$
2. Show that the area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.
3. Find the area of the region bounded by the curve $y = \sin x$ and x-axis between $x = 0$ and $x = 2\pi$.
4. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
5. Find the area enclosed between the line $y = x$ and the parabola $y^2 = 16x$.
6. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.
7. Show that the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$
8. Find the area bounded by curve $x^2y = a^2(a - y)$ and the x-axis. [2059 B.E.]
9. Find the area of the loop of the curve $y^2 = x^2(x + a)$
10. Find the area of the curve $y^2 = x(x - 1)^2$
11. Find the area of the loop of the curve $y^2(a + x) = x^2(a - x)$
12. Find the area between each of the following curve and its asymptotes.
    - i. $y^2(a - x) = x^3$
    - ii. $a^2x^2 = y^2(a^2 - x^2)$
    - iii. $x^2y^2 + a^2b^2 = a^2y^2$
    - iv. $a(y^2 - x^2) = x(x^2 + y^2)$

v. $x^2(x^2+y^2)=a^2(y^2-x^2)$

13. Find the area of a loop of the curve $r^2=a^2\cos2\theta$

14. Find the area of the entire region bounded by the curve $r=a\cos2\theta$

15. Find the area bounded by the curve $r=a(1-\cos\theta)$

16. Find the area of a loop $r=a\sin3\theta$ [2057/060 B.E.]

17. Find the area common to the circle $r=a$ and the cardioid $r=a(1+\cos\theta)$ [2062 B.E.]

18. Find the area enclosed by the following curves

i. $x=a(1-t^2),\ y=at(1-t^2),\ -1\le t\le 1$

ii. $x=\frac{1-t^2}{1+t^2},\quad y=t\frac{1-t^2}{1+t^2},\quad -1\le t\le 1$

Answers

1. $\frac{16}{5}$ 3. 4 4. πab

5. $\frac{128}{3}$ 6. $\frac{9}{8}$ 8. $a\pi^2$

9. $\frac{8}{15}a^{5/2}$ 10. $\frac{8}{15}$ 11. $a^2\left(2-\frac{\pi}{2}\right)$

12. i. $\frac{3\pi a^2}{4}$ ii. $4a^2$ iii. $2\pi ab$ iv. $2a^2\left(1+\frac{\pi}{4}\right)$

v. $a^2(\pi+2)$ 13. $\frac{a^2}{2}$ 14. $\frac{\pi a^2}{2}$

15. $\frac{3\pi a^2}{2}$ 16. $\frac{\pi a^2}{4}$ 17. $a^2\left(\frac{5\pi}{4}-2\right)$

18. i. $\frac{8a^2}{15}$ ii. $2-\frac{\pi}{2}$

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Chapter - 12

Rectification

12.1 Rectification

The process of finding the Arc-length of the plane curves whose equations are given in Cartesian, parametric or polar from is called *Rectification.*

12.2 Arc Length in Cartesian Form

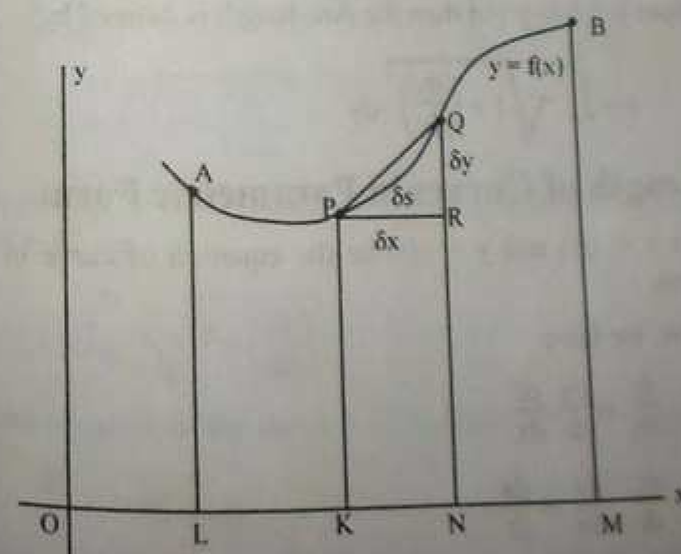
Let AB an arc of the curve $y=f(x)$ bounded by ordinates AL and BM where $OL=a$ and $OM=b$.

Let $P(x, y)$ be any point on the curve such that

Arc $AP=S$

Take $Q(x+\delta x, y+\delta y)$ be another point near to P such that

Arc $PQ=\delta s$, $KN=PR=\delta x$, $QR=\delta y$



$$= 96\pi a^3 \int_0^{\pi} \sin^7\frac{\theta}{2}\cos^3\frac{\theta}{2}\, d\theta - 128\pi a^3 \int_0^{\pi} \sin^7\frac{\theta}{2}\cos^5\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t$, then $\frac{1}{2}\, d\theta = dt$.

When $\theta = 0, t = 0,\ \theta = \pi, t = \frac{\pi}{2}$.

So, $v = 96\pi a^3 \int_0^{\pi/2} 2\sin^7 t\cos^3 t\, dt - 128\pi a^3 \int_0^{\pi/2} 2\sin^7 t\cos^5 t\, dt$

$$= \pi a^3 \frac{2\Gamma(4)\Gamma(2)}{2\Gamma(6)} - 128\pi a^3 \frac{2\Gamma(4)\Gamma(3)}{2\Gamma(7)}$$

$$= \pi a^3\left(\frac{96.3.2.1}{5.4.3.2} - \frac{128.3.2.2.}{6.5.4.3.2}\right)$$

$$= \pi a^3\left(\frac{24}{5} - \frac{32}{15}\right) = \pi a^3\frac{40}{15} = \frac{8}{3}\pi a^3.$$

Ex. 10: Find the volume of the solid formed by the revolution of the Cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex. [2057 B.E.]

Solution:

The tangent at the vertex of the cycloid is the x-axis.

The required volume = 2 times the volume generated by the revolution of the portion OAX

$$= 2\pi \int_0^{a\pi} y^2\, dx$$

$$= 2\pi \int_0^{\pi} a^2(1-\cos\theta)^2 . a(1+\cos\theta)\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} 4\sin^4\frac{\theta}{2}\, 2\cos^2\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t, \frac{1}{2}\, d\theta = dt$.

When $\theta = 0, t = 0$ and when $\theta = \pi, t = \frac{\pi}{2}$.

$\therefore v = 16\pi a^3 . 2\int_0^{\pi/2} \sin^4 t\cos^2 t\, dt$

$$= 32\pi a^3 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)} = \frac{32\pi a^3 \frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2.3.2.1}$$

$= \pi a^3 . \sqrt{\pi}.\sqrt{\pi} = \pi^2 a^3.$

Exercise -17

1. Find the volume of ellipsoid formed by the revolution of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.
2. Prove that the volume and surface of a sphere of radius a is $\frac{4}{3}\pi a^3$ and $4\pi a^2$ respectively.
3. An arc of a parabola is bounded at both ends by the latus rectum of length 4a. Find the volume generated when the arc is rotated about the latus rectum.
4. Find the volume and surface area of the solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the axis of x.
5. The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the volume and the area of the curved surface thus generated.
6. Find the volume of the solid formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.
7. Find the area of the surface of the solid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.
8. Find the volume of the solids formed by the revolution of the curve $y^2 = x^2(a - x)$ about the x-axis.
9. Find the surface area of solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.
10. Find the volume of the solid formed by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base. [2058/062 B.E.]

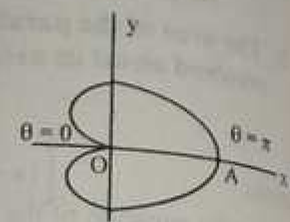
Ex. 8: Find the surface area of the solid generated by the revolution of the Cardioid $r = a(1 + \cos\theta)$ about the initial line. [2060 B.E.]

Solution:

Here, OA = 2a so that the x-coordinate of A is 2a.

The equation Cardioid is

$$r = a(1 + \cos\theta)$$



We have,

$$x = r\cos\theta = a(1 + \cos\theta)\cos\theta$$

$$y = r\sin\theta = a(1 + \cos\theta)\sin\theta$$

The required surface area

$$s = 2\pi\int_0^{\pi} y\frac{ds}{d\theta}\,d\theta$$

$$= 2\pi\int_0^{\pi} y\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

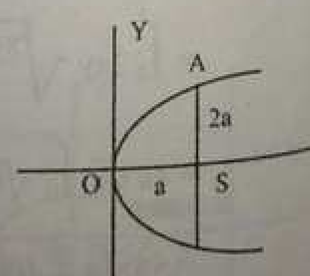
$$= 2\pi\int_0^{\pi} y\sqrt{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= 2\pi a\int_0^{\pi} y\sqrt{2(1 + \cos\theta)}\,d\theta$$

$$= 2\pi\int_0^{\pi} a(1 + \cos\theta)\sin\theta.\ a\sqrt{2(1 + \cos\theta)}\,d\theta$$

$$= 2\pi a^2\int_0^{\pi} 2\cos^2\frac{\theta}{2}\ 3\sin\frac{\theta}{2}\ \cos\frac{\theta}{2}.\ 2\cos\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\int_0^{\pi} \cos^4\frac{\theta}{2}\ \sin\frac{\theta}{2}\,d\theta.$$



Put $\frac{\theta}{2} = t$, $\frac{1}{2}\,d\theta = dt$.

When $\theta = 0$, $t = 0$, when $\theta = \pi$, $t = \frac{\pi}{2}$

$$\therefore \text{surface area} = 16\pi a^2\int_0^{\pi/2} 2\cos^4 t\sin t\,dt$$

$$= 32\pi a^2\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(1)}{2\Gamma\left(\frac{7}{2}\right)}$$

$$= 16\pi a^2\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{32}{5}\pi a^2.$$

Ex. 9: Find the volume of the solid generated by the revolution of the Cardioid $r = a(1 - \cos\theta)$ about the initial line.

Solution:

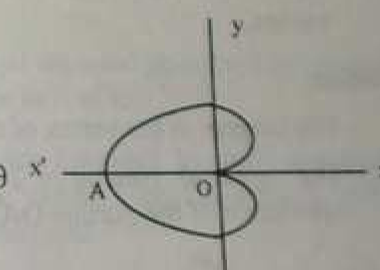
Here, OA = 2a so that the x-coordinate of A is - 2a.

The required volume is

$$v = \pi\int_{-2a}^{0} y^2\,dx$$

Here, the equation of Cardioid is

$$r = a(1 - \cos\theta)$$



Put, $x = r\cos\theta = a(1 - \cos\theta)\cos\theta$

and $y = r\sin\theta = a(1 - \cos\theta)\sin\theta$

Differentiating,

$$dx = (-a\sin\theta + 2a\sin\theta\cos\theta)\,d\theta.$$

When $x = 0$, $\theta = 0$ and when $x = -2a$, $\theta = \pi$

So, $$v = \pi\int_{-2a}^{0} a^2(1 - \cos\theta)^2\sin^2\theta(-a\sin\theta + 2a\sin\theta\cos\theta)\,d\theta$$

$$= \pi a^3\int_{\pi}^{0}(1 - \cos\theta)^2\sin^3\theta(1 - 2\cos\theta)\,d\theta$$

$$= \pi a^3\int_0^{\pi} 4\sin^4\frac{\theta}{2}.\ 8\sin^3\frac{\theta}{2}\ \cos^3\frac{\theta}{2}\left(3 - 4\cos^2\frac{\theta}{2}\right)d\theta$$

$$= \frac{\pi c^2}{4}\left[\frac{c}{2}e^{2x/c} + 2x - \frac{c}{2}e^{-2x/c}\right]_0^a$$

$$= \frac{\pi c^2}{4}\left[\frac{c}{2}e^{2a/c} + 2a - \frac{c}{2}e^{-2a/c} - \frac{c}{2} + \frac{c}{2}\right]$$

$$= \frac{\pi c^2}{4}\left[\frac{c}{2}(e^{a/c} + e^{-a/c})(e^{a/c} - e^{-a/c}) + 2a\right]$$

$$= \frac{\pi c^2}{4}\left[x\frac{(e^{a/c} + e^{-a/c})}{2}\frac{(e^{a/c} - e^{-a/c})}{2} + 2a\right]$$

$$= \frac{\pi c^2}{4}\left[2c\cosh\frac{a}{c}\sin h\frac{a}{c} + 2a\right]$$

$$= \frac{\pi c^2}{2}\left[\cos h\frac{a}{c}\sin h\left(\frac{a}{c}\right) + a\right]$$

$$= \frac{\pi c^2}{2}\left[\cos h\frac{a}{c}\sin h\left(\frac{a}{c}\right) + a\right]$$

$$v = \frac{\pi c^2}{2}\left[c\cosh\frac{a}{c}\ \sinh\left(\frac{a}{c}\right) + a\right] \text{cubic unit}$$

and the surface of the given curve is defined as

$$s = 2\pi\int_0^a y\, ds = 2\pi\int_0^a y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\int_0^a y\sqrt{1 + \sin h^2(x/c)}\, dx = 2\pi\int_0^a c\cosh^2\left(\frac{x}{c}\right) dx$$

$$= 2\pi c\int_0^a\left(\frac{e^{x/c} + e^{-x/c}}{2}\right)^2 dx = \frac{\pi c}{2}\int_0^a (e^{2x/c} + 2 + c^{-2x/c})dx$$

$$= \frac{\pi c}{2}\left[\frac{c}{2}e^{2x/c} + 2x - \frac{c}{2}e^{-2x/c}\right]_0^a = \frac{\pi c}{2}\left[\frac{c}{2}e^{2a/c} + 2a - \frac{c}{2}e^{-2a/c}\right]$$

$$= \frac{\pi c}{2}\left[\frac{c}{2}(e^{a/c} + e^{-a/c})(e^{a/c} - e^{-a/c}) + 2a\right]$$

$$= \frac{\pi c}{2}\left[2\frac{(e^{a/c} + e^{-a/c})}{2}\frac{(e^{a/c} - e^{-a/c})}{2} + 2a\right]$$

$$= \frac{\pi c}{2}\left[2\cosh\frac{a}{c}\sinh\frac{a}{c} + 2a\right]$$

$$= \pi c\left[c\cosh\frac{a}{c}\sinh\frac{a}{c} + a\right] \text{square unit.}$$

Ex. 3: Find the volume and area of the surface generated by the revolution of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about its base i.e. the line $y = 0$ **[2061 B.E. Back]**

Solution:

Here, the equation cycloid is

$x = (t - \sin t)$, $\quad y = a(1 - \cos t)$

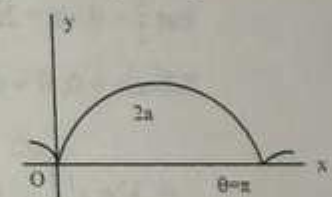
Differentiating,

$\frac{dx}{dt} = a(1 - \cos t)$, $\quad \frac{dy}{dt} = a\sin t$

So, $\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$.

The volume generated by revolution of cycloid is given by

$$v = \int \pi y^2\, dx$$



$$= 2\pi\int_0^\pi a^2(1 - \cos t)^2 a(1 - \cos t)\, dt$$

$$= 2\pi a^3\int_0^\pi\left(2\sin^2\frac{t}{2}\right)^2 2\sin^2\frac{t}{2}\, dt$$

$$= 8\pi a^3 . 2\int_0^\pi \sin^6\frac{t}{2}\, dt$$

Put $\frac{t}{2} = \theta$, $dt = 2d\theta$

When $t = 0$, $\theta = 0$, when $t = \pi$, $\theta = \pi/2$

$$= 16\pi a^3 . 2\int_0^{\pi/2}\sin^6 d\theta = 16a^3.2\pi\frac{\sqrt{\pi}\,\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{6+2}{2}\right)}$$

$$= 16\pi a^3 . \frac{2\sqrt{\pi}\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{2.3.2.1} = 5\pi^2 a^3 \text{ cubic unit.}$$

Eliminating 'h' between (1) and (2),

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)=0$$

$\therefore\ 2xy\frac{dy}{dx}+x^2-y^2=0$ is the required differential equation.

Exercise - 18

1. Determine the order and degree of each of the following differential equations.
   i. $(x+3y-2)\,dx+(2x-3y+5)\,dy=0$
   ii. $y=x\frac{d^2y}{dx^2}+\frac{k}{d^2y/dx^2}$
   iii. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=k\frac{d^2y}{dx^2}$
   iv. $x^2\frac{d^2y}{dx^2}+2xy\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3=0$
2. Form the differential equations from the following equations
   i. $y=a\log x+b$ ii. $xy=a+bx$
   iii. $xy=Ae^x+Be^{-x}$ iv. $y=ax^3+bx^2$
   v. $a\cos(\log x)+b\sin(\log x)$
3. Obtain the differential equation of all circles of radius a and centre (h, k).
4. Form a differential equation of simple harmonic motion given by $x=A\cos(nt+\alpha)$

Answers

1. i. Fist order, first degree. ii. Second order second degree
   iii. Second order, second degree. iv. Second order first degree
2. i. $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$ ii. $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$
   iii. $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-y=0$ iv. $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=0$
   v. $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$ 3. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$ 4. $\frac{d^2x}{dt^2}+n^2x=0$

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Chapter - 15

First Order and First Degree Differential Equation

15.1 Introduction

The differential of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ where M and N are functions of x and y or constant is called *First Order and First Degree Differential Equation.*

15.2 Separation of the Variables

In the differential equation $M\,dx + N\,dy = 0$, the variables are separated in such way that the coefficient of dx must be function x only or constant and the coefficient of dy must be function of y only or constant.

Worked out Examples

Ex. 1 Solve: $\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0$

Solution:

Here, the equation is

$$\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0.$$

Separating the variables,

$$\frac{dy}{\sqrt{1-y^2}}+\frac{dx}{\sqrt{1-x^2}}=0$$

Integrating,

$$\int\frac{dy}{\sqrt{1-y^2}}+\int\frac{dx}{\sqrt{1-x^2}}=0$$

$\therefore\ \sin^{-1}y+\sin^{-1}x=c$ is the required solution.

Ex. 2 Solve: $(\sin x+\cos x)\,dy=(\cos x-\sin x)\,dx$

Solution:

Here, the equation is

$$(\sin x+\cos x)\,dy=(\cos x-\sin x)\,dx.$$

Separating the variables,

$$dy=\frac{(\cos x-\sin x)}{(\sin x+\cos x)}dx$$

Integrating,

$$y = \log(\sin x + \cos x) + \log c$$

or $y = \log c(\sin x + \cos x)$

$\therefore$ $e^y = c(\sin x + \cos x)$ is the required solution.

Ex. 3: Solve: $(1 + x)\, y\, dx + (1 + y)\, x\, dy = 0$

Solution:

Here, the equation is

$$(1 + x)\, y\, dx + (1 + y)\, x\, dy = 0$$

Separating the variables,

$$\frac{1 + x}{x} dx + \frac{1 + y}{y} dy = 0$$

or $\dfrac{(1 + x)}{x} dx + \dfrac{(1 + y)}{y} dy = 0,$

or $\left(\dfrac{1}{x} + 1\right) dx + \left(\dfrac{1}{y} + 1\right) dy = 0$

Integrating

$$\log x + x + \log y + y = c$$

$\therefore$ $\log(xy) + x + y = c$ is the required solution.

Ex. 4: Solve: $\log\left(\dfrac{dy}{dx}\right) = ax + by$

Solution:

Here, the equation is

$$\log\left(\frac{dy}{dx}\right) = ax + by$$

or $\dfrac{dy}{dx} = e^{ax+by}$

Separating the variables

$$\frac{dy}{e^{by}} = e^{ax}\, dx,$$

or $e^{-by}\, dy = e^{ax}\, dx$

or $e^{ax}\, dx - e^{-by}\, dy = 0$

Integrating,

$$\frac{e^{ax}}{a} + \frac{e^{-by}}{b} + c = 0$$

$\therefore$ $be^{ax} + ae^{-by} + abc = 0$ is the required solution.

Ex. 5: Solve: $(1 - x^2)(1 - y)\, dx = xy(1 + y)\, dy$

Solution:

Here, the equation is

$$(1 - x^2)(1 - y)\, dx = xy(1 + y)\, dy.$$

Separating the variables,

$$\frac{1 - x^2}{x} dx = y\frac{(1 + y)}{1 - y} dy$$

or $\dfrac{1 - x^2}{x} dx = \dfrac{y(1 + y)}{1 - y} dy$

or $\dfrac{dx}{x} - x\, dx = \left(-y + \dfrac{2}{1 - y} - 2\right) dy$

Integrating

$$\log x - \frac{x^2}{2} = -\frac{y^2}{2} - 2\log(1 - y) - 2y + c$$

$\therefore$ $\log x - \dfrac{x^2}{2} + \dfrac{y^2}{2} + 2y + 2\log(1 - y) = c$ is the required solution.

Ex. 6: Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Solution:

Here, the equation is

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0.$$

Separating the variables,

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

or $\dfrac{\sec^2 x}{\tan x} dx + \dfrac{\sec^2 y}{\tan y} dy = 0$

Integrating,

$$\log(\tan x) + \log(\tan y) = \log c$$

or $\log(\tan x \tan y) = \log c$

$\therefore$ $\tan x \tan y = c$ is the required solution.

Ex. 7: Solve: $\dfrac{dy}{dx} + \dfrac{1 + \cos 2y}{1 - \cos 2x} = 0$

Solution:

Here, the equation is

$$\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0.$$

Separating the variables,

$$\frac{dy}{1 + \cos 2y} + \frac{dx}{1 - \cos 2x} = 0$$

or $\dfrac{dy}{2\cos^2 y} + \dfrac{dx}{2\sin^2 x} = 0$

or $\sec^2 y.\, dy + \operatorname{cosec}^2 x\, dx = 0$

Integrating

$\tan y - \cot x = c$ is the required solution.

Ex. 8: Solve: $\dfrac{dy}{dx} = \dfrac{x(2\log x + 1)}{\sin y + y\cos y}$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

Separating the variables,

$$(\sin y + y\cos y)\, dy = (2x\log x + x)\, dx$$

or $\sin y\, dy + y\cos y\, dy = 2x\log x\, dx + x dx$

Integrating,

$$-\cos y + y\sin y - \sin y\, dy = x^2\log x - \frac{1}{x}\, x^2\, dx + \frac{x^2}{2}$$

or $-\cos y + y\sin y + \cos y = x^2\log x - \dfrac{x^2}{2} + \dfrac{x^2}{2} + c$

$\therefore$ $y\sin y = x^2\log x + c$ is the required solution.

Ex.9: Solve: $\cos y\, dx + (1 + 2e^{-x})\sin y\, dy = 0$ given $y = \dfrac{\pi}{4}$ when $x = 0$

Solution:

Here, the equation is

$$\cos y\, dx + (1 + 2e^{-x})\sin y\, dy = 0$$

Separating the variables,

$$\frac{dx}{(1 + 2e^{-x})} + \frac{\sin y}{\cos y}\, dy = 0$$

or $\dfrac{e^x}{e^x + 2}\, dx + \dfrac{\sin y}{\cos y}\, dy = 0$

Integrating,

$$\log(e^x + 2) + \log \sec y = \log c$$

or $(e^x + 2)\sec y = c$ ....(1)

Using the given condition, $y = \dfrac{\pi}{4}$ when $x = 0$,

Then (1) becomes,

$3\sqrt{2} = c$, $\therefore$ $c = 3\sqrt{2}$

$\therefore$ $(e^x + 2)\sec y = 3\sqrt{2}$ is the required solution.

Ex.10: Find the particular solution of the following differential equation $\log\left(\dfrac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$, when $x = 0$

Solution:

Here, the equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y,$$

or $\dfrac{dy}{dx} = e^{3x - 4y}$

Separating the variables,

$$e^{-4y}\, dy = e^{3x}\, dx$$

or $e^{3x}dx - e^{-4y}\, dy = 0$

Integrating

$$\frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = c \quad \text{..........(1)}$$

Given that $y = 0$ when $x = 0$ then (1) becomes,

$$\frac{1}{3} + \frac{1}{4} = c$$

$$\therefore c = \frac{7}{12}.$$

Thus the general solution is $4e^{3x} + 3e^{-4y} = 7$.

Exercise-19

Solve the following differential equations

1. $x\sqrt{1 + y^2}\, dx + y\sqrt{1 + x^2}\, dy = 0$
2. $(x^2 + 1)\dfrac{dy}{dx} = 1$
3. $y\, dx = (e^x + 1)\, dy$
4. $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0$
5. $\tan y\, dx + \tan x\, dy = 0$
6. $\left(y - x\dfrac{dy}{dx}\right) = a\left(y^2 + \dfrac{dy}{dx}\right)$
7. $(1 + x)(1 + y^2)\, dx + (1 + y)(1 + x^2)\, dy = 0$
8. $(e^x + 1)\, y\, dy = (y + 1)\, e^x\, dx$
9. $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$
10. $e^{x-y}\, dx + e^{y-x}\, dy = 0$

Ex. 8: Solve: $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

Separating the variables,

$$(\sin y + y\cos y)\,dy = (2x\log x + x)\,dx$$

or $\sin y\,dy + y\cos y\,dy = 2x\log x\,dx + x\,dx$

Integrating,

$$-\cos y + y\sin y - \sin y\,dy = x^2\log x - \frac{1}{x}\,x^2\,dx + \frac{x^2}{2}$$

or $-\cos y + y\sin y + \cos y = x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + c$

$\therefore$ $y\sin y = x^2\log x + c$ is the required solution.

Ex.9: Solve: $\cos y\,dx + (1 + 2e^{-x})\sin y\,dy = 0$ given $y = \frac{\pi}{4}$ when $x = 0$

Solution:

Here, the equation is

$$\cos y\,dx + (1 + 2e^{-x})\sin y\,dy = 0$$

Separating the variables,

$$\frac{dx}{(1 + 2e^{-x})} + \frac{\sin y}{\cos y}\,dy = 0$$

or $\frac{e^x}{e^x + 2}\,dx + \frac{\sin y}{\cos y}\,dy = 0$

Integrating,

$$\log(e^x + 2) + \log\sec y = \log c$$

or $(e^x + 2)\sec y = c$ ....(1)

Using the given condition, $y = \frac{\pi}{4}$ when $x = 0$,

Then (1) becomes,

$3\sqrt{2} = c$, $\quad\therefore\ c = 3\sqrt{2}$

$\therefore$ $(e^x + 2)\sec y = 3\sqrt{2}$ is the required solution.

Ex.10: Find the particular solution of the following differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$, when $x = 0$

Solution:

Here, the equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y,$$

or $\frac{dy}{dx} = e^{3x + 4y}$

Separating the variables,

$$e^{-4y}\,dy = e^{3x}\,dx$$

or $e^{3x}dx - e^{-4y}\,dy = 0$

Integrating

$$\frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = c \qquad \text{..........(1)}$$

Given that $y = 0$ when $x = 0$ then (1) becomes,

$$\frac{1}{3} + \frac{1}{4} = c$$

$$\therefore c = \frac{7}{12}.$$

Thus the general solution is $4e^{3x} + 3e^{-4y} = 7$.

Exercise-19

Solve the following differential equations

1. $x\sqrt{1 + y^2}\,dx + y\sqrt{1 + x^2}\,dy = 0$
2. $(x^2 + 1)\frac{dy}{dx} = 1$
3. $y\,dx = (e^x + 1)\,dy$
4. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$
5. $\tan y\,dx + \tan x\,dy = 0$
6. $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$
7. $(1 + x)(1 + y^2)\,dx + (1 + y)(1 + x^2)\,dy = 0$
8. $(e^x + 1)\,y\,dy = (y + 1)\,e^x\,dx$
9. $3e^x\tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$
10. $e^{x-y}\,dx + e^{y-x}\,dy = 0$

11. $(a^2+y^2)x\,dx + y(x^2-a^2)\,dy = 0$
12. $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$
13. Find the particular solution of $2xy' = 3y$ given that $y=4$ when $x=4$
14. Find the particular solution of $y' = \sec y$, given that $y=0$ when $x=0$
15. Find the particular solution of $\frac{dy}{dx} = e^{x+y}$ and it is given that for $x=1, y=1$ find y as $x=-1$
16. Find the equation of the curve which passes through the point (1, 2) and has at every point, $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$
17. Find the particular solution of $y(1-x^2)\frac{dy}{dx} + x(1-y^2) = 0$ given that $y=1$ when $x=0$
18. Find the equation of the curve represented by $(y-yx)\,dx + (x+xy)dy = 0$ and passes the point (1, 1)

Answers

1. $\sqrt{1+x^2} + \sqrt{1+y^2} = c$
2. $y = \tan^{-1}x + c$
3. $y(1+e^{-x}) = c$
4. $(1+x^2)(1+y^2) = c$
5. $\sin x \sin y = c$
6. $(a+x)(1-ay) = cy$
7. $2(\tan^{-1}x + \tan^{-1}y) + \log(1+x^2)(1+y^2) = c$
8. $y - c = \log[(1+y)(e^x+1)]$
9. $\tan y = c(1-e^x)^3$ 10. $e^{2x} + e^{2y} = c$
11. $(x^2-a^2)(a^2+y^2) = c$ 12. $\sin x(e^y+1) = c$
13. $y = 4x^{3/2}$ 14. $\sin y = x$
15. $-e^{-y} = e^x - e - e^{-1}$, $y=-1$ 16. $y(x^2+1) = 4$
17. $(1-y^2)(1-x^2) = 0$ 18. $\log xy = x - y$

15.3 Change a Variable

If the first order differential equation $M(x, y)dx + N(x, y)\,dy = 0$ of the form $\frac{dy}{dx} = f(ax+by+c)$.

Then it can be solved by putting

$ax + by + c = v$,

or $a + b\frac{dy}{dx} = \frac{dv}{dx}$,

$\frac{dy}{dx} = \frac{1}{b}\frac{dv}{dx} - \frac{a}{b}$.

So, the differential equation can be written as

$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = f(v)$.

Separating the variables and integrating we get the required solution.

Worked Out Examples

Ex. 1: Solve $\frac{dy}{dx} = \cos(x+y)$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \cos(x+y)$$

Put $x + y = v$,

or $1 + \frac{dy}{dx} = \frac{dv}{dx}$

or $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Thus the equation reduces to

$$\frac{dv}{dx} - 1 = \cos v,$$

or $\frac{dv}{dx} = 1 + \cos v$

Separating the variables,

$$\frac{dv}{1+\cos v} = dx,$$

or $\frac{dv}{2\cos^2\frac{v}{2}} = dx$

or $\frac{1}{2}\sec^2\frac{v}{2}\,dv = dx$

Integrating,

$$\tan\left(\frac{v}{2}\right) = x + c$$

$$\therefore \quad \tan\left(\frac{x+y}{2}\right) = x + c$$ is the required solution.

Ex. 2: Solve $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

Put $x + y = v$,

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

So the given equation reduces to

$$\frac{dv}{dx} - 1 = \sin v + \cos v$$

or $$\frac{dv}{dx} = 1 + \sin v + \cos v,$$

or $$\frac{dv}{dx} = 2\cos^2\frac{v}{2} + 2\sin\frac{v}{2}\cos\frac{v}{2}$$

Separating the variables

$$\frac{dv}{2\cos^2\frac{v}{2} + 2\sin\frac{v}{2}\cos\frac{v}{2}} = dx$$

or $$\frac{\sec^2\left(\frac{v}{2}\right)dv}{2 + 2\tan\frac{v}{2}} = dx$$

Integrating

$$\log\left(1 + \tan\frac{v}{2}\right) = x + c$$

$$\therefore \quad \log\left(1 + \tan\frac{x+y}{2}\right) = x + c$$ is the required solution.

Ex. 3: Solve: $\left(\frac{x + y - a}{x + y - b}\right)\frac{dy}{dx} = \frac{x + y + a}{x + y + b}$

Solution:

Here, the equation is

$$\frac{dy}{dx} = \frac{(x + y + a)(x + y - b)}{(x + y - a)(x + y + b)}$$

Put $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

So, the equation changes into,

$$\frac{dv}{dx} - 1 = \frac{(v + a)(v - b)}{(v - a)(v + b)}$$

or $$\frac{dv}{dx} = \frac{v^2 + (a - b)v - ab}{v^2 + (b - a)v - ab} + 1$$

or $$\frac{dv}{dx} = \frac{v^2 + (a - b)v - ab + v^2 - (a - b)v - ab}{v^2 - (a - b)v - ab}$$

or $$\frac{dv}{dx} = \frac{2v^2 - 2ab}{v^2 - (a-b)v - ab}$$

Separating the variables

$$\frac{v^2 - (a - b)v - ab}{(v^2 - ab)}\,dv = 2dx$$

or $$\left[1 - \frac{(a-b)v}{(v^2 - ab)}\right]dv = 2dx$$

or $$\left(2 - \frac{(a - b)2v}{v^2 - ab}\right)dv = 4\,dx$$

Integrating

$$2v - (a-b)\log(v^2 - ab) = 4x + c$$

or $$2x + 2y + (b - a)\log\{(x + y)^2 - ab\} = 4x + c$$

$$\therefore \quad (b - a)\log\{(x + y)^2 - ab\} = 2(x - y + c)$$

is the required solution.

Ex. 4: Solve: $\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$

Solution:

Here, the equation is

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - x^2 - y^2}{x^2 + y^2}}$$

or $$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - (x^2 + y^2)}{x^2 + y^2}}$$

Put $x = r\cos\theta$, $\quad y = r\sin\theta$

$x^2 + y^2 = r^2$, $\quad \tan\theta = \frac{y}{x}$

or $x\,dx + y\,dy = r\,dr$, $\quad \sec^2\theta\,d\theta = \frac{x\,dy - y\,dx}{x^2}$

So, the equation changes into

$$\frac{r\,dr}{x^2 \sec^2\theta\, d\theta} = \sqrt{\frac{1-r^2}{r^2}}$$

or $$\frac{r\,dr}{r^2\cos^2\theta \sec^2\theta\, d\theta} = \frac{\sqrt{1-r^2}}{r}$$

or $$\frac{dr}{\sqrt{1-r^2}} = d\theta$$

Integrating,

$$\sin^{-1} r = \theta + c$$

or $$\sin^{-1}\sqrt{x^2+y^2} = \tan^{-1}\left(\frac{y}{x}\right) + c$$

or $$\sqrt{x^2+y^2} = \sin\left[\tan^{-1}\left(\frac{y}{x}\right)\right] + c$$

$\therefore$ $x^2 + y^2 = \sin^2\left[\tan^{-1}\left(\frac{y}{x}\right)\right] + c$ is the required solution.

Exercise-20

Solve the following differential equations

1. $(x+y)^2 \frac{dy}{dx} = a^2$
2. $\cos(x+y)\, dy = dx$
3. $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$
4. $\frac{dy}{dx} + 1 = e^{x+y}$
5. $\frac{dy}{dx} + 1 = e^{x-y}$
6. $\frac{dy}{dx} - x\tan(y-x) = 1$
7. $\frac{dy}{dx} = (4x+y+1)^2$
8. $(x^2 + y^2 + 2xy + 1)\, dy = (x+y)\, dx$
9. $(x+y+1)\frac{dy}{dx} = 1$
10. $\frac{dy}{dx} = \sqrt{y-x}$
11. $x^2(xdx + y\,dy) + 2y(x\,dy - ydx) = 0$

Answers

1. $x + y = a\tan\frac{(y-c)}{a}$

2. $\tan\frac{x+y}{2} = y - c$
3. $\tan(x+y) - \sec(x+y) = x + c$
4. $(x+c)e^x + e^{-y} = 0$
5. $e^y = \frac{1}{2}e^x + ce^{-x}$
6. $\sin(y-x) = ce^{x^2/2}$
7. $(4x+y+1) = 2\tan(2x+c)$
8. $y - \frac{1}{2}\log\{(x+y)^2 + x + y + 1\} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+2y+1}{\sqrt{3}} = c$
9. $y - \log(x+y+2) = c$
10. $\sqrt{y-x} + \log(\sqrt{y-x} - 1) = \frac{x}{2} + c$
11. $\sqrt{x^2+y^2}\left(1 + \frac{2}{x}\right) = c$

15.4 Homogeneous Differential Equation of First Order

If the first order differential equation is of the form

$$\frac{dy}{dx} = \frac{\psi(x,y)}{\phi(x,y)}$$

where $\psi(x, y)$ and $\phi(x, y)$ are homogeneous function of same degree n, then it can be solved by putting $y = vx$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So, the differential equation reduces to

$$v + x\frac{dv}{dx} = f(v).$$

Separating the variables and integrating we get the required solution.

Worked Out Examples

Ex. 1: Solve $\frac{dy}{dx} = \frac{x^2y^4}{x^3+y^3}$

Solution:

Here, $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$

This is homogenous. Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get, $v + \frac{xdv}{dx} = \frac{x^2.vx}{x^3+v^3x^3}$

$\frac{xdv}{dx} = \frac{v}{1+v^3} - v = \frac{v - v - v^4}{1+v^3}$

or $x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$

Separating the variables

$\frac{dx}{x} + \frac{(1+v^3)}{v^4}\, dv = 0$

or $\frac{dx}{x} + \frac{1}{v^4}\, dv + \frac{1}{v}\, dv = 0$

Integrating

$\log x - \frac{1}{3v^3} + \log v + \frac{c}{3} = 0$

or $\log (xv) + \frac{c}{3} = \frac{1}{3v^3}$

or $3 \log y + c = \frac{x^3}{y^3}$ is the required solution.

Ex. 2: Solve $(x^2 + y^2)\, dx + 2xy\, dy = 0$

Solution:

Here, $(x^2 + y^2)\, dx + 2xy\, dy = 0$,

or $\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$

This is homogenous. Put $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get,

$v + \frac{xdv}{dx} = -\frac{x^2+v^2.x^2}{2x.vx}$,

or $\frac{xdv}{dx} = -v - \frac{1+v^2}{2v}$

or $x\frac{dv}{dx} = -\frac{1+3v^2}{2v}$

or $\frac{dx}{x} + \frac{2v\, dv}{1+3v^2} = 0$

or $3\frac{dx}{x} + \frac{6v\, dv}{1+3v^2} = 0$

Integrating

$3 \log x + \log (1 + 3v^2) = \log c$, or $x^3 (1 + 3v^2) = c$

$x(x^2 + 3y^2) = c$ is the required solution.

Ex. 3: Solve $y^2\, dx + (xy + x^2)\, dy = 0$

Solution:

Here, $y^2\, dx + (xy + x^2)\, dy = 0$,

$\frac{dy}{dx} = -\frac{y^2}{xy+x^2}$

This is homogenous. Put $y = vx$,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get $v + x\frac{dv}{dx} = -\frac{v^2x^2}{x.vx+x^2}$

$\frac{xdv}{dx} = -\frac{v^2}{v+1} - v = \frac{-v^2 - v^2 - v}{v+1}$

or $x\frac{dv}{dx} + \frac{2v^2+v}{v+1} = 0$.

Separating the variables

$\frac{dx}{x} + \frac{v+1}{2v^2+v}\, dv = 0$

or $2\frac{dx}{x} + 2\left(\frac{1}{v} - \frac{1}{2v+1}\right) dv = 0$

Integrating

$2 \log x + 2 \log v - \log (2v + 1) = \log c$

or $\log x^2v^2 = \log c(2v + 1)$

or $xy^2 = c(2y + x)$ is the required solution.

Ex. 4: Solve $(x + y)\, dx + (y - x)\, dy = 0$

Solution:

Here, $(x + y)\, dx + (y - x)\, dy = 0$, or $\frac{dy}{dx} = -\frac{x+y}{y-x}$

This is homogenous. Put, y = vx, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get $v + x\frac{dv}{dx} = -\frac{x+vx}{vx-x}$.

or $\frac{xdv}{dx} = -\frac{1+v}{v-1} - v$

or $\frac{xdv}{dx} = \frac{-1-v-v^2+v}{v-1}$

or $x\frac{dv}{dx} = -\frac{v^2+1}{v-1}$.

Separating the variables

$$\frac{dx}{x} + \frac{v-1}{v^2+1}\,dv = 0$$

or $2\frac{dx}{x} + \frac{2v}{v^2+1}\,dv - \frac{2}{v^2+1}\,dv = 0$

Integrating

$$2\log x + \log(v^2+1) - 2\tan^{-1} v = c$$

or $-2\tan^{-1}\frac{y}{x} + \log x^2\left(\frac{y^2+x^2}{x^2}\right) = c$

∴ $\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log(y^2+x^2) = c$ is the required solution.

Ex. 5: Solve: $\frac{dy}{dx} = -\frac{x-2y}{2x-y}$

Solution:

Here, $\frac{dy}{dx} = -\frac{x-2y}{2x-y}$

This is homogenous. Put y = vx,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

We get, $v + \frac{xdv}{dx} = -\frac{x-2vx}{2x-vx}$

$\frac{xdv}{dx} = -\frac{1-2v}{2-v} - v = \frac{-1+2v-2v+v^2}{2-v}$

or $x\frac{dv}{dx} = -\frac{1-v^2}{2-v}$

Separating the variables

$$\frac{dx}{x} + \frac{2-v}{1-v^2}\,dv = 0,$$

or $\frac{dx}{x} + \frac{1+1-v}{1-v^2}\,dv = 0$

or $\frac{dx}{x} + \frac{1}{1-v^2}\,dv + \frac{1}{1+v}\,dv = 0$

Integrating

$$\log x + \frac{1}{2}\log\frac{1+v}{1-v} + \log(1+v) = \log c$$

or $\left\{\log x\frac{(1+v)^{1/2}}{(1-v)^{1/2}}(1+v)\right\} = \log c$

∴ $(x+y)^3 = c(y-x)$ is the required solution.

Ex. 6: Solve $\frac{dy}{dx} + \frac{3xy+y^2}{x^2+xy} = 0$

Solution:

Here, $\frac{dy}{dx} + \frac{3xy+y^2}{x^2+xy} = 0$

or $\frac{dy}{dx} = -\frac{3xy+y^2}{x^2+xy}$

This is homogenous. Put y = vx, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

We get

$$v + x\frac{dv}{dx} = -\frac{3x.vx+v^2x^2}{x^2+x.vx}$$

or $x\frac{dv}{dx} = -\frac{3v+v^2}{1+v} - v = \frac{-3v-v^2-v-v^2}{1+v}$

or $x\frac{dv}{dx} = -\frac{2v^2+4v}{1+v}$.

Separating the variables

$$\frac{dx}{x} + \frac{1+v}{2v(v+2)}\,dv = 0$$

or $\frac{dx}{x} + \frac{1}{2}\left(\frac{1}{2v} + \frac{1}{2(v+2)}\right)dv = 0$

Integrating

$$\log x + \frac{1}{4}\log v + \frac{1}{4}\log(v+2) = \log c$$

or $\log\{x\,v^{1/4}(v+2)^{1/4}\} = \log c$

or $\{x\,v^{1/4}(v+2)^{1/4}\} = c$

or $x^4 v(v+2) = c^4$

∴ $x^2 y(y+2x) = c$ is the required solution.

Exercise-21

Solve the following differential equations

1. $x + y\frac{dy}{dx} = 2y$
2. $(x^2 - y^2)\,dx + 2xy\,dy = 0$
3. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$
4. $(x^2 + y^2)\,dx = (x^2 + xy)\,dy$
5. $x(x - y)\,dy = y\,(x + y)\,dx$
6. $(x^2 + y^2)\,dy = xy\,dx$
7. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ [2059 B.E.]
8. $x^2dy + y(x + y)\,dx = 0$
9. $x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$
10. $x\sin\frac{y}{x}\,dy = \left(y\sin\frac{y}{x} - x\right)dx$
11. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$ [2061/062 B.E.]
12. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$
13. $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$
14. $(x^2 + 2xy^2)\,dx + (2x^2y + y^2)\,dy = 0$

Answers

1. $\log(y - x) = c + \frac{x}{y - x}$
2. $y^2 + x^2 - cx = 0$
3. $2x - y = cx^2y$
4. $(x - y)^2 = c\,xe^{-y/x}$
5. $xy = e^{x/y}$
6. $y = c\,e^{x^2/2y^2}$
7. $x = c\sin\frac{y}{x}$
8. $x^2y = c(y + 2x)$
9. $y + \sqrt{x^2 + y^2} = c$
10. $\log x = \cos\left(\frac{y}{x}\right) + c$
11. $(y - cx^2)^2 = x^2 + y^2$
12. $x + ye^{x/y} = c$
13. $3x + y\log x = cy$
14. $x^3 + y^3 + 3x^2y^2 = c$

15.5 First Order Differential Equations Reducible to Homogenous form

If the first order differential equation is of the form

$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$, then it can be solved by the following ways:

I. If the first order differential equation is of the form

$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$ and $\frac{a}{A} = \frac{b}{B} = \frac{1}{l}$, then the equations can easily be reduced to the form

$$\frac{dy}{dx} = \frac{ax + by + c}{l\,(ax + by) + C} \text{ and}$$

Put $\quad ax + by = v,$

$$\therefore \quad a + b\frac{dy}{dx} = \frac{dv}{dx}.$$

Separating the variables and integrating we get the required solution.

II. If $\frac{a}{A} \neq \frac{b}{B}$ then the equation can be reduced to the homogeneous form by putting $x = X+h,\ y = Y+k$, where h, k are constants which will be determined by solving the two equations in terms of h and k, the equations are to be chosen in such a way that the differential equation should be homogeneous.

Worked Out Examples

Ex. 1: Solve $\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 5}$

Solution:

Here, $\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 5} = \frac{x - y + 3}{2(x - y) + 5}$

Put $x - y = v$,

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

The equation becomes

$$1 - \frac{dv}{dx} = \frac{v + 3}{2v + 5}$$

$$\frac{dv}{dx} = 1 - \frac{v + 3}{2v + 5} = \frac{2v + 5 - v - 3}{2v + 5} = \frac{v + 2}{2v + 5}$$

Separating the variables

$$dx = \frac{(2v+5)\,dv}{v+2} = \frac{2(v+2)+1}{v+2}\,dv$$

or $dx = \left(2 + \frac{1}{v+2}\right) dv$

Integrating

$$x + c = 2v + \log(v+2)$$

or $x + c = 2x - 2y + \log(x - y + 2)$

$x - 2y + \log(x - y + 2) = c$ is the required solution.

Ex. 2: Solve: $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$

Solution:

Here, $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$

$$\frac{dy}{dx} = -\frac{(2x+y)+1}{2(2x+y)-1}$$

Put $2x + y = v$,

$$2 + \frac{dy}{dx} = \frac{dv}{dx},$$

or $\frac{dy}{dx} = \frac{dv}{dx} - 2$

So, the equation becomes

$$\frac{dv}{dx} - 2 = -\frac{v+1}{2v-1} = -\frac{v+1}{2v-1} + 2$$

$$= \frac{-v+1+4v-2}{2v-1} = \frac{3v-3}{2v-1}$$

or $\frac{dv}{dx} = \frac{3(v-1)}{(2v-1)}$

Separating the variables

$$\frac{2v-1}{v-1}\,dv = 3\,dx$$

or $\left(2 + \frac{1}{v-1}\right) dv = 3\,dx$

Integrating,

$$2v + \log(v-1) = 3x + c$$

or $4x + 2y + \log(2x + y - 1) = 3x + c$

$\therefore x + 2y + \log(2x + y - 1) = c$ is the required solution.

Ex. 3: Solve: $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

Solution:

Here, $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

$$\frac{dy}{dx} = -\frac{3y - 7x + 7}{7y - 3x + 3}$$

Put $x = X + h$, $y = Y + k$ where h, k are constants.

$$\frac{dY}{dX} = -\frac{3Y - 7X + (3k - 7h + 7)}{7Y - 3X + (7k - 3h + 3)}$$

Choose h, k such that $3k - 7h + 7 = 0$

and $7k - 3h + 3 = 0$

Solving these we get $h = 1$, $k = 0$

So, $\frac{dY}{dX} = -\frac{3Y - 7X}{7Y - 3X}$

This is homogenous. Put $Y = vX$,

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

The equation becomes

$$v + X\frac{dv}{dX} = -\frac{3vX - 7X}{7vX - 3X}$$

or $X\frac{dv}{dX} = -\frac{3v-7}{7v-3} - v$

or $X\frac{dv}{dX} = -\frac{(7v^2 - 7)}{7v - 3}$

Separating the variables

$$\frac{dX}{X} + \frac{(7v-3)\,dv}{7(v^2-1)} = 0$$

or $2\frac{dX}{X} + \frac{2v\,dv}{v^2-1} - \frac{6}{7}\frac{1}{v^2-1}\,dv$

Integrating

$$2\log X + \log(v^2 - 1) - \frac{6}{7}\cdot\frac{1}{2}\log\frac{v-1}{v+1} = \log c$$

or $\log X^2 \frac{(v^2-1)}{(v-1)^{3/7}}(v+1)^{3/7} = \log c$

or $X^2 \frac{(v+1)(v-1)(v+1)^{3/7}}{(v-1)^{3/7}} = c$

or $X^2 (v-1)^{4/7} (v+1)^{10/7} = c$

or $(Y - X)^4 (Y + X)^{10} = c^7$

or $(y - 0 - x + 1)^4 (y - 0 + x - 1)^{10} = c^7$

or $(y - x + 1)^2 (x + y - 1)^5 = c^{7/2}$

$\therefore (y - x + 1)^2 (x + y - 1)^5 = c$ is the required solution.

Ex. 4: Solve: (2x + 3y - 5) dy + (3x + 2y - 5) dx = 0

Solution:

Here, $(2x + 3y - 5)\,dy + (3x + 2y - 5)\,dx = 0$

$$\frac{dy}{dx} = -\frac{(3x + 2y - 5)}{2x + 3y - 5}$$

Put $x = X + h$, $y = Y + k$ where h and k are constants.

$$\frac{dy}{dx} = \frac{dY}{dX}.$$

The given equation becomes

$$\frac{dY}{dX} = -\frac{3X + 2Y + (3h + 2k - 5)}{2X + 3Y + (2h + 3k - 5)}$$

Choose h, k such that $3h + 2k - 5 = 0$

and $2h + 3k - 5 = 0$,

Solving these, we get $h = 1, k = 1$,

So, $$\frac{dY}{dX} = -\frac{3X + 2Y}{2X + 3Y}$$

This is homogenous. Put $Y = vX$,

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

So the equation reduces to

$$v + X\frac{dv}{dX} = -\frac{3X + 2vX}{2X + 3vX} = -\frac{3 + 2v}{2 + 3v}$$

or $$X\frac{dv}{dX} = -\frac{3v^2 + 4v + 3}{2 + 3v}$$

Separating the variables

$$\frac{dX}{X} + \frac{(2 + 3v)\,dv}{(3v^2 + 4v + 3)} = 0$$

or $$2\frac{dX}{X} + \frac{(4 + 6v)}{3v^2 + 4v + 3} = 0$$

Integrating

$2\log X + \log(3v^2 + 4v + 3) = \log c$

or $\log X^2 + \log(3v^2 + 4v + 3) = \log c$

or $X^2(3v^2 + 4v + 3) = c$

or $X^2\left(3\frac{Y^2}{X^2} + \frac{4Y}{X} + 3\right) = c$

or $(3Y^2 + 4XY + 3X^2) = c$

or $3(y - 1)^2 + 4(x - 1)(y - 1) + 3(x - 1)^2 = c$

or $3y^2 + 4xy + 3x^2 - 10x - 10y = c$

$\therefore$ $3x^2 + 3y^2 + 4xy - 10x - 10y = c$ is the required solution.

Ex. 5: Solve: $\dfrac{dy}{dx} + \dfrac{2x - y + 1}{2y - x - 1} = 0$

Solution:

Here, $$\frac{dy}{dx} + \frac{2x - y + 1}{2y - x - 1} = 0 \quad \text{.........(1)}$$

Put $x = X + h$, $y = Y + k$ where h and k are constants.

$$\frac{dy}{dx} = \frac{dY}{dX}$$

The given equation becomes

$$\frac{dY}{dx} = -\frac{2X - Y + (2h - k + 1)}{2Y - X + (2k - h - 1)}$$

Choose h, k such that $2h - k + 1 = 0$

and $2k - h - 1 = 0$

Solving these, we get $h = -\frac{1}{3}$, $k = \frac{1}{3}$

The equation (1) reduces to

$$\frac{dY}{dX} = -\frac{2X - Y}{2Y - X} \quad \text{.........(2)}$$

This is homogenous. Put $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dx}$$

Thus the equation (2) becomes

$$v + X\frac{dv}{dX} = \frac{2X - vX}{2vX - X}$$

or $$X\frac{dv}{dX} = -\frac{2 - v}{2v - 1} - v$$

or $$X\frac{dv}{dx} = -\frac{2v^2 - 2v + 2}{2v - 1}$$

Separating the variables

$$2\frac{dX}{X} + \frac{(2v - 1)\,dv}{(v^2 - v + 1)} = 0$$

or $2\log X + \log(v^2 - v + 1) = \log c$

or $X^2(v^2 - v + 1) = c$

or $X^2\left\{\left(\frac{Y}{X}\right)^2 - \frac{Y}{X} + 1\right\} = c$ or $X^2\frac{(Y^2 - XY + X^2)}{X^2} = c$

or $\left\{\left(y - \frac{1}{3}\right)^2 - \left(x + \frac{1}{3}\right)\left(y - \frac{1}{3}\right) + \left(x + \frac{1}{3}\right)^2\right\} = c$

$\therefore$ $3x^2 + 3y^2 - 3xy - 3y + 3x = c$ is the required solution.

Exercise - 22

Solve the differential equations

1. $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$ [2058 B.E.]
2. $(x+y+1)\,dx - (2x+2y+1)\,dy = 0$
3. $(4x+6y+5)\,dy = (3y+2x+4)\,dx$
4. $(2x+2y+3)\,dy - (x+y+1)\,dx = 0$ [2057/2060 B.E.]
5. $\frac{dy}{dx} = \frac{x+y}{x+y-2}$
6. $(6x-5y+4)\,dy + (y-2x-1)\,dx = 0$
7. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$
8. $\frac{dy}{dx} + \frac{2x+3y+1}{3x+5y-1} = 0$
9. $(x-3y+4)\,dy + (7y-5x)\,dx = 0$
10. $(x-y)\,dy - (x+y+1)\,dx = 0$

Answers

1. $y - x - c = \log(x+y)$
2. $6y - 3x = \log(3x+3y+2) + c$
3. $7(6y-3x) - 9\log(14x+21y+22) = c$
4. $6y - 3x + \log(3x+3y+4) = c$
5. $y - x - \log(x+y-1) = c$
6. $(5y-2x-3)^4 = c(4y-4x-3)$ 7. $c^2(x-y)^3 = (x+y-2)$
8. $5y^2 + 2x^2 + 6xy - 2y + 2x = c$ 9. $(3y-4x+10)^2 = c(y-x+1)$
10. $\tan^{-1}\left(\frac{2y+1}{2x+1}\right) = \frac{1}{2}\log\left[c\left\{\left(x+\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2\right\}\right]$

15.6 First Order Linear Differential Equation

If the first order, differential equation is of the form

$\frac{dy}{dx} + Py = Q$ where P and Q are function of x or constant, is called *First Order Linear Differential Equation.*

To solve these types of differential equations, we multiply $e^{\int P dx}$ on both sides of the given equation.

So the equation becomes

$$e^{\int P dx}\frac{dy}{dx} + Py\,e^{\int P dx} = Q\,e^{\int P dx}$$

or $\frac{d}{dx}(y \times e^{\int P dx}) = Q\,e^{\int P dx}$

Integrating

$y \times e^{\int P dx} = \int\left(Q\,e^{\int P dx}\,dx\right) + c$ is the required solution.

So its formula is remembered as

$$y \times e^{\int P dx} = \left(Q\,e^{\int P dx}\,dx\right) + c.$$

The factor $e^{\int P dx}$ is called *Integrating Factor* and is sometimes shortly written as $(I.F.) = e^{\int P dx}$.

So, $y \times I.F. = \int (Q \times I.F.)\,dx + c$

Similarly

The first order differential equation is of the form

$\frac{dx}{dy} + Px = Q$ where P and Q are function of y or constants is called linear in x and its $I.F. = e^{\int P dy}$ then its solution is

$$x \times I.F. = \int (Q \times I.F.)\,dy + c$$

Worked Out Examples

Ex.1 Solve: $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solution:

Here, $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\,y = \frac{4x^2}{1+x^2}$$

This is linear form, $P = \frac{2x}{1+x^2}$, $Q = \frac{4x^2}{1+x^2}$

So, $I.F. = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$

Its general solution is

$$y \times (I.F.) = \int Q \times (I.F.)\,dx$$

or $y(1+x^2) = \int \frac{4x^2}{(1+x^2)} (1+x^2)\,dx + c$

$\therefore\ y(1+x^2) = \frac{4x^3}{3} + c$ is the required solution.

Ex. 2 Solve: $x(x-1)\frac{dy}{dx} - y = \frac{x^2}{(x-1)^2}$

Solution:

Here, $x(x-1)\frac{dy}{dx} - y = \frac{x^2}{(x-1)^2}$

or $\frac{dy}{dx} - \frac{1}{x(x-1)}y = \frac{x}{(x-1)^3}$

This is linear form, $P = -\frac{1}{x(x-1)}$, $Q = \frac{x}{(x-1)^3}$

So, I.F. $= e^{\int P\,dx} = e^{\int \frac{1}{x(x-1)} dx} = e^{\int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx}$

$= e^{\log x - \log(x-1)} = e^{\log \frac{x}{x-1}} = \frac{x}{x-1}$

Its general solution is

$y \times (\text{I.F}) = \int Q \times (\text{I.F.})\,dx$

or $y \times \frac{x}{x-1} = \int \frac{x}{(x-1)^3} \frac{x}{(x-1)} dx + c$

or $y\frac{x}{(x-1)} = \int \frac{x^2}{(x-1)^4} dx + c$

Put $x - 1 = t$, $dx = dt$

$= \int \frac{(t+1)^2}{t^4} dt + c = \int \frac{t^2+2t+1}{t^4} dt + c$

or $\frac{yx}{(x-1)} = \int \left(\frac{1}{t^2} + \frac{2}{t^3} + \frac{1}{t^4}\right) dt = -\frac{1}{t} - \frac{1}{t^2} - \frac{1}{3t^3} + c$

or $\frac{yx}{(x-1)} = -\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{1}{3}\frac{1}{(x-1)^3} + c$

or $3xy = -3 - \frac{3}{(x-1)} - \frac{1}{(x-1)^2} + 3c(x-1)$

$\therefore\ 3xy = -\frac{3x}{x-1} - \frac{1}{(x-1)^2} + 3c(x-1)$ is the required solution.

Ex. 3 Solve: $\cos x \frac{dy}{dx} + y \sin x = \sec^2 x$

Solution:

Here, $\cos x \frac{dy}{dx} + y \sin x = \sec^2 x$

or $\frac{dy}{dx} + \tan x\, y = \sec^3 x$

This is linear form, $P = \tan x$, $Q = \sec^3 x$

So, I.F. $= e^{\int P dx} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

Its general solution is

$y \times \text{I.F.} = \int Q \times (\text{I.F.})\,dx$

or $y \sec x = \int \sec^4 x + c$

or $y \sec x = \int (1 + \tan^2 x) \sec^2 x\, dx + c$

Put $\tan x = t$, $\sec^2 x dx = dt$

$= \int (1 + t^2)\,dt + c = t + \frac{t^3}{3} + c$

or $y \sec x = \tan x + \frac{\tan^3 x}{3} + c$

$\therefore\ 3y \sec x = c + 3\tan x + \tan^3 x$ is the required solution.

Ex. 4: $x\frac{dy}{dx} + 2y = x^2 \log x$

Solution:

Here, $x\frac{dy}{dx} + 2y = x^2 \log x$

or $\frac{dy}{dx} + \frac{2}{x}y = x \log x$

This is linear form, $P = \frac{2}{x}$, $Q = x \log x$

So, I.F. $= e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$

Its general solution is

$y \times \text{I.F.} = \int Q \times (\text{I.F.})\,dx$

or $yx^2 = \int x^3 \log x\, dx + c$

or $yx^2 = \log x \frac{x^4}{4} - \int \frac{1}{x}\frac{x^4}{4} dx + c$

or $yx^2 = \frac{x^4}{4} \log x - \frac{x^4}{16} + c$

$\therefore\ 16x^2y = 4x^4 \log x - x^4 + c$ is the required solution.

Ex. 5: Solve: $(x^3+1)\frac{dy}{dx} + 3x^2 y = \sin^2 x$ [2057 R.E.]

Solution:

Here, $(x^3+1)\frac{dy}{dx}+3x^2y=\sin^2x$

or $\frac{dy}{dx}+\frac{3x^2}{x^3+1}y=\frac{\sin^2x}{x^3+1}$

This is linear form, $P=\frac{3x^2}{x^3+1}$, $Q=\frac{\sin^2x}{x^3+1}$

So, $I.F.=e^{\int Pdx}=e^{\int\frac{3x^2}{x^3+1}dx}=e^{\log(x^3+1)}=x^3+1$

Its general solution is

$$y\times I.F.=\int Q\times(I.F.)\,dx$$

or $y(x^3+1)=\int\frac{\sin^2x}{x^3+1}(x^3+1)\,dx$

$y(x^3+1)=\frac{x}{2}-\frac{\sin 2x}{4}+c$

$\therefore$ $4y(x^3+1)=2x-\sin 2x+c$ is the required solution.

Ex. 6: $(x+y+1)\frac{dy}{dx}=1$

Solution:

Here, $(x+y+1)\frac{dy}{dx}=1$

or $\frac{dy}{dx}=\frac{1}{x+y+1}$,

or $\frac{dx}{dy}=x+y+1$

or $\frac{dx}{dy}-x=y+1$

This is linear form, $P=-1$, $Q=y+1$

$I.F.=e^{\int Pdy}=e^{-1\int dy}=e^{-y}$

Its general solution is

$$x\times I.F.=\int Q\times(I.F.)\,dx$$

or $xe^{-y}=\int(e^{-y}y+e^{-y})\,dy+c$

or $xe^{-y}=-ye^{-y}+\int e^{-y}\,dy+\int e^{-y}\,dy=-ye^{-y}-2e^{-y}+c$

$\therefore$ $(x+y+2)=ce^y$ is the required solution.

Exercise - 23

Solve the following differential equations

1. $\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{1}{(1+x^2)^2}$
2. $\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^2}$
3. $(1-x^2)\frac{dy}{dx}-xy=1$
4. $\frac{dy}{dx}+2y=4x$
5. $(1+x)\frac{dy}{dx}-xy=1-x$
6. $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$
7. $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$
8. $\cos^2x\frac{dy}{dx}+y=\tan x$
9. $\sin x\frac{dy}{dx}+y\cos x=x\sin x$
10. $\frac{dy}{dx}+y\cot x=2\cos x$
11. $\frac{dy}{dx}+y\tan x=\sec x$
12. $x\log x\frac{dy}{dx}+y=2\log x$ [2061 B.E.]
13. $\frac{dy}{dx}+y=\cos x$
14. $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$
15. $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$
16. Solve: $\frac{dy}{dx}+\frac{y}{x}=x^2$ if $y=1$ when $x=1$

Answers

1. $y(1+x^2) = \tan^{-1}x + c$	2. $y = 1 + ce^{\frac{1}{x}}$
3. $y\sqrt{1-x^2} = \sin^{-1}x + c$	4. $y = 2x - 1 + ce^{-2x}$
5. $y(1+x) = x + ce^{x}$	6. $2y\, e^{\tan^{-1}x} = e^{2\tan^{-1}x} + c$
7. $yx \sec x = \tan x + c$	8. $(y - \tan x + 1)\, e^{\tan x} = c$
9. $y \sin x + x \cos x - \sin x = c$	10. $2y \sin x = c - \cos 2x$
11. $y = \sin x + c \cos x$	12. $y \log x = (\log x)^2 + c$
13. $2y = \cos x + \sin x + c\, e^{-x}$	14. $x = (\tan^{-1}y - 1) + c\, e^{-\tan^{-1}y}$
15. $2x\, e^{\tan^{-1}y} + c = e^{2\tan^{-1}y}$	16. $4xy = x^4 + 3$

15.7 Bernoulli's Equation

The first order differential equation of the form

$\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone is called *Bernoulli's Equation*. It can be solved by reducing it to linear equation.

For this

$$\frac{dy}{dx} + Py = Qy^n$$

Dividing both sides by y^n,

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

Put $\frac{1}{y^{n-1}} = v,$

$$y^{-n+1} = v,$$

Differentiating it with respect to x

$$(-n+1)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

So the equation reduces to

$$\frac{1}{(-n+1)}\frac{dv}{dx} + Pv = Q$$

or $\frac{dv}{dx} + P(-n+1)\, v = (-n+1)\, Q$

This is linear in v, so its integrating factor (I.F.) = $e^{\int P(-n+1)\,dx}$ and its general solution is

$$v \times \text{I.F.} = \int (-n+1)\, Q \times (\text{I.F.})\, dx + c$$

Worked Out Examples

Ex. 1: Solve: $\frac{dy}{dx} + \frac{2y}{x} = \frac{y^3}{x^3}$

Solution:

Here, $\frac{dy}{dx} + \frac{2y}{x} = \frac{y^3}{x^3}$

Dividing by y^3,

$$y^{-3}\frac{dy}{dx} + \frac{2}{x}\, y^{-2} = \frac{1}{x^3}$$

Put $y^{-2} = v$, $\quad -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

So the equation reduces to

$$-\frac{1}{2}\frac{dv}{dx} + \frac{2}{x}v = \frac{1}{x^3}$$

or $\frac{dv}{dx} - \frac{4}{x}v = \frac{-2}{x^3}$

This is linear form, $P = \frac{-4}{x}$, $\quad Q = -\frac{2}{x^3}$

So, I.F. $= e^{\int P dx} = e^{-4\log x} = \frac{1}{x^4}$

Its general solution is,

$$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $v\frac{1}{x^4} = -\int \frac{2}{x^7}\, dx + c$

or $\frac{1}{y^2}\frac{1}{x^4} = \frac{1}{3x^6} + c,$

$\therefore$ $3x^2 = y^2 + cx^6y^2$ is the required solution.

Ex. 2: Solve: $\frac{dy}{dx} + \frac{x}{1-x^2}\, y = x\sqrt{y}$

Solution:

Here, $\frac{dy}{dx} + \frac{x}{1-x^2}\, y = x\sqrt{y}$

Dividing by $\sqrt{y}$, $\quad y^{-1/2}\frac{dy}{dx} + \frac{x}{1-x^2}\, y^{1/2} = x$

Put $y^{1/2} = v$, $\quad \frac{1}{2}\, y^{-1/2}\frac{dy}{dx} = \frac{dv}{dx}$

So the equation reduces to

$$2\frac{dv}{dx} + \frac{x}{1-x^2}\, v = x$$

or $\frac{dv}{dx} + \frac{x}{2(1-x^2)} v = \frac{x}{2}$

This is linear form, $P = \frac{x}{2(1-x^2)}$, $Q = \frac{x}{2}$

So I.F. $= e^{\int P dx} = e^{\int \frac{x}{2(1-x^2)} dx}$

$= e^{\int -\frac{(-2x)}{4(1-x^2)} dx} = e^{-\frac{1}{4}\log(1-x^2)} = \frac{1}{(1-x^2)^{1/4}}$

Its general solution is

$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$

or $\frac{v}{(1-x^2)^{1/4}} = \int \frac{x}{2(1-x^2)^{1/4}} dx + c$

Put $1 - x^2 = t$, $-2x\,dx = dt$, $x dx = \frac{-dt}{2}$

$= -\int \frac{dt}{4t^{1/4}} = -\frac{t^{3/4}}{3}$

or $\frac{y^{1/2}}{(1-x^2)^{1/4}} = -\frac{1}{3}(1-x^2)^{3/4} + c$

$\therefore$ $\sqrt{y} = -\frac{1}{3}(1-x^2) + c(1-x^2)^{1/4}$ is the required solution.

Ex. 3: Solve: $x\frac{dy}{dx} + y \log y = xye^x$ 2059/062 B.E.

Solution:

Here, $x\frac{dy}{dx} + y \log y = xye^x$

Dividing by xy,

$y^{-1}\frac{dy}{dx} + \frac{\log y}{x} = e^x$

Put $\log y = v$, $y^{-1}\frac{dy}{dx} = \frac{dv}{dx}$

So the equation reduces to,

$\frac{dv}{dx} + \frac{v}{x} = e^x$

This is linear form, $P = \frac{1}{x}$, $Q = e^x$

$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Its general solution is,

$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$

or $v\,x = \int x e^x dx + c$

$\therefore$ $x \log y = xe^x - e^x + c$ is the required solution.

Ex. 4: Solve: $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$

Solution:

Here, $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$.

Dividing by y^2

$y^{-2}\frac{dy}{dx} - 2y^{-1} \tan x = \tan^2 x$

Put $y^{-1} = v$, $-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

So the equation reduces to

$-\frac{dv}{dx} - 2v \tan x = \tan^2 x$

or $\frac{dv}{dx} + 2 \tan x.\, v = -\tan^2 x$

This is linear form, $P = 2 \tan x$, $Q = -\tan^2 x$.

So, I.F. $= e^{\int P dx} = e^{2\int \tan x\, dx} = e^{2\log \sec x} = \sec^2 x$

Its general solution is

$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$

or $v \sec^2 x = -\int \tan^2 x \sec^2 x\, dx + c$

Put $\tan x = t$, $\sec^2 x\, dx = dt$

or $v \sec^2 x = -\int t^2 dt = -\frac{t^3}{3} - c = -\frac{\tan^3 x}{3} - c$

or $v \sec^2 x = -\frac{\tan^3 x}{3} + c$

$\therefore$ $3\sec^2 x + y \tan^3 x + cy = 0$ is the required solution.

Ex. 5: Solve: $\frac{dy}{dx} + \frac{1}{x} \tan y = \frac{1}{x^2} \tan y \sin y$

Solution:

Here, $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y$.

Dividing tany siny

$$\text{cosec}y \cot y \frac{dy}{dx} + \frac{\text{cosec}y}{x}$$

Put cosecy = v,

$$-\text{cosec}y \cot y \frac{dy}{dx} = \frac{dv}{dx}$$

The equation reduces to

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2}, \text{ or } \frac{dv}{dx} - \frac{v}{x} = -\frac{1}{x^2}$$

This is linear form, $P = \frac{-1}{x}$, $Q = -\frac{1}{x^2}$

So, I.F. $= e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

Its general solution is,

$$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $v\frac{1}{x} = -\int \frac{1}{x^3}\, dx + c$

or $\frac{\text{cosec}y}{x} = \frac{1}{2x^2} + c$

$\therefore$ $2x = (1 + 2cx^2)\sin y$ is the required solution.

Ex. 6: Solve: $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Solution:

Here, $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Dividing by secy,

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)\, e^x$$

Put siny = v, $\cos y \frac{dy}{dx} = \frac{dv}{dx}$

So, the equation reduces to,

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)\, e^x$$

This is linear form, $P = -\frac{1}{1+x}$ $Q = (1+x)\, e^x$

So, I.F. $= e^{\int P dx} = e^{-\int \frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}$

Its general solution is,

$$v \times \text{I.F.} = \int Q \times (\text{I.F.})\, dx$$

or $v \cdot \frac{1}{1+x} = \int e^x\, dx + c$,

or $\frac{\sin y}{1+x} = e^x + c$

or $\sin y = (1+x)(e^x + c)$ is the required solution.

Exercise - 24

Solve the following differential equations

1. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$
2. $(1-x^2)\frac{dy}{dx} + xy = xy^2$
3. $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$
4. $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$ [2067]
5. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ [2061 Back B.E.]
6. $\cos x\frac{dy}{dx} = y(\sin x - y)$
7. $x\frac{dy}{dx} + y = y^2 \log x$
8. $\frac{dy}{dx} = y\tan x - y^2 \sec x$ [2060 B.E.]
9. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\, e^x \sec y$.
10. $\frac{dy}{dx} + \frac{1}{x}\sin 2y = x^3\cos^2 y$

Answers

1. $2x = cx^2y + y$
2. $\sqrt{1-x^2}\,(1-y) = cy$
3. $2x = (cx^2+1)\,e^y$
4. $x = \log y\left(cx^2 + \frac{1}{2}\right)$
5. $e^{x^2}\tan y = \frac{1}{2}e^{x^2}(x^2-1) + c$
6. $\sec x = y(\tan x + c)$
7. $c\,xy + y(\log x + 1) = 1$
8. $\sec x = y\tan x + cy$
9. $\sin y = (1+x)(e^x + c)$
10. $6x^2\tan y = x^6 + c$

15.8 Exact Differential Equation

The first order differential equation $Mdx + Ndy = 0$ where M and N are functions of x, and y is said to be *exact* if there exist a function U(x, y) such that $Mdx + Ndy = d(U(x, y))$

i.e. if $Mdx + Ndy$ is perfect differential.

An exact differential equation can always be obtained by differentiating of its primitive.

For example,

$x^2\,dy + 2xy\,dx = 0$ is *exact* because

$x^2dy + 2xy\,dx = d(x^2y)$ and $U = x^2y$ is the *Primitive* of the given differential equation.

A necessary and sufficient condition for differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Let $Mdx + Ndy = 0$ is exact, we have to show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

For this,

Let U(x, y) = c be its primitive.

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = 0$$

By definition of exactness, it should be equal to given equation.

Thus $M = \frac{\partial U}{\partial x}$ and $N = \frac{\partial U}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y\partial x} \text{ and } \frac{\partial N}{\partial y} = \frac{\partial^2 U}{\partial x\partial y}$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Hence ,the necessary condition.

To prove the sufficient condition.

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, we have to show that $Mdx + Ndy = 0$ is exact.

For this,

Let $M = \frac{\partial U}{\partial x}$,

or $\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y\partial x}$

or $\frac{\partial^2 U}{\partial y\partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

or $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$

Integrating

$$N = \frac{\partial U}{\partial y} + f(y)$$

Thus

$$Mdx + Ndy = \frac{\partial U}{\partial x}dx + \left(\frac{\partial U}{\partial y} + f(y)\right)dy$$

$$= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + f(y)\,dy$$

$$= d\left[U + \int f(y)\,dy\right] = d[U + F(y)].$$

It shows that the equation $Mdx + Ndy = 0$ is exact.

15.9 Rules for Solving Exact Differential Equation

The rules for solving the first order exact differential equation $M\,dx + N\,dy = 0$ are as follows.

I. Integrate M with respect to x treating y as constant.

II. Take those terms of N which are free from x and integrate them with respect to y.

III. Add the above results and equate the sum to some constant, gives the required solution.

15.10 Integrating Factor

If the first order differential equation $Mdx + Ndy = 0$ is linear but not exact then it becomes exact if we multiply it by suitable factor called *Integrating Factor* (I.F.). In many cases, the integrating factors are found by inspection as given below:.

Equation	*I.F.*	*Exact equation*
1. $x\,dy + y\,dx$	1	$x\,dy + y\,dx = d(xy)$
$x\,dy + y\,dx$	$\frac{1}{xy}$	$\frac{x\,dy + y\,dx}{xy} = d(\log xy)$
$xdy + ydx$	$(xy)^n$	$(xy)^n(x\,dy + y\,dx) = \frac{1}{n+1}\,d(xy)^{n+1}$
2. $x\,dy - y\,dx$	$\frac{1}{x^2}$	$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$
$x\,dy - y\,dx$	$\frac{1}{y^2}$	$\frac{x\,dy - y\,dx}{y^2} = -\,d\left(\frac{x}{y}\right)$
$x\,dy - y\,dx$	$\frac{1}{xy}$	$\frac{x\,dy - y\,dx}{xy} = d\left(\log\frac{y}{x}\right)$
$x\,dy - y\,dx$	$\frac{1}{x^2+y^2}$	$\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$
$x\,dy - y\,dx$	$\frac{1}{x^2+y^2}$	$-\frac{y\,dx - x\,dy}{x^2+y^2} = -\,d\left(\tan^{-1}\frac{x}{y}\right)$
3. $x\,dx + y\,dy$	$\frac{1}{2}$	$\frac{x\,dx + y\,dy}{2} = \frac{1}{2}\,d(x^2+y^2)$
$x\,dx + y\,dy$	$\frac{1}{x^2+y^2}$	$\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{1}{2}\,d\left(\log(x^2+y^2)\right)$
4. $dx + dy$	$\frac{1}{x+y}$	$\frac{dx+dy}{x+y} = d\,[\log(x+y)]$
5. $2xy\,dx - x^2dy$	$\frac{1}{y^2}$	$\frac{2xydx - x^2dy}{y^2} = d\left(\frac{x^2}{y}\right)$
6. $2xy\,dy - y^2dx$	$\frac{1}{x^2}$	$\frac{2xy\,dy - y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$

Worked Out Examples

Ex. 1: Solve: $(x^2 + 2xy^2)\,dx + (2x^2y + y^2)\,dy = 0$

Solution:

Here, $(x^2 + 2xy^2)\,dx + (2x^2y + y^2)\,dy = 0$

$$M = x^2 + 2xy^2, \quad N = 2x^2y + y^2$$

$$\frac{\partial M}{\partial y} = 4xy, \quad \frac{\partial N}{\partial x} = 4xy.$$

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Now, $\int M dx$. keeping y is constant

$$= \int (x^2 + 2xy^2)\,dx = \frac{x^3}{3} + x^2y^2$$

Taking the term free from x in N, i.e., y^2

So, $\int y^2 dy = \frac{y^3}{3}$

Hence the required solution is,

$$\frac{x^3}{3} + \frac{y^3}{3} + x^2y^2 = k$$

or $x^3 + y^3 + 3x^2y^2 = c.$

Ex. 2: Solve: $x\,dy + (x + 1)\,dx = 0$

Solution:

Here, $xdy + (x + 1)\,dx = 0$

$$M = (x + 1), \quad N = x$$

$$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = 1.$$

So, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact.

So, the given differential equation can be written as

$$xdy + (x + 1)\,ydx = 0.$$

Separating the variables

$$\frac{dy}{y} + \frac{x+1}{x}\,dx$$

Integrating

$$\log y + x + \log x = c$$

$\log(xy) + x = c$, is the required solution.

Ex. 3: Solve: $(x + y)\,(dx - dy) = dx + dy$

Solution:

Here, $(x + y)\,(dx - dy) = dx + dy.$

The equation can be written as

$$dx - dy = \frac{dx + dy}{x + y}$$

or $\int dx - \int dy = \int d\left(\log(x + y)\right)$

or $c + x - y = \log(x + y)$

$\therefore$ $\log(x + y) = c + x - y$ is the required solution

Ex. 4: Solve: $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

Solution:

Here, $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

or $(\cos^2 x\, dy - 2y \sin x \cos x\, dx) + y^2\, dy = 0$

or $\int d(\cos^2 x\, y) + \int y^2\, dy = 0$ or $\cos^2 x\, y + \frac{y^3}{3} = c$

or $y \cos^2 x + y^3 = c$ or $\frac{3y(1 + \cos 2x)}{2} + y^3 = c,$

$\therefore$ $3y \cos 2x + 2y^3 + 3y = c$ is the required solution.

Ex. 5: Solve: $(x^3y^2 - y)\, dx - (x^2y^3 + x)\, dy = 0$

Solution:

Here, $(x^3y^2 - y)\, dx - (x^2y^3 + x)\, dy = 0$

The equation can be written as

$x^2y^2 (x\, dy - y\, dy) = (y\, dx + x\, dy)$

or $(x\, dx - y\, dy) - \frac{(x\, dy + y\, dx)}{x^2y^2} = 0$

or $\int x\, dx - \int y\, dy + \int d\left(\frac{1}{xy}\right) = 0$

or $\frac{x^2}{2} - \frac{y^2}{2} + \frac{1}{xy} = c$

$\therefore$ $x^3y - xy^3 + 2 = cxy$ is the required solution.

Ex. 6: Solve: $x\, dy - y\, dx = \sqrt{x^2 - y^2}\, dx$ 2062 B.E.

Solution:

Here, $x\, dy - y\, dx = \sqrt{x^2 - y^2}\, dx$

$$\frac{x dy - y dx}{x^2\sqrt{1 - y^2/x^2}} = \frac{dx}{x}$$

or $\int d\left(\sin^{-1}\frac{y}{x}\right) = \int \frac{dx}{x}$

or $\sin^{-1}\frac{y}{x} = \log x + \log c$

or $\sin^{-1}\frac{y}{x} = \log cx$

or $\frac{y}{x} = \sin(\log cx)$

$\therefore y = x \sin(\log cx)$ is the required solution.

Ex. 7: Solve: $y(axy + e^x)\, dx - x\, dy = 0$

Solution:

Here, $y(axy + e^x)\, dx - x\, dy = 0$

or $ax\, dx + \frac{y e^x dx - e^x dy}{y^2} = 0$

or $\int ax\, dx + \int d\left(\frac{e^x}{y}\right) = 0$

or $\frac{ax^2}{2} + \frac{e^x}{y} = c$

$\therefore$ $ax^2y + 2e^x = cy$ is the required solution.

Exercise - 25

Solve the following equations

1. $(2ax + by)\, y\, dx + (ax + 2by)\, x\, dy = 0$
2. $(x^2 - ay)\, dx - (ax - y^2)\, dy = 0$
3. $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$
4. $(x + y)\, dy + (y - x)\, dx = 0$
5. $2xy\, dx - (x^2 - y^2)\, dy = 0$
6. $(x^2 + y^2 + 2x)\, dx + xy\, dy = 0$
7. $x\frac{dy}{dx} + y = y^2 \log x$
8. $xdy - ydx + a(x^2 + y^2)\, dx = 0$ 2058 B.E.
9. $xdx + ydy + (x^2 + y^2)\, dy = 0$
10. $(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$
11. $\sin x\, dy - y \cos x\, dx + y^2\, dx = 0$
12. $x\frac{dy}{dx} = y + x^2 \log x$
13. $x \cos\left(\frac{y}{x}\right)(y\, dx + x\, dy) = y \sin\left(\frac{y}{x}\right)(x\, dy - y\, dx)$
14. $\cos x (\cos x - \sin y)\, dx + \cos y (\cos y - \sin x)\, dy = 0$
15. $(x + 2y^3)\, dy = y\, dx$

16. $x^3y^3\,dx + 3x^2y\,dy + 2y\,dx = 0$

17. $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \dfrac{\sqrt{a^2 - x^2 - y^2}}{x^2 + y^2}$

18. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

19. $x^2\,dy + xy\,dx + 2\sqrt{1 - x^2y^2}\,dx = 0$

20. $x\,dx + y\,dy = a^2\dfrac{(x\,dy - y\,dx)}{x^2 + y^2}$

21. $(x^2 + y^2)\,dx - 2xy\,dy = 0$

Answers

1. $ax^2y + bxy^2 = c$
2. $x^3 + y^3 - 3axy = c$
3. $y^2 - x^2 + xy - 3y - x = c$
4. $x^2 - y^2 - 2xy = c$
5. $x^2 + y^2 = cy$
6. $3x^4 + 8x^3 + 6x^2y^2 = c$
7. $cxy + 1 = y(1 + \log x)$
8. $\tan^{-1}\left(\dfrac{y}{x}\right) + ax = c$
9. $x^2 + y^2 = c\,e^{-2y}$
10. $\log\left(\dfrac{x}{y}\right) = c + \dfrac{1}{xy}$
11. $\sin x = y\,(x + c)$
12. $y = x^2\log x - x^2 + c$
13. $xy\cos\left(\dfrac{y}{x}\right) = c$
14. $2x + 2y + \sin 2x + \sin 2y - 4\sin y\sin x = c$
15. $y^3 + cy = x$
16. $x(xy - 2)^3 = c\,(xy - 1)$
17. $\sqrt{x^2 + y^2} = a\sin\left(\tan^{-1}\left(\dfrac{y}{x}\right) + c\right)$
18. $x + \log(x^2 + y^2) = c$
19. $\sin^{-1}(xy) + 2\log x = c$
20. $x^2 + y^2 = 2a^2\tan^{-1}\left(\dfrac{y}{x}\right) + c$
21. $y^2 = x\,(x + c)$

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Chapter - 16

First Order but not First Degree Differential Equations

16.1 Introduction

Let $p = \dfrac{dy}{dx}$. The first order differential equation of the form $f(x, y, p) = 0$ is called *First Order but not First Degree* differential equation. The solution of such equations contains only one arbitrary constant.

16.2 Solvable for p

If differential equation $f(x, y, p) = 0$ can be factorized into linear factors such as $\{p - f_1(x, y)\}\ \{p - f_2(x, y)\}\ \ldots\ \{p - f_n(x, y)\}$, then it is called *Solvable* for p. Each factor equated to zero, we get the solution of the form

$$F_1(x, y, c_1) = 0,\ F_2(x, y, c_2) = 0,\ \ldots\ldots\ F_n(x, y, c_n) = 0.$$

Its general solution is

$$F_1(x, y, c)\ F_2(x, y, c)\ \ldots\ldots\ F_n(x, y, c) = 0.$$

Worked Out Examples

Ex. 1: Solve $p^2 - xy = y^2 - px$

Solution:

Here, $p^2 - y^2 - xy + px = 0$

or $(p - y)(p + y) + x(p - y) = 0$

or $(p - y)(p + y + x) = 0$

Either,

$$p - y = 0 \Rightarrow \frac{dy}{dx} - y = 0$$

$$\Rightarrow \int\frac{dy}{y} - \int dx = 0$$

$$\Rightarrow \log y - x = \log c$$

$$\Rightarrow \frac{y}{c} = e^x$$

or $\quad p + y + x = 0 \Rightarrow \dfrac{dy}{dx} + y = -x$

Which is linear form, $P = 1,\ Q = -x$

So, $I.F. = e^{\int P dx} = e^{\int dx} = e^x$

Its general solution is

$$y \times I.F. = \int Q \times (I.F.)\, dx$$

$$\Rightarrow y.x = \int (-x).e^x dx = -x e^x - \int (-1) e^x dx = -x e^x + e^x + c$$

$$\Rightarrow xy = -x e^x + e^x + c$$

$$\Rightarrow y = -x + 1 + ce^{-x}$$

$$\Rightarrow y + x - 1 - ce^{-x} = 0$$

Hence, the general solution of given differential equation is

$$(y - ce^x)(y + x - ce^{-x} - 1) = 0$$

Ex. 2: Solve $p^2 - 2p - 3 = 0$

Solution:

Here, $p^2 - 2p - 3 = 0$

or $(p - 3)(p + 1) = 0$

Either, $\frac{dy}{dx} - 3 = 0 \Rightarrow dy - 3dx = 0$

$$\Rightarrow y - 3x = c$$

$$\Rightarrow y - 3x - c = 0$$

or $p + 1 = 0 \Rightarrow \frac{dy}{dx} + 1 = 0$

$$\Rightarrow dy + dx = 0$$

$$\Rightarrow y + x = c$$

$$\Rightarrow y + x - c = 0$$

Hence, the general solution is $(y - 3x + c)(y + x + c) = 0$

Ex. 3: Solve: $(p - xy)(p - x^2)(p - y^2) = 0$

Solution:

Here, $(p - xy)(p - x^2)(p - y^2) = 0$

Either, $p - xy = 0 \Rightarrow \frac{dy}{dx} - xy = 0,$

$$\Rightarrow \int \frac{dy}{y} - \int x\, dx = 0$$

$$\Rightarrow \log y - \frac{x^2}{2} + \log c = 0$$

$$\Rightarrow \log (yc) = \frac{x^2}{2}$$

$$\Rightarrow yc = e^{x^2/2}$$

$$\Rightarrow e^{x^2/2} - yc = 0$$

or $p - x^2 = 0 \Rightarrow \frac{dy}{dx} - x^2 = 0$

$$\Rightarrow dy - x^2 dx = 0$$

$$\Rightarrow y - \frac{x^3}{3} = c$$

$$\Rightarrow x^3 - 3y + c = 0$$

or $p - y^2 = 0 \Rightarrow \frac{dy}{dx} - y^2 = 0$

$$\Rightarrow \int \frac{dy}{y^2} - \int dx = 0$$

$$\Rightarrow \frac{-1}{y} - x = c$$

$$\Rightarrow (xy + cy + 1) = 0$$

Hence, the general solution is

$$(e^{x^2/2} - cy)(x^3 - 3y + c)(xy + cy + 1) = 0$$

Ex. 4: Solve: $p^3 + 3xp^2 - y^3 p^2 - 3xy^3 p = 0$

Solution:

Here, $p^3 + 3xp^2 - y^3 p^2 - 3xy^3 p = 0$

or $(p + 3x)(p^2 - y^3 p)$

or $p^2 (p + 3x) - y^3 p (p + 3x) = 0$

or $p(p + 3x)(p - y^3) = 0$

Either, $p = 0, \Rightarrow \frac{dy}{dx} = 0,$

$$\Rightarrow \int dy = 0$$

$$\Rightarrow y = c$$

$$\Rightarrow y - c = 0$$

or $p + 3x = 0 \Rightarrow \frac{dy}{dx} + 3x = 0$

$$\Rightarrow \int dy + \int 3x\, dx = 0$$

$$\Rightarrow y + \frac{3x^2}{2} = c$$

$$\Rightarrow 2y + 3x^2 - 2c = 0$$

or $p - y^3 = 0 \Rightarrow \frac{dy}{dx} - y^3 = 0,$

$$\Rightarrow \int \frac{dy}{y^3} - \int dx = 0$$

$$\Rightarrow \frac{-1}{2y^2} - x + c = 0$$

$$\Rightarrow 2xy^2 + 1 - 2cy^2 = 0$$

Hence, the general solution is

$$(y - c)\left(y + \frac{3}{2}x^2 - c\right)(2xy^2 + 1 - 2cy^2) = 0$$

Exercise - 26

Solve the following equations

1. $p^2 + p - 6 = 0$
2. $p^2 + 2px - 3x^2 = 0$
3. $p^2 - p(e^x + e^{-x}) + 1 = 0$
4. $p(p^2 + xy) = p^2(x + y)$
5. $yp^2 + (x - y)p - x = 0$
6. $p^2 + 2px + py + 2xy = 0$
7. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$
8. $p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$
9. $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$
10. $\left(\frac{dy}{dx}\right)^2 + 2y\cot x\frac{dy}{dx} = y^2$

Answers

1. $(y + 3x - c)(y - 2x - c) = 0$
2. $(2y + 3x^2 - c)(2y - x^2 - c) = 0$
3. $(y - e^x - c)(y + e^{-x} - c) = 0$
4. $(y - c)(2y - x^2 - c)(y - ce^x) = 0$
5. $(y^2 + x^2 - c)(y - x - c) = 0$
6. $(y + x^2 - c)(x + \log y - c)$
7. $(y - c)(x^2 + y - c)(xy + 1 + cy) = 0$
8. $(2y - x^2 - c)(y - ce^x)(y + x - 1 - ce^{-x}) = 0$
9. $(yx^3 - c)(y - cx^2) = 0$
10. $(y + y\cos x - c)(y - y\cos x - c) = 0$

16.3 Solvable for y

If the first order but not first degree differential equation is of the form $y = f(x, p)$ then it can be solved by the method of *Solvable for y*.

Let $y = f(x, p)$ ....(1)

Differentiating with respect to x,

$$\frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

or $$p = \phi\left(x, p, \frac{dp}{dx}\right) \quad ....(2)$$

Which is differential equation in the two variables x and p. Suppose its solution is $F(x, p, c) = 0$ ....(3)

Eliminating p from (1) and (3) gives the required solution. If p can not be eliminated from given two equations (1) and (3), then the equations (1) and (3) together give required solution.

16.4 Solvable for x

If the first order but not first degree differential equation is of the form $x = f(y, p)$ then it can be solved by the method of *Solvable* for x.

Let $x = f(y, p)$ ....(1)

Differentiating with respect to y, we get

$$\frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

or $$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \quad ....(2)$$

Which is a differential equation of two variables y and p. Suppose its solution be $F(y, p, c) = 0$ ....(3)

Eliminating p from (1) and (3) gives the required solution.

If p can not be eliminated from the given two equations it is customary to express x and y in terms of p, give the solution of the equation.

Worked Out Examples

Ex. 1: Solve $y = 2p^2 + 2px$ [2059 B.E.]

Solution:

Here, $y = 2p^2 + 2px$,

or $x = \frac{y}{2p} - \frac{2p^2}{2p}$

or $x = \frac{1}{2}\frac{y}{p} - \frac{1}{2}yp$ ....(1)

Which is solvable for x, differentiate it with respect to y,

$$\frac{dx}{dy} = \frac{1}{2p^2}\left[p - y\frac{dp}{dy}\right] - \frac{1}{2}\left(y\frac{dp}{dy} + p\right)$$

or $\frac{1}{p} = \frac{1}{2p} - \frac{1}{2}y\frac{dp}{dy}\left(\frac{1}{p^2} + 1\right) - \frac{p}{2}$

or $\frac{1}{p} - \frac{1}{2p} + \frac{p}{2} = -\frac{y}{2}\frac{dp}{dy}\left(\frac{1}{p^2+1}\right)$

or $\frac{(p^2+1)}{2p} = -\frac{y}{2}\frac{dp}{dy}\frac{(p^2+1)}{p^2}$

or $\int\frac{dy}{y} + \int\frac{dp}{p} = 0$

or $\log y + \log p = \log c$

or $p = \frac{c}{y}$ ....(2)

Eliminating p from (1) and (2), we get

$$y = y\frac{c^2}{y^2} + 2\frac{c}{y}x$$

or $y^2 = c^2 + 2cx$ is the required general solution.

Ex. 2: Solve: $y - 2px + ayp^2 = 0$ [2061 B.E.]

Solution:

Here, $y - 2px + ayp^2 = 0$,

or $x = \frac{y}{2p} + \frac{ayp^2}{2p}$

or $x = \frac{y}{2p} + \frac{a}{2}yp$

Which is solvable for x, differentiate it with respect to y,

$$\frac{dx}{dy} = \frac{1}{2}\frac{\left(p - y\frac{dp}{dy}\right)}{p^2} + \frac{a}{2}\left(y\frac{dp}{dy} + p\right)$$

or $\frac{1}{p} = \frac{1}{2p} + \frac{ap}{2} + \frac{y}{2}\frac{dp}{dy}\left(a - \frac{1}{p^2}\right)$

or $-\frac{(ap^2-1)}{2p} = \frac{y}{2p^2}\frac{dp}{dy}(ap^2-1)$

or $\int\frac{dy}{y} + \int\frac{dp}{p} = 0$

or $\log y + \log p = \log c$

or $yp = c$

or $p = \frac{c}{y}$ ....(2)

Eliminating p from (1) and (2), we get

$$y - 2\frac{c}{y}x + ay\frac{c^2}{y^2} = 0$$

or $y^2 + ac^2 = 2cx$ is the required solution.

Ex. 3: Solve: $p^3 - 4xyp + 8y^2 = 0$

Solution:

Here, $p^3 - 4xyp + 8y^2 = 0$

or $x = \frac{p^3}{4yp} + \frac{8y^2}{4yp}$

or $x = \frac{p^2}{4y} + \frac{2y}{p}$ ....(1)

Which is solvable for x, differentiate with respect to y,

$$\frac{dx}{dy} = \frac{1}{4}\frac{y\,2p\frac{dp}{dy} - p^2}{y^2} + 2\left(\frac{p - y\frac{dp}{dy}}{p^2}\right)$$

or $\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2}{p} + \frac{y}{2}\frac{dp}{dy}\left(\frac{p}{y^2} - \frac{4}{p^2}\right)$

or $\frac{1}{p} + \frac{p^2}{4y^2} - \frac{2}{p} = \frac{ydp}{2dy}\left(\frac{p}{y^2} - \frac{4}{p^2}\right)$

or $\frac{(p^3-4y^2)}{4py^2} = \frac{y}{2}\frac{dp}{dy}\frac{(p^3-4y^2)}{y^2p^2}$

or $\frac{1}{2} = y\frac{dp}{pdy}$

or $\int\frac{dy}{y} = 2\int\frac{dp}{p}$

or $2\log p = \log y + \log c$

or $p^2 = yc$ ....(2)

Eliminating p from (1) and (2), we get

$$x = \frac{yc}{4y} + \frac{2y}{\sqrt{yc}}$$

or $x = \frac{c}{4} + \frac{2\sqrt{y}}{\sqrt{c}}$

or $c(4x - c)^2 = 64y$ is the required solution.

Ex. 4: Solve: $xp^2 - 2yp + ax = 0$

Solution:

Here, $xp^2 - 2yp + ax = 0$

$$y = \frac{ax}{2p} + \frac{px}{2} \quad \text{......(1)}$$

This is solvable for y, differentiating it with respect to x.

$$\frac{dy}{dx} = \frac{a}{2}\frac{\left(p - x\frac{dp}{dx}\right)}{p^2} + \frac{1}{2}\left(p + x\frac{dp}{dx}\right)$$

or $p = \frac{a}{2p} + \frac{p}{2} + \frac{1}{2}x\frac{dp}{dx}\left(1 - \frac{a}{p^2}\right)$

or $p - \frac{a}{2p} - \frac{p}{2} = \frac{x}{2}\frac{dp}{dx}\left(1 - \frac{a}{p^2}\right)$

or $x\frac{dp}{dx}\left(1 - \frac{a}{p^2}\right) = \left(p - \frac{a}{p}\right)$

or $x\frac{dp}{dx}\frac{(p^2 - a)}{p^2} = \frac{(p^2 - a)}{p}$

or $\frac{dp}{dx}\frac{x}{p} = 1$

or $\int\frac{dp}{p} = \int\frac{dx}{x}$

or $\log p = \log x + \log c,$

or $p = cx$ ......(2)

Eliminating p from (1) and (2), we get

$c^2x^3 - 2ycx + ax = 0$

or $2y = cx^2 + \frac{a}{c}$ is the required general solution.

Ex. 5: Solve: $y = p^2 + x$

Solution:

Here, $y = p^2 + x$ ......(1)

Which is solvable for y, differentiate it with respect to x,

or $\frac{dy}{dx} = 2p\frac{2p}{dx} + 1$

or $(p - 1) = 2p\frac{dp}{dx}$

or $dx = \frac{2p}{p-1}\,dp$

or $\int dx = \int\left(2 + \frac{2}{p-1}\right)dp$

$x = 2p + 2\log(p - 1) - c$ ......(2)

Since p can not easily be eliminated from (1) and (2)

Thus $x = 2p + 2\log(p - 1) - c$

$y = p^2 + 2p + 2\log(p - 1) - c$ are the required solution.

Ex. 6: Solve: $y = 2px + p^2$

Solution:

Here $y = 2px + p^2$ ......(1)

This is solvable for y, differentiating it with respect to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or $p = 2p + 2(x + p)\frac{dp}{dx}$

or $-p = 2(x + p)\frac{dp}{dx}$

or $\frac{dx}{dp} + \frac{2x}{p} = -2$

Which is linear form, $P = \frac{2}{p}$, $Q = -2$

So, I.F. $= e^{\int P dp} = e^{2\int\frac{1}{p}dp} = e^{2\log p} = e^{\log p^2} = p^2$

So, its general solution is

$x \times \text{I.F.} = \int Q \times (\text{I.F.})\,dp$

or $x.\,p^2 = -2\int p^2\,dp + c$

or $xp^2 = -\frac{2}{3}p^3 + c$ ......(2)

Eliminating p from (1) and (2)

From (1), $p = -x \pm \sqrt{x^2 + y}$

From (2), $x\left[-x + \sqrt{x^2 + y}\right]^2 + \frac{2}{3}\left[-x + \sqrt{x^2 + y}\right]^3 + c = 0$

or $x\left[x^2 - 2x\sqrt{x^2 + y} + x^2 + y\right]$
$+ \frac{2}{3}\left[-x^3 + (x^2 + y)^{3/2} + 3x^2\sqrt{x^2 + y} - 3x(x^2 + y)\right] + c = 0$

or $2x^3 - 2x^2\sqrt{x^2 + y} + xy - \frac{2}{3}x^3 + \frac{2}{3}(x^2 + y)^{3/2}$
$+ 2x^2\sqrt{x^2 + y} - 2x(x^2 + y) + c = 0$

or $3xy - 2x^3 - 6xy + 3c = -2(x^2 + y)^{3/2}$

Squaring

$(2x^3+3xy+c)^2 - 4(x^2+y)^3 = 0$ is the required solution.

Ex. 7: Solve: $xy^2(p^2+2) = 2py^3 + x^3$ **[2060/062 B.E.]**

Solution:

Here $xy^2p^2 + 2xy^2 = 2py^3 + x^3$

or $xy^2p^2 + 2xy^2 - 2py^3 - x^3 = 0$

or $xy^2p^2 - x^3 + 2xy^2 - 2py^3 = 0$

or $x(y^2p^2 - x^2) + 2y^2(x - py) = 0$

or $x(yp + x)(yp - x) - 2y^2(py - x) = 0$

or $(py - x)(xyp + x^2 - 2y^2) = 0$

Either

$py - x = 0 \quad \Rightarrow y\dfrac{dy}{dx} - x = 0$

$\Rightarrow y\,dy - x\,dx = 0$

Integrating $\quad y^2 - x^2 = c$

or $xyp + x^2 - 2y^2 = 0$

or $xy\dfrac{dy}{dx} = 2y^2 - x^2$

or $\dfrac{dy}{dx} = \dfrac{2y^2 - x^2}{xy}$

Put $y = vx$, $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

So, $v + x\dfrac{dv}{dx} = \dfrac{2v^2x^2 - x^2}{xvx} = \dfrac{x^2(2v^2-1)}{x^2v}$

or $x\dfrac{dv}{dx} = \dfrac{2v^2-1}{v} - v = \dfrac{2v^2-1-v^2}{v}$

or $x\dfrac{dv}{dx} = \dfrac{v^2-1}{v}$

or $\dfrac{dx}{x} = \dfrac{v}{v^2-1}dv$

Integrating, we get

$\log x = \dfrac{1}{2}\log(v^2-1) + \dfrac{1}{2}\log c$

or $2\log x = \log(v^2-1)c$

or $x^2 = (v^2-1)c$

or $x = \sqrt{(v^2-1)c}$

or $x = \sqrt{\left(\dfrac{y^2}{x^2} - 1\right)c}$

or $x^2 = \dfrac{y^2 - x^2}{x^2}c$

or $x^4 = (y^2 - x^2)c$

Thus the required solution is

$(y^2 - x^2 - c)(x^4 - cy^2 + cx^2) = 0$

Exercise-27

Find the general solution of the following differential equations

1. $p^3x - p^2y - 1 = 0$
2. $y = 2px + p^3y^2$
3. $x = 4p + 4p^3$
4. $\sin y \cos px - \cos y \sin px = p$
5. $y + px = x^4p^2$
6. $p^2 - py + x = 0$
7. $e^y - p^3 - p = 0$
8. $4(xp^2 + yp) = y^4$
9. $x + \dfrac{p}{\sqrt{1-p^2}} = a$
10. $xp^3 = a + bp$
11. $y = 2px + p^2$
12. $p^3 - p(y+3) + x = 0$
13. $x + yp = ap^2$
14. $y = \sin p - p\cos p$
15. $p^2 - 2px + 1 = 0$
16. $y = (1+p)x + ap^2$

Answers

1. $c^3x - c^2y - 1 = 0$
2. $y^2 = 2cx + c^3$
3. $x = 4p + 4p^3,\ y = 2p^2 + 3p^4 + c$
4. $\sin(y - cx) = c$
5. $xy + c = c^2x$
6. $x = py - p^2,\ x = \dfrac{p\left[\log\left(p + \sqrt{p^2-1}\right) + c\right]}{\sqrt{p^2-1}}$
7. $y = \log(p^3+p),\ x = 2\tan^{-1}p - \dfrac{1}{p} + c$
8. $y = 4c(cxy + 1)$
9. $(y+c)^2 - (x-a)^2 = 1$
10. $x = \dfrac{a}{p^3} + \dfrac{b}{p^2},\quad y = \dfrac{3a}{2p^2} + \dfrac{2b}{p} + c$
11. $(3xy + 2x^3 + c)^2 - 4(x^2+y)^3 = 0$
12. $x = p(y+3) - p^3,\ y\sqrt{1-p^2} + (1-p^2)^{3/2} = c$
13. $x(1+p^2)^{1/2} = p\left\{a\log\left(p + \sqrt{1+p^2}\right) + c\right\},\ yp + x = ap^2$
14. $x = c - \cos\left[\dfrac{\sqrt{1-(c-x)^2} - y}{c - x}\right]$
15. $x = \dfrac{1}{2}\left(p + \dfrac{1}{p}\right),\quad y = \dfrac{1}{4}p^2 - \dfrac{1}{2}\log p + c$
16. $y = 2a - ap^2 + (1+p)e^{-p}\quad x = 2a(1-p) + ce^{-p}$

18.5 Clairaut's Equation

A first order but not first-degree differential equation of the form $y = px + f(p)$ is called *Clairaut's Equation.*

It can solved by the following method.

Let, $y = px + f(p)$ ......(1)

Differentiating with respect to x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or $p = p + [x + f'(p)]\frac{dp}{dx}$

or $0 = [x + f'(p)]\frac{dp}{dx}$

Either $\frac{dp}{dx} = 0 \Rightarrow dp = 0$

$\Rightarrow p = c$ ......(2)

Eliminating p from (1) and (2) we get

$y = cx + f(c)$ is the required general solution.

or $[x + f'(p)] = 0 \Rightarrow f'(p) = -x$ ......(3)

Eliminating p from (1) and (3) gives the required singular solution.

Worked Out Examples

Ex. 1: Solve: $y = px + p^n$

Solution:

Here, $y = px + p^n$ ......(1)

This is Clairaut's equation

Differentiating it with respect to x,

or $\frac{dy}{dx} = p + x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx}$,

or $p = p + x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx}$

or $p = p + (x + np^{n-1})\frac{dp}{dx}$

or $\frac{dp}{dx}(x + np^{n-1}) = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow dp = 0$ ......(2)

$\Rightarrow p = c$

Eliminating p between (1) and (2) we get

$y = cx + c^n$ is the required *general* solution.

or $x + np^{n-1} = 0$ ......(3)

Eliminating p from (1) and (3)

From (3), $np^{n-1} = -x \Rightarrow p^n = -\frac{px}{n}$

From (1), $y = px - \frac{px}{n}$,

or $y = \frac{p(n-1)x}{n} \Rightarrow p = \frac{ny}{(n-1)x}$

From (3), $x + n\left(\frac{ny}{(n-1)x}\right)^{n-1} = 0$

or $x^n(n-1)^{n-1} + n^n y^{n-1} = 0$ is the required *singular* solution.

Ex. 2 Solve: $y = px + \sqrt{a^2p^2 + b^2}$

Solution:

Here, $y = px + \sqrt{a^2p^2 + b^2}$ ......(1)

This is Clairaut's equation.

Differentiating it with respect to x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{1}{2}\frac{2a^2p}{\sqrt{a^2p^2 + b^2}}\frac{dp}{dx}$$

or $p = p + \left(x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}}\right)\frac{dp}{dx}$

or $\frac{dp}{dx}\left(x + \frac{a^2}{\sqrt{a^2p^2+b^2}}\right) = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow dp = 0$

$\Rightarrow p = c$ ......(2)

Eliminating p from (1) and (2), we get

$y = cx + \sqrt{a^2c^2 + b^2}$ is the required *general* solution.

or $x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}} = 0$ ......(3)

Eliminating p from (1) and (3)

From (3), $\frac{a^2p}{\sqrt{a^2p^2 + b^2}} = -x$

Squaring

$a^4p^2 = x^2(a^2p^2 + b^2) \Rightarrow a^2(a^2 - x^2)p^2 = b^2x^2$

$\Rightarrow p^2 = \frac{b^2x^2}{a^2(a^2 - x^2)}$

$$\Rightarrow p = \pm \frac{bx}{a\sqrt{a^2 - x^2}}$$

From (1), $y = \frac{-bx}{a\sqrt{a^2 - x^2}}\, x + \sqrt{\frac{a^2b^2x^2}{a^2(a^2 - x^2)} + b^2}$

or $y = \frac{-bx^2}{a\sqrt{a^2 - x^2}} + \sqrt{\frac{b^2x^2 + a^2b^2 - b^2x^2}{a^2(a^2 - x^2)}}$

or $y = \frac{-bx^2}{a\sqrt{a^2 - x^2}} + \frac{ab}{\sqrt{a^2 - x^2}}$

or $y = \frac{-bx^2 + a^2b}{a\sqrt{a^2 - x^2}} = \frac{b}{a}\frac{a^2 - x^2}{\sqrt{a^2 - x^2}}$

or $y = \frac{b}{a}\sqrt{a^2 - x^2}$

or $a^2y^2 = b^2a^2 - b^2x^2$

or $b^2x^2 + a^2y^2 = a^2b^2$

or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the required *singular* solution.

Ex. 3: Solve: $(x - a)p^2 + (x - y)p - y = 0$

Solution:

Here, $(x - a)p^2 + (x - y)p - y = 0$

or $xp^2 - ap^2 + xp - py - y = 0$

or $px(p + 1) - ap^2 - y(p + 1) = 0$

or $y(p + 1) = px(p + 1) - ap^2$

or $y = px - \frac{ap^2}{p + 1}$ ....(1)

This is Clairaut's equation, differentiating it with respect to x,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{(p + 1)\, 2ap\frac{dp}{dx} - ap^2\frac{dp}{dx}}{(p + 1)^2}$$

or $\frac{dp}{dx}\left(x - \frac{ap^2 + 2ap}{(p + 1)^2}\right) = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow dp = 0$

$\Rightarrow p = c$ ......(2)

Eliminating p from (1) and (2), we get

$y = cx - \frac{ac^2}{c + 1}$ is the required *general* solution.

or $x - \frac{ap^2 + 2ap}{(p + 1)^2} = 0 \quad \Rightarrow x = \frac{ap^2 + 2ap}{(p + 1)^2}$ ......(3)

Eliminating p from (1) and (3),

Putting the value of x from (3) in (1)

$$y = p\frac{(ap^2 + 2ap)}{(p + 1)^2} - \frac{ap^2}{(p + 1)}$$

$$= \frac{1}{(p + 1)}\frac{(ap^3 + 2ap^2 - ap^3 - ap^2)}{(p + 1)}$$

$$= \frac{ap^2}{(p + 1)}$$ ......(4)

Adding (3) and (4)

$$x + y = p\frac{ap^2 + 2ap}{(p + 1)^2} - \frac{(ap^2 + 2ap)}{(p + 1)^2}$$

$$= (p + 1)\frac{ap^2 + 2ap}{(p + 1)^2} - \frac{ap^2}{(p + 1)}$$

or $x + y = \frac{2ap}{p + 1}$

or $(x + y)^2 = \frac{4a^2p^2}{(p + 1)^2}$

$= 4a\frac{ap^2}{(p + 1)^2} = 4ay$ [from (4)]

$(x + y)^2 = 4ay$ is the required *singular* solution.

Ex. 4: Solve: $y = px - \sqrt{m^2 + p^2}$

Solution:

Here, $y = px - \sqrt{m^2 + p^2}$ ....(1)

This is Clairaut's equation, differentiating it with respect to x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{2\sqrt{m^2 + p^2}}\, 2p\frac{dp}{dx}$$

or $p = p + \left(x + \frac{p}{\sqrt{m^2 + p^2}}\right)\frac{dp}{dx}$

or $0 = \left(x + \frac{p}{\sqrt{m^2 + p^2}}\right)\frac{dp}{dx}$

Either $\frac{dp}{dx} = 0 \quad \Rightarrow p = c$ ......(2)

Eliminating p from(1) and (2), we get

$y = cx - \sqrt{m^2 + c^2}$ is the required *general* solution.

or $x - \frac{p}{\sqrt{m^2+p^2}} = 0$ ...(3)

Eliminating p from (1) and (3),

From (3), $x^2 = \frac{p^2}{m^2+p^2}$

$\Rightarrow m^2x^2 + p^2x^2 = p^2 \quad \Rightarrow m^2x^2 = p^2(1-x^2)$

$\Rightarrow p^2 = \frac{m^2x^2}{1-x^2} \quad \Rightarrow p = \frac{mx}{\sqrt{1-x^2}}$

From (1), $y = \frac{mx}{\sqrt{1-x^2}} . x - \sqrt{m^2 + \frac{m^2x^2}{1-x^2}}$

$\Rightarrow y = \frac{mx^2}{\sqrt{1-x^2}} - \sqrt{\frac{m^2 - m^2x^2 + m^2x^2}{1-x^2}}$

$\Rightarrow y = \frac{mx^2}{\sqrt{1-x^2}} - \frac{m}{\sqrt{1-x^2}} \quad \Rightarrow y = \frac{m(x^2-1)}{\sqrt{1-x^2}}$

$\Rightarrow y = -\frac{m(1-x^2)}{\sqrt{1-x^2}} \quad \Rightarrow y = -m\sqrt{1-x^2}$

$\Rightarrow m^2(1-x^2) = y^2$

$y^2 + m^2x^2 = m^2$ is the required *singular* solution.

Exercise - 28

1. $y = px + p - p^2$
2. $y = px + \frac{a}{p}$
3. $py = p^2(x-b) + a$
4. $y = px + ap - ap^2$
5. $(y-px)^2(1+p^2) = a^2p^2$
6. $p = \log(px - y)$
7. $(y+1)p - xp^2 + 2 = 0$
8. $(xp-y)^2 = p^2 - 1$

Answers

1. $y = cx + c - c^2$, $4y = (x+1)^2$
2. $y = cx + \frac{a}{c}$, $y^2 = 4ax$
3. $y^2 = 4a(x-b)$
4. $y = cx + ac - ac^2$, $(x+a)^2 = 4ay$
5. $(y-cx)^2(1+c^2) = a^2c^2$, $x^{2/3} + y^{2/3} = a^{2/3}$
6. $y = cx - e^c$, $y = x\log x - x$
7. $(y+1)c - xc^2 + 2 = 0$, $8x + (y+1)^2 = 0$
8. $(xc-y)^2 = c^2 - 1$, $x^2 - y^2 = 1$

✦✦✦

Chapter - 17

Second Order Linear Differential Equations with Constant Coefficient

17.1 Introduction

The second order differential equation of the form

$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = Q$, where P_1 and P_2 are constants and Q function of x, is called *Second Order Linear Differential Equation with Constant Coefficient.*

We write $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$. We will consider the following second order differential equations of the form

$(D^2 + P_1D + P_2)y = 0$ and

$(D^2 + P_1D + P_2)y = Q$.

Theorem: *If $y = f_1(x)$ and $y = f_2(x)$ are two independent solutions of the equation $(D^2 + P_1D + P_2)y = 0$ i.e. $f(D)y = 0$, then $y = c_1f_1(x) + c_2f_2(x)$ will be the general solution of the equation.*

Proof:

Here, we are given that $y = f_1(x)$ and $y = f_2(x)$ be two independent solutions of $f(D)y = 0$

$\therefore f(D)f_1(x) = 0$ ....(1)

and $f(D)f_2(x) = 0$ ....(2)

Now,

$$f(D)[c_1f_1(x) + c_2f_2(x)] = f(D)c_1f_1(x) + f(D)c_2f_2(x) = c_1f(D)[f_1(x)] + c_2f(D)[f_2(x)] = c_1.0 + c_2.0 = 0$$

So, $f(D)[c_1f_1(x) + c_2f_2(x)] = 0$

It shows that $y = c_1f_1(x) + c_2f_2(x)$ is the general solution of given differential equation.

17.2 Solution of $\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0$

If $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$ then the differential equation can be written as $(D^2 + P_1D + P_2)\,y = 0$ ....(1)

Let $y = e^{mx}$ be a solution of (1), then (1) becomes,

$$m^2 e^{mx} + P_1m\,e^{mx} + P_2e^{mx} = 0$$

or $e^{mx}(m^2 + P_1m + P_2) = 0$

Since $e^{mx} \neq 0$, $m^2 + P_1m + P_2 = 0$

Which is called *Auxiliary Equation* of the given differential equation and it is quadratic equation in m. So m has two roots.

Case I

When m has two different real roots, say $m = \alpha, \beta$ then the equation (1) can be written as

$$(D - \alpha)(D - \beta)\,y = 0 \quad \text{.......(2)}$$

Put $(D - \beta)y = v$ ....(3)

So from (2)

$$(D - \alpha)\,v = 0$$

or $\frac{dv}{dx} - \alpha v = 0$

or $\frac{dv}{v} - \alpha dx = 0$

Integrating

$$\log v - \alpha x = \log k$$

or $\log\frac{v}{k} = e^{\alpha x}$

$\therefore v = ke^{\alpha x}$

So from (3)

$$(D - \beta)y = k\,e^{\alpha x}$$

or $\frac{dy}{dx} - \beta y = ke^{\alpha x}$

Which is linear, I.F. $= e^{-\int \beta dx} = e^{-\beta x}$.

So its general solution is

$$y \times e^{-\beta x} = \int (ke^{\alpha x} \times e^{-\beta x})\,dx + c_2$$

or $y \times e^{-\beta x} = k\frac{e^{(\alpha - \beta)x}}{\alpha - \beta} + c_2$

or $y = \frac{k}{\alpha - \beta}\,e^{\alpha x} + c_2\,e^{\beta x}$ where $c_1 = \frac{k}{\alpha - \beta}$

$\therefore y = c_1\,e^{\alpha x} + c_2e^{\beta x}$ is the general solution of the given equation.

Case II

When m has two equal real roots, say $m = \alpha, \alpha$

So the equation (1) can be written as

$$(D - \alpha)(D - \alpha)y = 0$$

Put $(D - \alpha)y = v$ ....(4)

So from (4), $(D - \alpha)\,v = 0$ ....(5)

or $\frac{dv}{dx} - \alpha v = 0$ or $\frac{dv}{v} - \alpha dx = 0$

Integrating,

$$\log v - \alpha x = \log c_1$$

or $\log\frac{v}{c_1} = e^{\alpha x}$ $\therefore v = c_1\,e^{\alpha x}$

So from (5),

$$(D - \alpha)y = c_1\,e^{\alpha x}$$

or $\frac{dy}{dx} - \alpha y = c_1\,e^{\alpha x}$

This is linear. Here, $P = -\alpha$, $Q = c_1e^{\alpha x}$

$$\text{I.F.} = e^{-\int \alpha dx} = e^{-\alpha x}.$$

So its general solution is

$$y \times e^{-\alpha x} = \int (c_1e^{\alpha x} \times e^{-\alpha x})\,dx + c_2$$

or $y \times e^{-\alpha x} = c_1\int dx + c_2 = c_1x + c_2$

$\therefore y = (c_1x + c_2)\,e^{\alpha x}$ is the general solution of the differential equation.

Case III

When m has two imaginary roots, say $m = \alpha + i\beta, \alpha - i\beta$.

As we know that the general solution of the given equation is

$$y = c_1e^{(\alpha + i\beta)x} + c_2e^{(\alpha - i\beta)x}$$

or $y = e^{\alpha x}(c_1\,e^{i\beta x} + c_2\,e^{-i\beta x})$

or $y = e^{\alpha x}\,[c_1(\cos\beta x + i\sin\beta x) + c_2\,(\cos\beta x - i\sin\beta x)]$

or $y = e^{\alpha x}\,[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x]$

$\therefore$ $y = e^{\alpha x}\,[A\cos\beta x + B\sin\beta x]$ where $A = c_1 + c_2$, $B = i(c_1 - c_2)$ is the general solution of the given equation.

Worked Out Examples

Ex. 1 Solve: $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

Solution:

Here, $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

or $(D^2 - 7D + 12)\, y = 0$

So, its auxiliary equation is,

$m^2 - 7m + 12 = 0$

or $(m - 3)(m - 4) = 0$, or $m = 3, 4$

Thus $y = c_1 e^{3x} + c_2 e^{4x}$ is the general solution.

Ex. 2 Solve: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

Solution:

Here, $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 5y = 0$

or $(D^2 + 2D + 5)\, y = 0$

So, its auxiliary equation is

$m^2 + 2m + 5 = 0$

or $m = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$

Thus $y = e^{-x}(A \cos 2x + B \sin 2x)$ is the general solution.

Ex. 3 Solve: $(D^2 - 4D + 4)\, y = 0$

Solution:

Here, $(D^2 - 4D + 4)\, y = 0$

So, its auxiliary equation is

$m^2 - 4m + 4 = 0$

or $(m - 2)(m - 2) = 0$

or $m = 2, 2$

Thus $y = (c_1 + c_2x)\, e^{2x}$ is the general solution.

Ex. 4: Find the value of y which satisfies the equation

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ given that when $x = 0$, $y = 3$ and $\frac{dy}{dx} = 0$

Solution:

Here, $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

or $(D^2 + D - 2)\, y = 0$

So, its auxiliary equation is

$m^2 + m - 2 = 0$

or $(m + 2)(m - 1) = 0$

or $m = -2, 1$

Thus $y = c_1 e^{x} + c_2 e^{-2x}$ ....(1)

Differentiating,

$\frac{dy}{dx} = c_1e^{x} - 2c_2 e^{-2x}$ ....(2)

Using, $y = 3$ when $x = 0$, then (1) becomes

$3 = c_1 e^0 + c_2 e^0$

or $3 = c_1 + c_2$ ....(3)

Also using, $\frac{dy}{dx} = 0$ when $x = 0$. The equation (2) becomes

$0 = c_1 e^0 - 2c_2 e^0$

or $0 = c_1 - 2c_2$ ....(4)

Solving (3) and (4)

$c_1 = 2$ and $c_2 = 1$

Thus $y = 2e^{x} + e^{-2x}$ is the required solution.

Exercise - 29

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$
2. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$
3. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$
4. $(D^2 + D)y = 0$
5. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$
6. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$
7. $16\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 9y = 0$
8. $(D + 3)^2\, y = 0$
9. $(D^2 + 3aD - 4a^2)\, y = 0$
10. Solve $\frac{d^2x}{dt^2} + \mu x = 0, \mu > 0$ given that $x = a$, and $\frac{dx}{dt} = 0$ when $t = \frac{\pi}{2\sqrt{\mu}}$
11. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ given that $x = 1$ when $t = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.
12. $\frac{d^2y}{dx^2} + y = 0$ given that $y = 4$ when $x = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

Answers

1. $y = c_1 e^{-x} + c_2 e^{-2x}$
2. $y = c_1e^{-x} + c_2e^{-4x}$
3. $y = (c_1 + xc_2)\, e^{-x}$
4. $y = c_1 + c_2e^{-x}$
5. $y = e^{-3x}(A \cos 4x + B \sin 4x)$
6. $y = e^{-2x}(A\cos 3x + B\sin 3x)$

7. $y = (c_1 + c_2x)\, e^{-3x/4}$
8. $y = (c_1 + xc_2)\, e^{-3x}$
9. $y = c_1 e^{-4ax} + c_2 e^{ax}$
10. $x = a\sin\sqrt{\mu}\, t$
11. $x = 2e^t - e^{2t}$
12. $y = 4\cos x$

17.3 Particular Integral

The differential equation is of the form

$$(D^2 + P_1D + P_2)\, y = Q$$

i.e. $f(D)\, y = Q$, is called second order linear differential equation with constant coefficient, where, P_1, P_2 are constants and Q is function of x or constant.

Clearly, this equations is satisfied by, $y = \frac{1}{f(D)} Q$

So that $\frac{1}{f(D)} Q$ is the *Particular Integral* (P.I) of the given equation.

$$\therefore \text{P.I.} = \frac{1}{f(D)} Q$$

Where as the general solution of $(D^2 + P_1D + P_2)y = 0$ is called *Complementary Function.* (C.F.)

Theorem 2:

If $y = f(x)$ is the complete solution of $f(D)y = 0$ and $y = g(x)$ is a particular solution of the equation $f(D)y = Q$ then the complete solution of the equation $f(D)\, y = Q$ is $y = f(x) + g(x)$.

Proof:

Since $y = f(x)$ is the complete solution of the equation $f(D)y = 0$,

$$\therefore \quad f(D)\, f(x) = 0 \qquad \text{.........(1)}$$

Also $y = g(x)$ is a particular solution of the equation

$$f(D)\, y = Q$$

$$\text{So, } f(D)\, g(x) = Q \qquad \text{.........(2)}$$

Adding (1) and (2), we have,

$$f(D)\, [f(x) + g(x)] = Q$$

Clearly, $y = f(x) + g(x)$ satisfies the equation

$$f(D)y = Q$$

and hence $y = f(x) + g(x)$ is the general solution because it contains two arbitrary constants.

The part $y = f(x)$ is called *Complementary Function* (C.F.) and the part $y = g(x)$ is called *Particular Integral* (P.I.) of the equation.

Thus by the theorem, the general solution of the given differential equation is given by $y = \text{C.F} + \text{P.I.}$

Where, $\text{P.I.} = \frac{1}{f(D)} Q$

Theorem 3:

If Q be function of x, then $\frac{1}{D} Q$ operates the integration of Q with respect to x.

i.e. $\frac{1}{D} Q = \int Q\, dx$

Proof:

Let $\frac{1}{D} Q = u$

Operating both sides by D,

$$D\left(\frac{1}{D} Q\right) = Du$$

or $\quad Q = Du$

or $\quad Q = \frac{du}{dx}$

Integrating both sides with respect to x,

$$u = \int Q\, dx$$

Since, $\frac{1}{D} Q$ contains no arbitrary constant, no arbitrary constant being added.

$$\therefore \quad \frac{1}{D} Q = \int Q dx$$

Theorem 4:

$\frac{1}{D - a} Q = e^{ax} \int Q\, e^{-ax}\, dx$, where a is any constant.

Proof:

Let $\frac{1}{D - a} Q = u$

Operating on both sides by (D - a),we get

$$(D - a)\left(\frac{1}{D - a} Q\right) = (D - a)\, u$$

or $\quad Q = \frac{du}{dx} - au$

or $\quad \frac{du}{dx} - au = Q$

Which is linear .Here, $P = -a$, $Q = Q$

$$I.F. = e^{\int P dx} = e^{-\int a dx} = e^{-ax}.$$

So its general solution is

$$u \times e^{-ax} = \int Q e^{-ax}\, dx$$

or $u = e^{ax} \int Q e^{-ax}\, dx$

$$\therefore \frac{1}{D-a} Q = e^{ax} \int Q e^{-ax}\, dx$$

17.4 Rules for Finding Particular Integral

1. When $Q = e^{ax}$ where a is any constant,

$$\textbf{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} Q, \quad f(a) \neq 0.$$

Since $D e^{ax} = a e^{ax}$

$D^2 e^{ax} = a^2 e^{ax}$

So $f(D) e^{ax} = (D^2 + P_1 D + P_2) e^{ax}$

$= (a^2 + P_1 a + P_2) e^{ax}$

or $f(D) e^{ax} = f(a) e^{ax}$

Operating both sides by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)} [f(D) e^{ax}] = \frac{1}{f(D)} [f(a) e^{ax}]$$

or $e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$

or $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ provided $f(a) \neq 0$

$\therefore$ P.I. $= \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ provided $f(a) \neq 0$

Case of failure:

If $f(a) = 0$, then the above method fails .

Since $f(a) = 0$,D-a is a factor of f(D).

Let $f(D) = (D-a)\,\phi(D)$ where $\phi(a) \neq 0$. ......(1)

Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\,\phi(D)} e^{ax}$$

$$= \frac{1}{(D-a)} \frac{1}{\phi(a)} e^{ax} = \frac{1}{\phi(a)} \frac{1}{(D-a)} e^{ax} \text{ (By the theorem)}$$

$$= \frac{1}{\phi(a)} e^{ax} \int e^{ax} e^{-ax}\, dx = \frac{1}{\phi(a)} e^{ax} \int dx$$

$$= x \frac{1}{\phi(a)} e^{ax}$$

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{\phi(a)} e^{ax} \quad \text{....(2)}$$

But from (1)

$f(D) = (D-a)\,\phi(D)$

Differentiating with respect to D, we get

$f'(D) = (D-a)\,\phi'(D) + \phi(D)$

or $f'(a) = 0 + \phi(a)$

$\therefore$ $\phi(a) = f'(a)$

From (2), we have

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax} \quad \text{provided } f'(a) \neq 0.$$

Note

If $f'(a) = 0$, then $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$

For example

Here, the equation is

$(D^2 - 9)\, y = 9e^{3x}$

So its auxiliary equation is

$m^2 - 9 = 0$

or $m^2 = 9$

$\therefore$ $m = \pm 3$

So C.F. $= c_1 e^{-3x} + c_2 e^{3x}$

Now,

$$\text{P.I.} = \frac{1}{D^2 - 9} 9e^{3x} = x \frac{1}{2D} 9e^{3x} = \frac{9x}{2 \times 3} e^{3x} = \frac{3}{2} x\, e^{3x}$$

Thus y = C.F. + P.I.

$y = c_1 e^{-3x} + c_2 e^{3x} + \frac{3}{2} x\, e^{3x}$ is the general solution.

2. When Q = Sin ax or Cosax.

We have

D(Sin ax) = a Cosax

D^2(Sin ax) = $-a^2$ Sin ax

D^3(Sin ax) = $-\,a^3$Cos ax

$D^4(\text{Sin } ax) = (-a^2)^2 \text{ Sin } ax$

In general $(D^2)^n \sin ax = (-a^2)^n \sin ax$

$\therefore \quad f(D^2) \sin ax = f(-a^2) \sin ax$

Operating both sides by $\frac{1}{f(D^2)}$, we get

$$\frac{1}{f(D^2)}[f(D^2)\sin ax] = \frac{1}{f(D^2)} f(-a^2) \sin ax$$

or $\quad \sin ax = f(-a^2) \frac{1}{f(D^2)} \sin ax$

Dividing both sides by $f(-a^2)$

$$\frac{1}{f(-a^2)} \sin ax = \frac{1}{f(D^2)} \sin ax \qquad \text{provided } f(-a^2) \neq 0$$

Hence $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax \qquad$ provided $f(-a^2) \neq 0$

Similarly, $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax \qquad$ provided $f(-a^2) \neq 0$

Case of failure:

If $f(-a^2) = 0$ then the above method fails.

We have,

$$e^{iax} = \cos ax + i \sin ax$$

$$\frac{1}{f(D^2)} (\cos ax + i \sin ax) = \frac{1}{f(D^2)} e^{iax}$$

Put $D = ia$,

Then $f(D^2) = f(-a^2) = 0$, so that it is a case of failure.

$$\therefore \quad \frac{1}{f(D^2)} (\cos ax + i \sin ax) = x \frac{1}{f'(D^2)} e^{iax}$$

$$= x \frac{1}{f'(D^2)} (\cos ax + i \sin ax)$$

Equating the real parts,

$$\frac{1}{f(D^2)} \cos ax = x \frac{1}{f'(D^2)} \cos ax$$

$$= x \frac{1}{f'(-a^2)} \cos ax \qquad \text{provided } f(-a^2) \neq 0$$

Equating the imaginary parts,

$$\frac{1}{f(D^2)} \sin ax = x \frac{1}{f'(D^2)} \sin ax$$

$$= x \frac{1}{f'(-a^2)} \sin ax \qquad \text{provided } f(-a^2) \neq 0$$

Similarly,

When $Q = \cos ax$, then $\text{P.I.} = \frac{1}{f(D)} Q$

$$= \frac{1}{f(D)} \cos ax = \frac{1}{f(D^2)} \cos ax$$

$$= \frac{1}{f(-a^2)}, \qquad \text{provided } f(-a^2) \neq 0$$

If $f(-a^2) = 0$, then $\text{P.I.} = x \frac{1}{f'(D^2)} \cos ax$

$$= x \frac{1}{f'(a^2)} \cos ax, \qquad \text{provided } f'(-a^2) \neq 0$$

Note 1:

If $f'(-a)^2 = 0$, then

$$\frac{1}{f(D^2)} \sin ax = x^2 \frac{1}{f''(-a^2)} \sin ax$$

and $\frac{1}{f(D^2)} \cos ax = x^2 \frac{1}{f''(-a^2)} \cos ax \qquad$ provided $f''(-a^2) \neq 0$

Note 2:

If $Q = \sin(ax + b)$ or $\cos(ax + b)$, then the particular integral is obtained by the above method also,

$$\text{P.I.} = \frac{1}{f(D^2)} \sin(ax + b)$$

$$= \frac{1}{f(-a^2)} \sin(ax + b) \qquad \text{provided } f(-a^2) \neq 0$$

When $Q = \cos(ax + b)$

and $\text{P.I.} = \frac{1}{f(D^2)} \cos(ax + b)$

$$= \frac{1}{f(-a^2)} \cos(ax + b) \qquad \text{provided } f(-a^2) \neq 0$$

For Examples,

i. **Solve: $(D^2 - 4)y = \sin 2x$**

So its auxiliary equation is $m^2 - 4 = 0$

or $m^2 = 4$, or $m = 2, -2$

So, $\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$

Now, $\text{P.I.} = \frac{1}{D^2 - 4} \sin 2x = \frac{1}{-2^2 - 4} \sin 2x$

$= \frac{1}{-4-4} \sin 2x = -\frac{1}{8} \sin 2x$

Thus y = C.F. + P.I.

or $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} \sin 2x$ is the general solution.

ii. Solve $(D^2 + 1) y = \cos 2x$

So its auxiliary equation is $m^2 + 1 = 0$

or $m^2 = -1$, or $m = \pm i$

So C.F. = A cosx + B sinx

Now

$$P.I. = \frac{1}{D^2+1} \cos 2x = \frac{1}{-2^2+1} \cos 2x = -\frac{1}{3} \cos 2x$$

Thus y = C.F. + P.I.

or $y = A \cos x + B \sin x - \frac{1}{3} \cos 2x$ is the general solution.

3. If $Q = x^m$, where m is positive integer,

then $P.I. = \frac{1}{f(D)} Q = \frac{1}{f(D)} x^m$

$$= \frac{1}{D^2 + P_1 D + P_2} x^m = \frac{1}{P_2}\left(1 + \frac{D^2 + P_1 D}{P_2}\right)^{-1} x^m$$

$[f(D)]^{-1}$ can be expanded in ascending powers of D and then operate on x^m with each term of the expansions.

It can be expanded by the Binomial theorem as follow:

i. $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \ldots\ldots\ldots$

ii. $(1 - D)^{-1} = 1 + D + D^2 + D^3 + \ldots\ldots\ldots$

iii. $(1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots\ldots\ldots$

iv. $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots\ldots\ldots$

For example,

Solve : $\frac{d^2y}{dx^2} - 4y = x^2$

or $(D^2 - 4) y = x^2$

So, its auxiliary equation is $m^2 - 4 = 0$

or $m^2 = 4$, or $m = 2, -2$

So, C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

Now,

$$P.I. = \frac{1}{(D^2-4)} x^2 = -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} x^2$$

$$= -\frac{1}{4}\left[1 + \frac{D^2}{4} + \left(\frac{D^2}{4}\right)^2 + \left(\frac{D^2}{4}\right)^3 + \ldots\right] x^2 = -\frac{1}{4}x^2 - \frac{1}{8}$$

Thus y = C.F. + P.I.

or $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}x^2 - \frac{1}{8}$ is the general solution.

4. If $Q = e^{ax} V$, where V is the function of x, then

$$P.I. = \frac{1}{f(D)} Q = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

Let $V_1 = V_1(x)$ be function of x, then

$D(e^x V_1) = e^{ax} DV_1 + ae^{ax} V_1 = e^{ax} (D + a) V_1$

or $D^2 (e^{ax} V_1) = e^{ax} (D^2 + aD) V_1 + ae^{ax} (D + a)V_1$

$= e^{ax}(D^2 + aD + aD + a^2) V_1$

$D^2 (e^{ax} V_1) = e^{ax} (D + a)^2 V_1$

: :

: :

$D^r (e^{ax} V_1) = e^{ax} (D + a)^r V_1$

$\therefore$ $f(D) (e^{ax} V_1) = e^{ax} f(D + a) V_1$ ....(1)

Let $f(D + a) V_1 = V$

$$\frac{1}{f(D+a)} [f(D + a)V_1] = \frac{1}{f(D+a)} V$$

or $V_1 = \frac{1}{f(D+a)} V$

So (1) becomes

$$f(D)\left[e^{ax} \frac{1}{f(D+a)} V\right] = e^{ax} V$$

Operating both sides by $\frac{1}{f(D)}$

$$e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} (e^{ax} V)$$

$\therefore$ $P.I. = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$

For example,

Solve: $D^2 - 4D + 4)y = x^3 e^{2x}$

So, its auxiliary equation is $m^2 - 4m + 4 = 0$

or $(m - 2)^2 = 0$, or $m = 2, 2$

So, C.F. $= (c_1 + xc_2) e^{2x}$

Now $\text{P.I.} = \frac{1}{(D^2 - 4D + 4)} x^3 e^{2x}$

$= e^{2x} \frac{1}{\{(D+2)^2 - 4(D+2) + 4\}} x^3$

$= e^{2x} \frac{1}{(D^2 + 4D + 4 - 4D - 8 + 4)} x^3$

$= e^{2x} \frac{1}{D^2} x^3 = e^{2x} \frac{1}{D}\left(\frac{1}{D} x^3\right) = e^{2x} \frac{1}{D}\left(\frac{x^4}{4}\right)$

$= e^{2x} \frac{x^5}{20} = \frac{1}{20} x^5$

Thus y = C.F. + P.I.

or $y = (c_1 + xc_2) e^{2x} + \frac{1}{20} e^{2x} x^5$ is the general solution.

5. If $Q = xV$, where V is function of x,

then $\text{P.I.} = \frac{1}{f(D)} Q = \frac{1}{f(D)} xV = x \frac{1}{f(D)} V - \frac{f'(D)}{(f(D))^2} V$

For example

Solve: $(D^2 + 4) y = x \sin x$

Its auxiliary equation is

$m^2 + 4 = 0,$

or $m = 2i, -2i$

So, C.F. = Acos2x + Bsin2x

and $\text{P.I.} = \frac{1}{(D^2 + 4)} (x \sin x)$

$= \text{Imaginary part of} \frac{1}{D^2 + 4} x e^{ix}$

$= \text{I.P. of } e^{ix} \frac{1}{[(D+i)^2 + 4]} x$

$= \text{I.P. of } e^{ix} \frac{1}{(D^2 + 2iD - 1 + 4)} x$

$= \text{I.P. of } \frac{e^{ix}}{3} \left(1 + \frac{D^2 + 2iD}{3}\right)^{-1} x$

$= \text{I.P. of } \frac{e^{ix}}{3} \left[1 - \left(\frac{D^2 + 2iD}{3}\right) + \left(\frac{D^2 + 2iD}{3}\right)^2 - \ldots\ldots\right] x$

$= \text{I.P. of } \frac{e^{ix}}{3} \left(x - \frac{2i}{3}\right) = \text{I.P. of } \frac{(\cos x + i \sin x)}{3} \left(x - \frac{2i}{3}\right)$

$= \frac{x}{3} \sin x - \frac{2}{9} \cos x$

Thus y = C.F. + P.I.

or $y = A\cos 2x + B\sin 2x + \frac{x}{3} \sin x - \frac{2}{9} \cos x$

is the general solution.

6. If $Q = x^m \cos ax$ and $Q = x^m \sin ax$,

then $\text{P.I.} = \frac{1}{F(D)} Q = \frac{1}{F(D)} x^m \cos ax$

$= \frac{1}{F(D)} [\text{Real part of } x^m e^{iax}]$

and $\text{P.I.} = \frac{1}{F(D)} Q = \frac{1}{F(D)} x^m \sin ax$

$= \frac{1}{F(D)} [\text{Imaginary part of } x^m e^{iax}]$

Worked Out Examples

Ex. 1: Solve: $(D^2 - 3D + 2) y = e^{5x}$

Solution:

Here, $(D^2 - 3D + 2) y = e^{5x}$

So, its auxiliary equation is

$m^2 - 3m + 2 = 0$

or $(m - 2)(m - 1) = 0$

$\therefore$ $m = 1, 2$

So, $\text{C.F.} = c_1 e^x + c_2 e^{2x}$

Now, $\text{P.I.} = \frac{1}{(D^2 - 3D + 2)} e^{5x} = \frac{1}{5^2 - 3.5 + 2} e^{5x}$

$= \frac{1}{25 - 15 + 2} e^{5x} = \frac{1}{12} e^{5x}$

Thus y = C.F. + P.I.

or $y = c_1 e^x + c_2 e^{2x} + \frac{1}{12} e^{5x}$ is the general solution.

Ex. 2: Solve: $(D^2 - 3D + 2)y = e^x$

Solution:

Here, $(D^2 - 3D + 2)y = e^x$

So, its auxiliary equation is

$m^2 - 3m + 2 = 0$

or $(m - 1)(m - 2) = 0$

$\therefore$ $m = 1, 2$

So, C.F. $= c_1e^x + c_2e^{2x}$

Now, P.I. $= \frac{1}{D^2 - 3D + 2} e^x$ (Case of failure)

$= x \frac{1}{2D - 3} e^x = x \frac{1}{2.1 - 3} e^x = - xe^x$

Thus $y =$ C.F. + P.I.

or $y = c_1e^x + c_2e^{2x} - xe^x$ is the general solution.

Ex. 3: Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$

Solution:

Here, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$

or $(D^2 - 2D + 5)y = 10\sin x$, where $\frac{d}{dx} = D$

So, its auxiliary equation is

$m^2 - 2m + 5 = 0$

or $m = \frac{2 \pm \sqrt{4 - 20}}{2}$

$\therefore$ $m = 1 \pm 2i$

So, C.F. $= e^x(A \cos 2x + B\sin 2x)$

Now, P.I. $= \frac{1}{(D^2 - 2D + 5)} 10 \sin x = \frac{1}{(-1^2 - 2D + 5)} 10 \sin x$

$= \frac{1}{4 - 2D} 10 \sin x = - \frac{1}{2(D - 2)} 10\sin x$

$= - \frac{1}{2} \frac{(D + 2)}{(D - 2)(D + 2)} 10 \sin x = - \frac{10(D + 2)}{2(D^2 - 2)} \sin x$

$= \frac{-5}{-1^2 - 4}(D + 2) \sin x = \frac{-5}{-5}(D + 2) \sin x$

$= (D + 2) \sin x = \cos x + 2\sin x$

Thus $y =$ C.F. + P.I.

or $y = e^x(A \cos 2x + B \sin 2x) + \cos x + 2\sin x$

is the general solution

Ex. 4: Solve: $(D^2 + D + 1)y = \sin 2x$

Solution:

Here, $(D^2 + D + 1)y$

So, its auxiliary equation is,

$m^2 + m + 1 = 0$

or $m = \frac{-1 \pm \sqrt{1-4}}{2}$

$\therefore$ $m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

So C.F. $= e^{-x/2}\left(A\cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\right)$

Now P.I. $= \frac{1}{(D^2 + D + 1)} \sin 2x = \frac{1}{(-2^2 + D + 1)} \sin 2x$

$= \frac{1}{-4 + D + 1} \sin 2x = \frac{1}{(D - 3)} \sin 2x = \frac{(D + 3)}{D^2 - 9} \sin 2x$

$= \frac{(D + 3) \sin 2x}{(-2^2 - 9)} = - \frac{1}{13}(2 \cos 2x + 3\sin 2x)$

Thus $y =$ C.F. + P.I.

or $y = e^{-x/2}\left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$

is the general solution

Ex. 5 :Solve: $(D^2 + 2D + 1) y = e^x + x^2$

Solution:

Here, $(D^2 + 2D + 1) y = e^x + x^2$

Its auxiliary equation is

$m^2 + 2m + 1 = 0$

or $(m + 1)^2 = 0$

$\therefore$ $m = -1, -1$

So, C.F. $= (c_1 + xc_2) e^{-x}$

and P.I. $= \frac{1}{(D^2 + 2D + 1)} (e^x + x^2)$

$= \frac{1}{(D^2 + 2D + 1)} e^x + \frac{1}{(D^2 + 2D + 1)} x^2$

$= \frac{1}{(1 + 2 + 1)} e^x + [1 + (2D + D^2)]^{-1} x^2$

$= \frac{1}{4} e^x + [1 - (2D + D^2) + (2D + D^2)^2 + ...] x^2$

$= \frac{1}{4} e^x + [x^2 - 4x - 2 + 8] = \frac{1}{4} e^x + (x^2 - 4x + 6)$

Thus $y =$ C.F. + P.I.

$y = (c_1 + c_2x) e^{-x} + \frac{1}{4}e^x + (x^2 - 4x + 6)$ is the general solution.

Ex. 6: Solve : $(D^2 - 1) y = x e^{2x}$

Solution:

Here, $(D^2 - 1) y = x e^{2x}$

So, its auxiliary equation is

$$m^2 - 1 = 0$$

or $m^2 = 1,$

$\therefore \quad m = \pm 1$

So, C.F. $= c_1 e^x + c_2 e^{-x}$

Now, $\text{P.I.} = \dfrac{1}{(D^2 - 1)} xe^{2x} = e^{2x} \dfrac{1}{\{(D+2)^2 - 1\}} x$

$$= e^{2x} \frac{1}{(D^2 + 4D + 4 - 1)} x = e^{2x} \frac{1}{3 + 4D + D^2} x$$

$$= \frac{e^{2x}}{3} \left(1 + \frac{4D + D^2}{3}\right)^{-1} x$$

$$= \frac{e^{2x}}{3} \left[1 - \frac{4D + D^2}{3} + \left(\frac{4D + D^2}{3}\right) + \dots\dots\dots \right] x$$

$$= \frac{e^{2x}}{3} \left[x - \frac{4}{3}\right] = \frac{xe^{2x}}{3} - \frac{4e^{2x}}{9} = \frac{1}{9} e^{2x} (3x - 4)$$

Thus y = C.F. + P.I.

or $\quad y = c_1 e^x + c_2 e^{-x} + \dfrac{1}{9} e^{2x} (3x - 4)$ is the general solution.

Ex. 7: Solve: $\dfrac{d^2y}{dx^2} + 4y = \sin^2 x$

Solution:

Here, $\dfrac{d^2y}{dx^2} + 4y = \sin^2 x$

or $\quad (D^2 + 4) y = \sin^2 x$

So, its auxiliary equation is

$$m^2 + 4 = 0$$

or $\quad m^2 = -4,$

$\therefore \quad m = \pm 2i$

So, C.F. $= (A \cos 2x + B \sin 2x)$

Now, $\text{P.I.} = \dfrac{1}{D^2 + 4} \sin^2 x = \dfrac{1}{D^2 + 4} \left(\dfrac{1 - \cos 2x}{2}\right)$

$$= \frac{1}{2(D^2 + 4)} e^{0x} - \frac{1}{2(D^2 + 4)} \cos 2x = \frac{1}{8} - \frac{1}{4D} \cos 2x$$

(Case of failure for second term)

$$= \frac{1}{8} - \frac{x}{4} \frac{\sin 2x}{2} = \frac{1}{8} - \frac{1}{8} x \sin 2x$$

Thus y = C.F. + P.I.

or $\quad y = A\cos 2x + B\sin 2x + \dfrac{1}{8} - \dfrac{1}{8} x \sin 2x$ is the general solution.

Ex. 8: Solve: $(D^2 - 2D + 5)\, y = e^{2x} \sin x$ 2058/061 B.E

Solution:

Here, $(D^2 - 2D + 5)\, y = e^{2x} \sin x$

So, its auxiliary equation is

$$m^2 - 2m + 5 = 0$$

or $\quad m = \dfrac{+2 \pm \sqrt{4 - 20}}{2}$

$\therefore \quad m = 1 \pm 2i$

So, C.F. $= e^x (A\cos 2x + B\sin 2x)$

Now,

$$\text{P.I.} = \frac{1}{(D^2 - 2D + 5)} e^{2x} \sin x$$

$$= e^{2x} \frac{1}{\{(D+2)^2 - 2(D+2) + 5\}} \sin x$$

$$= e^{2x} \frac{1}{(D^2 + 4D + 4 - 2D - 4 + 5)} \sin x$$

$$= e^{2x} \frac{1}{(D^2 + 2D + 5)} \sin x = e^{2x} \frac{1}{(-1^2 + 2D + 5)} \sin x$$

$$= e^{2x} \frac{1}{(2D + 4)} \sin x = \frac{e^{2x}}{2} \frac{1}{(D + 2)} \sin x$$

$$= e^{2x} \frac{(D - 2)}{2(D + 2)(D - 2)} \sin x = \frac{e^{2x}}{2} \frac{(D - 2)}{D^2 - 4} \sin x$$

$$= \frac{e^{2x}}{2} \frac{(D - 2)}{(-1^2 - 4)} \sin x = -\frac{1}{10} e^{2x} (\cos x - 2\sin x)$$

Thus y = C.F. + P.I.

or $\quad y = e^x(A\cos 2x + B\sin 2x) - \dfrac{1}{10} e^{2x} (\cos x - 2\sin x)$

is the general solution.

Ex. 9: Solve: $(D^2 + 1)\, y = \sin x \sin 2x$

Solution:

Here, $(D^2 + 1)\, y = \sin x \sin 2x$

Its auxiliary equation is

$$m^2 + 1 = 0$$

$\therefore \quad m = \pm i$

So, C.F. $= A\cos x + B\sin x$

and $\text{P.I.} = \dfrac{1}{(D^2 + 1)} (\sin x \sin 2x) = \dfrac{1}{(D^2 + 1)} \dfrac{1}{2} (2 \sin x \sin 2x)$

$$= \frac{1}{2(D^2+1)}\,[\cos(2x - x) - \cos(x + 2x)]$$

$$= \frac{1}{2(D^2+1)}\,(\cos x - \cos 3x)$$

$$= \frac{1}{2(D^2+1)}\cos x - \frac{1}{2(D^2+1)}\cos 3x$$

(Case of failure for first term)

$$= x\frac{1}{4D}\cos x - \frac{1}{2(-3^2+1)}\cos 3x = \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

Thus y = C.F. + P.I.

or $y = A\cos x + B\sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$

is the general solution.

Ex. 10: Solve: $(D^2 + 1)\,y = x^2 \sin 2x$

Solution:

Here, $(D^2 + 1)\,y = x^2 \sin 2x$

Its auxiliary equation is

$m^2 + 1 = 0$

or $m^2 = -1$,

$\therefore m = i, -i$

So, C.F. = $A\cos x + B\sin x$

and P.I. $= \frac{1}{(D^2+1)}(x^2 \sin 2x)$

$$= \text{I.P. of } \frac{1}{D^2+1}(x^2 e^{i2x})$$

$$= \text{Imaginary part of } e^{i2x}\frac{1}{\{(D+2i)^2+1\}}x^2$$

$$= \text{I.P. of } e^{i2x}\frac{1}{(D^2+4iD-4+1)}x^2$$

$$= \text{I.P. of } e^{i2x}\frac{1}{D^2+4iD-3}x^2$$

$$= \text{I.P. of } \frac{e^{i2x}}{-3}\left[1-\frac{(4iD+D^2)}{3}\right]^{-1}x^2$$

$$= \text{I.P. of } \frac{e^{i2x}}{-3}\left[1+\frac{4iD+D^2}{3}+\left(\frac{4iD+D^2}{3}\right)^2+\ldots\ldots\right]x^2$$

$$= \text{I.P. of } \frac{1}{3}(\cos 2x + i\sin 2x)\left(x^2+\frac{8ix}{3}+\frac{2}{3}-\frac{32}{9}\right)$$

$$= -\frac{1}{3}\sin 2x\left(x^2 - \frac{26}{9}\right) - \frac{8}{9}x\cos 2x$$

$$= \frac{1}{27}\sin 2x\,(26 - 9x^2) - \frac{8}{9}x\cos 2x$$

Thus y = C.F. + P.I.

or $y = A\cos x + B\sin x - \frac{8}{9}x\cos 2x + \frac{1}{27}(26 - 9x^2)\sin 2x$

is the general solution.

Exercise - 30

Solve the following equations

1. $(D^2 - 1)\,y = 5e^{2x}$
2. $(D - 2)^2 y = e^{4x}$
3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2e^{3x}$
4. $\frac{d^2y}{dx^2} + 4y = \sin 2x$
5. $(D^2 + 16)\,y = \cos 4x$
6. $\frac{d^2y}{dx^2} + y = \cos^2 x$
7. $(D^2 - D - 2)\,y = \sin 2x + e^x$
8. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{2x}\sin x$ [2057/059/062 B.E.]
9. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$
10. $(D^2 + 2)y = \sin (x\sqrt{2})$
11. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$
12. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$
13. $(D^2 - 4D + 4)\,y = x^2 + e^{2x}$
14. $(D - 2)^2\,y = x^2 e^{2x}$ [2060 B.E.]
15. $(D^2 - 3D + 2)\,y = \cosh x$
16. $(D^2 - 1)\,y = \sinh x$
17. $(D^2 + 4)\,y = x\sin^2 x$
18. $(D^2 - 4)\,y = x\sinh x$
19. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$
20. $(D^2 + 2D + 1)\,y = x\cos x$

Answers

1. $y = c_1e^{x} + c_2e^{-x} + \frac{5}{3}e^{2x}$

2. $y = (c_1 + xc_2)\, e^{2x} + \frac{1}{4}e^{4x}$

3. $y = c_1e^{x} + c_2e^{3x} + xe^{3x}$

4. $y = A\cos 2x + B\sin 2x - \frac{x}{4}\cos 2x$

5. $y = A\cos 4x + B\sin 4x + \frac{x}{8}\sin 4x$

6. $y = A\cos x + B\sin x + \frac{1}{2} - \frac{1}{6}\cos 2x$

7. $y = c_1e^{-x} + c_2\, e^{2x} + \frac{1}{20}(\cos 2x - 3\sin 2x) - \frac{1}{2}e^{x}$

8. $y = c_1\, e^{-x} + c_2\, e^{-2x} + \frac{e^{2x}}{170}\,(11\sin x - 7\cos x)$

9. $y = (c_1 + xc_2)\, e^{x} + \frac{e^{3x}}{8}\,(2x^2 - 4x + 3)$

10. $y = A\cos(x\sqrt{2}) + B\sin(x\sqrt{2}) - \frac{1}{2\sqrt{2}}\, x\cos(x\sqrt{2})$

11. $y = e^{x}\left[A\cos\sqrt{3}\,x + B\sin\sqrt{3}\,x\right] + \frac{1}{2}e^{x}\cos x$

12. $y = c_1e^{-2x} + c_2\, e^{x} - \frac{1}{10}\,(\cos x + 3\sin x) - \frac{1}{4}\,(2x + 1)$

13. $y = (c_1 + xc_2)\, e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + \frac{1}{2}x^2e^{2x}$

14. $y = (c_1 + c_2x)\, e^{2x} + \frac{x^4e^{2x}}{12}$

15. $y = c_1e^{x} + c_2e^{2x} + \frac{1}{12}e^{-x} - \frac{1}{2}xe^{x}$

16. $y = c_1e^{x} + c_2e^{-x} + \frac{x}{2}\cosh x$

17. $y = A\cos 2x + B\sin 2x + \frac{x}{8} - \frac{x}{32}\cos 2x - \frac{1}{16}x^2\sin 2x$

18. $y = c_1e^{-2x} + c_2\, e^{2x} - \frac{x}{3}\sin hx - \frac{2}{9}\cos hx$

19. $y = (c_1 + xc_2)\, e^{x} - e^{x}\,(2\cos x + x\sin x)$

20. $y = (c_1 + c_2x)\, e^{-x} + \frac{x}{2}\sin x - \frac{1}{2}(\sin x - \cos x)$

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17.5 Homogeneous Linear Differential Equation

The differential equation of the form

$$x^n\frac{d^ny}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + + \ldots\ldots\ldots\ldots P_{n-1}x\frac{dy}{dx} + P_ny = Q$$

where $P_1, P_2, \ldots \ldots P_n$ are constants and Q the function of x alone is called Homogenous linear differential equation of nth order.

The differential equation of the form

$$x^2\frac{d^2y}{dx^2} + P_1x\frac{dy}{dx} + P_2y = Q$$

where P_1, P_2 are constants and Q the function of x is called *Second order homogeneous linear equation.*

Let the second order linear differential equation is

$$x^2\frac{d^2y}{dx^2} + P_1x\frac{dy}{dx} + P_2y = Q \qquad \ldots(1)$$

or $(x^2D^2 + P_1xD + P_2)y = Q.$

This can be reduced to the linear differential equation with constant coefficient by the substitution

$$x = e^z \text{ or } z = \log x \text{ and } \frac{dz}{dx} = \frac{1}{x}$$

For then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x} = \frac{1}{x}\frac{dy}{dz}$

So $x\frac{dy}{dx} = \frac{dy}{dz}$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right).$$

So $x^2\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

Thus, the equation (1) reduces to

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + P_1\frac{dy}{dz} + P_2 y = Q$$

Writing $\frac{d}{dz} = \delta$ we get

$$(\delta^2 - \delta + P_1\delta + P_2)y = Q$$

or $[\delta^2 + (P_1 - 1)\delta + P_2]\, y = Q$

This is second order linear differential equation with constant coefficient and can be solved by the methods of linear differential equation with constant coefficients.

Similarly, it can be proved that

$$x^2\frac{d^2y}{dx^2} = \delta(\delta-1)(\delta-2)\,y \text{ etc., so that}$$

$$x^n\frac{d^ny}{dx^n} = \delta(\delta-1)(\delta-2)\ldots\ldots(\delta-n+1)y \text{ where } \delta = \frac{d}{dz}.$$

Thus making these substitutions the nth order linear homogenous differential equation is reduced to the linear form.

17.6 Equation Reducible to the Homogeneous Linear Differential Equation

The differential equation of the form

$$(ax+b)^n\frac{d^ny}{dx^n} + P_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + P_{n-1}(ax+b)\frac{dy}{dx} + P_n y = Q$$

where $P_1, P_2, P_3, \ldots\ldots\ldots\ldots P_n$ are constants.

Here the substitution $ax + b = t$ will reduce this equation to the Homogenous linear differential equation and further substitution $t = e^z$ reduces it to the linear differential equation with constant coefficient.

Worked Out Examples

1. Solve $(x^2D^2 + xD - 1)y = x^3$

Solution:

Here, the differential equation is

$$(x^2D^2 + xD - 1)y = x^3 \qquad \ldots(1)$$

This is homogenous differential equation. So put

$$x = e^z, \qquad z = \log x$$

$$xDy = \delta y, \qquad x^2D^2y = (\delta^2 - \delta)y \text{ where } \delta = \frac{d}{dz} \text{ in (1)}$$

So the equation (1) reduces to

$$(\delta^2 - \delta + \delta - 1)y = e^{3z}$$

or $(\delta^2 - 1)y = e^{-3z}$

Its auxiliary equation is

$$m^2 - 1 = 0$$

or $m^2 = 1 \qquad \therefore m = \pm 1$

So, $\text{C.F.} = c_1e^{-z} + c_2e^{z} = \frac{c_1}{x} + c_2x$

$$\text{P.I.} = \frac{1}{\delta^2 - 1}e^{3z}$$

Putting $\delta = 3$

$$\text{P.I.} = \frac{1}{9-1}e^{3z} = \frac{1}{8}e^{3z} = \frac{1}{8}x^3$$

Its general solution is

$$y = \text{C.F.} + \text{P.I.} = \frac{c_1}{x} + c_2x + \frac{1}{8}x^3.$$

2. Solve : $(x^2D^2 - 2)y = x^2 + \frac{1}{x^2}$

Solution:

Here the differential equation is

$$(x^2D^2 - 2)y = x^2 + \frac{1}{x^2}.$$

This is homogenous differential equation .So put

$$x = e^z, z = \log x, \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x}$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} + \frac{dy}{dz}\left(-\frac{1}{x^2}\right)$

$= \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = \delta^2 y - \delta y$ where $\delta^2 = \frac{d^2}{dz^2}$

So (1) becomes

$\delta^2 y - \delta y - 2y = e^{2z} + e^{-2z}$

or $(\delta^2 - \delta - 2)y = 0$

This is differential equation with constant coefficient and its auxiliary equation is

$m^2 - m - 2 = 0$

or $(m-2)(m+1) = 0$

$\therefore$ $m = -1, 2$

$C.F. = c_1 e^{-z} + c_2 e^{2z} = \frac{c_1}{x} + c_2 x^2$

$P.I. = \frac{1}{\delta^2 - \delta - 2}(e^{2z} + e^{-2z}) = \frac{1}{\delta^2 - \delta - 2}e^{2z} + \frac{1}{\delta^2 - \delta - 2}e^{-2z}$

On putting $\delta = 2$ in the first term the denominator becomes zero, multiplying by z and differentiating denominator with respect to δ in the first term and putting $\delta = -2$ in the second term.

$P.I. = z\frac{1}{2\delta - 1}e^{2z} + \frac{1}{4+2-2}e^{-2z}$

$= z\frac{1}{4-1}e^{2z} + \frac{1}{4}e^{-2z} = z\frac{e^{2z}}{3} + \frac{1}{4}e^{-2z}$

$= \frac{x^2}{3}\log x + \frac{1}{4x^2}$

So the general solution is

$y = C.F. + P.I.$

$y = \frac{c_1}{x} + c_2 x^2 + \frac{x^2}{3}\log x + \frac{1}{4x^2}$.

3. Solve: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Solution:

Here the differential equation is

$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$ ...(1)

$(x^2D^2 + xD)y = 12\log x$

This is homogenous differential equation. So put

$x = e^z$, $z = \log x$

$xDy = \delta y$, $x^2D^2y = (\delta^2 - \delta)y$ where $\delta = \frac{d}{dz}$ in (1).

So, the equation (1) reduces to

$(\delta^2 - \delta + \delta)y = 12z$

or $\delta^2 y = 12z$

or $\frac{d^2y}{dz^2} = 12z$

Integrating successively

$\frac{dy}{dz} = 6z^2 + c_1$

$y = 2z^3 + c_1 z + c_2$

$\therefore$ $y = 2(\log x)^3 + c_1\log x + c_2$ is the general solution.

4. Solve: $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$.

Solution:

Here the differential equation is

$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$.

$(x^2D^2 - xD + 2)y = x\log x$. ...(1)

This is homogenous differential equation .So put

$x = e^z$, $z = \log x$

$xDy = \delta y$, $x^2D^2y = (\delta^2 - \delta)y$ where $\delta = \frac{d}{dz}$ in(1).

So the equation (1) reduces to

$(\delta^2 - \delta - \delta + 2)y = e^z z$

or $(\delta^2 - 2\delta + 2)y = e^z z$

Its auxiliary equation is

$m^2 - 2m + 2 = 0$

$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$.

$\therefore$ $m = 1 \pm i$.

So C.F. $= e^z(A\cos z + B\sin z) = x(A\cos\log x + B\sin\log x)$

$P.I. = \frac{1}{\delta^2 - 2\delta + 2}e^z z$

Writing e^z outside the operator and replacing δ by $\delta+1$,

$$= e^z \frac{1}{(\delta+1)^2 - 2(\delta+1)+2} z$$

$$= e^z \frac{1}{\delta^2+1} (z) = e^z (1+\delta^2)^{-1} (z)$$

$$= e^z (1 - \delta^2 + \delta^4 - \ldots\ldots\ldots\ldots) (z)$$

$$= e^z z = x \log x$$

Its general solution is

$$y = C.F. + P.I.$$

$$= A\, x\cos(\log x) + B\, x \sin(\log x) + x \log x$$

5. Solve: $(x+2)^2 \frac{d^2y}{dx^2} - 4(x+2)\frac{dy}{dx} + 6y = x.$

Solution:

Here the differential equation is

$$(x+2)^2 \frac{d^2y}{dx^2} - 4(x+2)\frac{dy}{dx} + 6y = x.$$

This is homogenous differential equation. Put $x+2 = e^z$

$$z = \log(x+2), \quad \frac{dz}{dx} = \frac{1}{x+2}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{x+2}$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x+2}\frac{dy}{dz}\right) = \frac{1}{x+2}\frac{d^2y}{dz^2}\frac{dz}{dx} + \frac{dy}{dz}\left(-\frac{1}{(x+2)^2}\right)$$

$$= \frac{1}{(x+2)^2}\frac{d^2y}{dz^2} - \frac{1}{(x+2)^2}\frac{dy}{dz} = \frac{1}{(x+2)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$(x+2)^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

So (1) becomes

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 4\frac{dy}{dz} + 6y = e^z - 2$$

or $\frac{d^2y}{dz^2} - 5\frac{dy}{dz} + 6y = e^z - 2$

or $(\delta^2 - 5\delta + 6)y = e^z - 2$

This is differential equation with constant coefficient. Its auxiliary equation is

$$m^2 - 5m + 6 = 0$$

or $(m-3)(m-2) = 0$

$\therefore$ $m = 2, 3$

$$C.F. = c_1 e^{2z} + c_2 e^{3z} = c_1 (x+2)^2 + c_2 (x+2)^3$$

$$P.I. = \frac{1}{\delta^2 - 5\delta + 6}(e^z - 2) = \frac{1}{\delta^2 - 5\delta + 6} e^z - \frac{1}{\delta^2 - 5\delta + 6} 2e^{0z}$$

$$= \frac{1}{1-5+6} e^z - \frac{1}{0-0+6} 2e^{0z} = \frac{e^z}{2} - \frac{2}{6} = \frac{e^z}{2} - \frac{1}{3}$$

$$= \frac{x+2}{2} - \frac{1}{3}$$

So the general solution is

$$y = C.F. + P.I.$$

$$y = c_1 (x+2)^2 + c_2 (x+2)^3 + \frac{x+2}{2} - \frac{1}{3}$$

6. Solve : $(2x-1)^2 \frac{d^2y}{dx^2} - 4(2x-1)\frac{dy}{dx} + 8y = 8x.$

Solution:

Here the differential equation is

$$(2x-1)^2 \frac{d^2y}{dx^2} - 4(2x-1)\frac{dy}{dx} + 8y = 8x. \qquad \ldots(1)$$

This is homogenous differential equation. So put

$$2x - 1 = e^z, \quad z = \log(2x-1)$$

$$\frac{dz}{dx} = \frac{2}{2x-1}$$

or $x = \frac{1}{2}(e^z + 1).$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{2}{2x-1}$$

$\therefore$ $(2x-1)\frac{dy}{dx} = \frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = \frac{dy}{dz}\cdot\left(\frac{-4}{(2x-1)^2}\right) + \frac{2}{2x-1}\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{4}{(2x-1)^2}\frac{dy}{dz} + \frac{4}{(2x-1)^2}\frac{d^2y}{dz^2}$$

$\therefore$ $(2x-1)^2 \frac{d^2y}{dx^2} = 4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$ in (1).

So the equation (1) reduces to

$$4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - 4\frac{dy}{dz} + 8y = \frac{1}{2}(e^z + 1).$$

or $\frac{d^2y}{dz^2} - \frac{dy}{dz} - \frac{dy}{dz} + 2y = \frac{1}{8}e^z + \frac{1}{8}$

or $\quad \frac{d^2y}{dz^2} - 2\frac{dy}{dz} + 2y = \frac{1}{8} e^z + \frac{1}{8}$

or $\quad (\delta^2 - 2\delta + 2)\, y = \frac{1}{8} e^z + \frac{1}{8}$

Its auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

$$\therefore \quad m = 1 \pm i.$$

So $\quad$ C.F. $= e^z (A \cos z + B \sin z) = x (A \cos \log x + B \sin \log x)$

$$\text{P.I.} = \frac{1}{\delta^2 - 2\delta + 2}\left(\frac{1}{8} e^z + \frac{1}{8}\right)$$

$$= \frac{1}{8} \frac{1}{\delta^2 - 2\delta + 2}(e^z) + \frac{1}{8} \frac{1}{\delta^2 - 2\delta + 2} e^{0z}$$

Putting $\delta = 1$ in the first term and $\delta = 0$ in the second item.

$$= \frac{1}{8} \frac{1}{1^2 - 2 + 2}(e^z) + \frac{1}{8} \frac{1}{0^2 - 20 + 2} e^{0z}$$

$$= \frac{1}{8} e^z + \frac{1}{8}\frac{1}{2} = \frac{e^z}{8} + \frac{1}{16} = \frac{1}{8}(2x - 1) + \frac{1}{16}$$

$$= \frac{x}{4} - \frac{1}{8} + \frac{1}{16} = \frac{x}{4} - \frac{1}{16}$$

Its general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= A x \cos(\log x) + B x \sin(\log x) + \frac{x}{4} - \frac{1}{16}.$$

Exercise -31

Solve the following differential equations.

1. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = \frac{1}{x}$
2. $(x^2 D^2 + x D - 1)y = x^2$
3. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 4x^3$
4. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^6$
5. $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 6x$

6. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4$
7. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 4y = x^2$
8. $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x.$
9. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$
10. $(x^2 D^2 + x D + 1)\, y = \sin(\log x^2)$
11. $(x^2 D^2 - 2)y = x^2 + \frac{1}{x}.$
12. $(x+p)^2 \frac{d^2y}{dx^2} - 4(x+p) \frac{dy}{dx} + 6y = x$
13. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1) \frac{dy}{dx} = (2x+3)(2x+4)$
14. $(2x+3)^2 \frac{d^2y}{dx^2} + 2(2x+3) \frac{dy}{dx} - 4y = 8x$
15. $(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1) \frac{dy}{dx} + 16y = 8(2x+1)^2$

Answers

1. $y = c_1 x + c_2 x^2 + \frac{1}{6x}$
2. $y = c_1 x + \frac{c_2}{x} + \frac{1}{3} x^2$
3. $y = c_1 x + c_2 x^2 + 2x^3$
4. $y = (c_1 + c_2 \log x)\, x^2 + \frac{x^6}{16}$
5. $y = c_1 + \frac{c_2}{x} + x^2$
6. $y = \frac{c_1}{x} + c_2 x^4 + \frac{1}{5} x^4 \log x$
7. $y = c_1 x^2 + c_2 x^{-2} + \frac{1}{4} x^2 \log x$

8. $y = (c_1 + c_2 \log x)x + \log x + 2$

9. $y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}$

10. $y = A\cos(\log x) + B\sin(\log x) - \frac{1}{3}\sin \log x^2.$

11. $y = \frac{c_1}{x} + c_2 x^2 + \frac{x^2}{3}\log x - \frac{1}{3x}$

12. $y = c_1(x+p)^2 + c_2(x+p)^3 + \frac{3x+2p}{6}$

13. $y = (x+1)^2 + 6(x+1) + [\log(x+1)]^2 + c_1 \log(x+1) + c_2$

14. $y = c_1 \log(2x+3) + \frac{c_2}{\log(2x+3)} + \frac{2x+3}{2}\log(2x+3) + 3$

15. $y = (2x+1)^2 [\{\log(2x+1)\}^2 + c_1 \log(2x+1) + c_2]$

Chapter - 18

Application of Differential Equation in Engineering field

18.1 Introduction

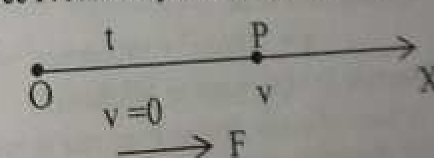
Mathematics is a language as tool of proving the inventions and makes easier to formulate in the mathematical model of the theoretical laws. Mathematics is modeling of physical mechanical, electrical, electronics, communications, and other system. There are so many applications of Mathematics in engineering field such as loaded beams, whirling of shafts, electrical transmission lines etc.

18.2 Application of First and Second Order Linear Differential Equation

The mathematical formulation of physical problems involving changing quantities leads to differential equation .We shall consider here, only those physical problems which leads us to a first *order and second order linear differential equations*. A summary of the fundamental principles requires for the formation differential equation are given below.

18.2.1 Physical applications

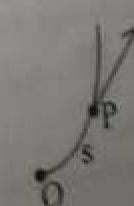
Let a body of mass start moving from O along the straight line OX under the action of force F. After any time t, it be moving at P where OP = x , then



(i) its velocity (v) = $\frac{dx}{dt}$

(ii) its acceleration (a) = $\frac{dv}{dt}$ or $v\frac{dv}{dx}$ or $\frac{d^2x}{dt^2}$

If the body be moving along a curve, then



(i) its velocity (v) = $\frac{ds}{dt}$

(ii) its acceleration (a) = $\frac{dv}{dt}$ or $v\frac{dv}{ds}$ or $\frac{d^2s}{dt^2}$.

The quantity mv is called *momentum*.

Chapter - 19

Transformation of Axes

19.1 Introduction :

In the two dimensional coordinate geometry, the equation of a curve or the position of a point are always considered with respect to two mutually perpendicular axes whereas point of intersection of the axes is taken as origin. Sometimes, it is more convenient to discuss about the equation of a curve or the coordinates of the point by changing the origin or direction of axes or both. Either of these process of changing the origin or direction of axes or both is known as *Transformation of Axes*.

19.2 Translation and Rotation of Axes :

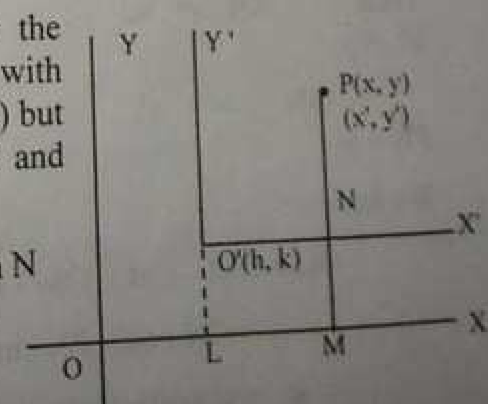
To change the coordinate of a point when the origin O(0, 0) is shifted to O' (h, k) without changing the direction of axes.

Let OX and OY be two mutually perpendicular axes and O'X' and O'Y' be the new axes parallel to the original axes.

Let O'(h, k) be the coordinate of new origin with respect to OX and OY so that

OL = h, O'L = k.

Consider P be any point in the plane and its coordinate with respect to OX and OY be (x, y) but the coordinate respect to O'X' and O'Y' be (x', y').



Draw PM⊥OX to meet O'X' in N

then, OM = x, PM = y,

O' N = x', PN = y'

Also we have,

$$OM = LM + OL = x' + h$$

$$\therefore \quad x = x' + h$$

and $PM = PN + NM$

$\therefore \quad y = y' + k$

It shows that the origin is transformed to the point (h, k) by substituting

$x = x' + h$

and $y = y' + k$

This transformation is known as *Translation*.

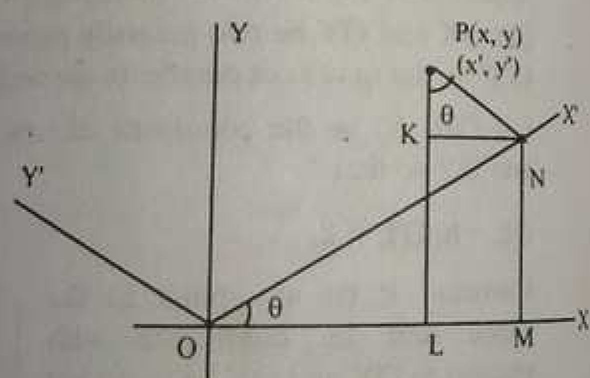
Thus the transformation of an equation f(x, y) = 0 with respect to new axes is f(x + h, y + k) = 0 and conversely.

To change the direction of axes without changing the origin.

Let OX and OY be the given rectangular axes being O at origin and OX' and OY' be new axes made by rotating the axes through an angle θ without changing the origin. So $\angle X'OX = \theta$.

Let P(x, y) and P(x', y') be the coordinates of a point P refereed to the original axes OX and OY and the new axes OX' and OY' respectively.

Draw PL and PN perpendiculars to OX and OX' respectively. Also draw NM and NK perpendiculars to OX and PL respectively.



Then OL = x, PL = y

ON = x', PN = y'

We have

$OL = OM - LM = OM - KN$

$= ON \cos\theta - PN \sin\theta$

$x = x' \cos\theta - y' \sin\theta$

$PL = PK + KL = PK + NM$

$= PN \cos\theta + ON \sin\theta$

$y = y' \cos\theta + x' \sin\theta$

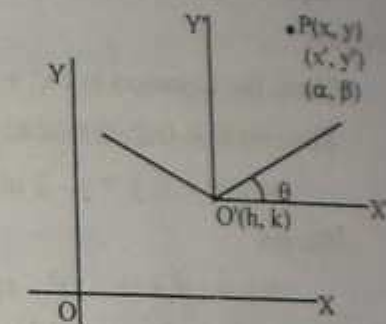
Thus the transformation formulae when the axes are rotated through an angle θ without changing the origin are

$x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta.$

To change the direction of axes along the change of origin.

Let OX and OY be two mutually perpendicular axes and the origin is transformed to the point O'(h, k) without changing the direction.



Let P be the point and its coordinates be (x, y) and (α, β) with respect to original and new axes.

So, $\left.\begin{aligned} x &= \alpha + h \\ y &= \beta + k \end{aligned}\right\}$ ....(1)

The new axes O'X' and O'Y' be rotated through an angle θ at the point O' and coordinate of P is (x', y').

Then $\alpha = x' \cos\theta - y' \sin\theta$

$\beta = y' \cos\theta + x' \sin\theta$

Substituting the value of α and β in (1),

$x = x' \cos\theta - y' \sin\theta + h$

$y = y' \cos\theta + x' \sin\theta + k.$

Worked out Examples

Ex. 1: Transform the equation $x^2 + y^2 - 4x + 8y - 17 = 0$ to parallel axes through the point (2, - 4).

Solution:

Here, the equation is $x^2 + y^2 - 4x + 8y - 17 = 0$ ....(1)

The origin is transferred to (2, - 4), so putting

$x = x + 2, \quad y = y - 4$ in (1)

We get

$$(x+2)^2 + (y-4)^2 - 4(x+2) + 8(y-4) - 17 = 0$$

or $x^2 + 4x + 4 + y^2 - 8y + 16 - 4x - 8 + 8y - 32 - 17 = 0$

$\therefore$ $x^2 + y^2 - 37 = 0.$

Hence the equation referred to new axes is $x^2 + y^2 = 37.$

Ex. 2: Transform to parallel axes through the point (1, - 2), the equation is $2x^2 + y^2 - 4x + 4y + 3 = 0$.

Solution:

Here, the equation is $2x^2 + y^2 - 4x + 4y + 3 = 0$ ....(1)

The origin is transferred to the point (1, - 2), so put

$x = x + 1,\ y = y - 2$ in (1),

We get

$$2(x+1)^2 + (y-2)^2 - 4(x+1) + 4(y-2) + 3 = 0$$

or $2(x^2 + 2x + 1) + y^2 - 4y + 4 - 4x - 4 + 4y - 8 + 3 = 0$

$\therefore$ $2x^2 + y^2 = 3$

Hence the equation referred to new axes is $2x^2 + y^2 = 3.$

Ex. 3: What does the equation $(a - b)(x^2 + y^2) - 2ax$ becomes if the origin is transferred to the point $\left(\frac{ab}{a-b}, 0\right)$.

Solution:

Here, the given equation is $(a - b)(x^2 + y^2) - 2ax = 0$ ....(1)

The origin is transferred to $\left(\frac{ab}{a-b}, 0\right)$. So put

$x = x + \frac{ab}{a-b},\ y = y + 0$ in (1),we get

$$(a-b)\left[\left(x + \frac{ab}{a-b}\right)^2 + (y-0)^2\right] - 2ab\left(x + \frac{ab}{a-b}\right) = 0$$

or $(a-b)\left(x^2 + \frac{2ab}{a-b}x + \frac{a^2b^2}{(a-b)^2} + y^2\right) - 2abx - \frac{2a^2b^2}{a-b} = 0$

or $(a-b((x^2+y^2) + 2abx + \frac{a^2b^2}{a-b} - 2abx - \frac{2a^2b^2}{a-b} = 0$

or $(a-b)(x^2+y^2) - \frac{a^2b^2}{a-b} = 0$

$\therefore$ $(a-b)^2(x^2+y^2) - a^2b^2 = 0$

Hence the transformed equation is $(a - b)^2 (x^2 + y^2) = a^2b^2$.

Ex. 4: If the axes be turned through an angle $\tan\theta = 2$, what does the equation $4xy - 3x^2 = a^2$ become ? [2061 B.E.]

Solution:

Here, the equation is

$4xy - 3x^2 = a^2$ ....(1)

Since the axes be turned through angle $\tan\theta = 2$

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{2}{\sqrt{1+4}} = \frac{2}{\sqrt{5}}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}$$

Then put $x = x\cos\theta - y\sin\theta = x\frac{1}{\sqrt{5}} - y\frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}(x - 2y)$

and $y = y\cos\theta + x\sin\theta = y\frac{1}{\sqrt{5}} + x\frac{2}{\sqrt{5}}$

$= \frac{1}{\sqrt{5}}(y + 2x)$ in (1), we get,

$$4\frac{1}{\sqrt{5}}(x-2y)\frac{1}{\sqrt{5}}(y+2x) - 3.\frac{1}{5}(x-2y)^2 = a^2$$

or $\frac{4}{5}(x-2y)(2x+y) - \frac{3}{5}(x-2y) = a^2$

or $4(2x^2 - 3xy - 2y^2) - 3(x^2 - 4xy + 4y^2) = 5a^2$

or $8x^2 - 12xy - 8y^2 - 3x^2 + 12xy - 12y^2 = 5a^2$

or $5x^2 - 20y^2 = 5a^2$

$\therefore$ $x^2 - 4y^2 = a^2$

Hence the transferred equation is

$x^2 - 4y^2 = a^2.$

Ex. 5: Find the angle through which the axes must be turned so that the equation $ax^2 + 2hxy + by^2 = 0$ may become an equation in which there is no term involving xy.

Solution:

Here the expression $= ax^2 + 2hxy + by^2$ ......(1)

Let the axes be rotated through an angle θ.

So put $x = x\cos\theta - y\sin\theta$

and $y = y\cos\theta + x\sin\theta$ in (1), we get

$$ax^2 + 2hxy + by^2$$
$$= a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(y\cos\theta + x\sin\theta) + b(y\cos\theta + x\sin\theta)^2$$
$$= a(x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta) + 2h(xy\cos^2\theta + x^2\cos\theta\sin\theta - y^2\sin\theta\cos\theta - xy\sin^2\theta) + b(y^2\cos^2\theta + 2xy\cos\theta\sin\theta + x^2\sin^2\theta) = 0$$

To remove the xy term the coefficient of xy in the expression must be zero.

i.e. $a(-2\cos\theta\sin\theta) + 2h(\cos^2\theta - \sin^2\theta) + b(2\cos\theta\sin\theta) = 0$

or $(b - a)\sin 2\theta + 2h\cos 2\theta = 0$

or $(a - b)\sin 2\theta = 2h\cos 2\theta$

or $\tan 2\theta = \dfrac{2h}{a - b}$

or $2\theta = \tan^{-1}\left(\dfrac{2h}{a - b}\right)$

$\therefore\ \theta = \dfrac{1}{2}\tan^{-1}\left(\dfrac{2h}{a - b}\right)$.

Ex. 6: Through what angle must the axes be rotated to remove the term containing xy in $11x^2 + 4xy + 14y^2 = 5$.

Solution:

Here the equation is

$$11x^2 + 4xy + 14y^2 = 5 \quad ......(1)$$

Let the axes be turned through an angle θ.

So put

$x = x\cos\theta - y\sin\theta$

and $y = y\cos\theta + x\sin\theta$ in (1), we get

$$11(x\cos\theta - y\sin\theta)^2 + 4(x\cos\theta - y\sin\theta)(y\cos\theta + x\sin\theta) + 14(y\cos\theta + x\sin\theta)^2 = 5$$

or $(11\cos^2\theta + 4\cos\theta\sin\theta + 14\sin^2\theta)x^2 + (-22\cos\theta\sin\theta + 4\cos^2\theta - 4\sin^2\theta + 28\cos\theta\sin\theta)xy + (11\sin^2\theta - 4\sin\theta\cos\theta + 14\cos^2\theta)y^2 = 5$

To remove the term xy the coefficient of xy = 0

or $6\cos\theta\sin\theta + 4(\cos^2\theta - \sin^2\theta) = 0$

or $6\cos\theta\sin\theta + 4(\cos^2\theta - \sin^2\theta) = 0$

or $3\sin 2\theta + 4\cos 2\theta = 0$

or $\dfrac{\sin 2\theta}{\cos 2\theta} = -\dfrac{4}{3}$

or $\tan 2\theta = -\dfrac{4}{3}$

or $\dfrac{2\tan\theta}{1 - \tan^2\theta} = -\dfrac{4}{3}$ or $\dfrac{\tan\theta}{1 - \tan^2\theta} = -\dfrac{2}{3}$

or $-2 + 2\tan^2\theta = 3\tan\theta$

or $2\tan^2\theta - 3\tan\theta - 2 = 0$

or $(2\tan\theta + 1)(\tan\theta - 2) = 0$

$\therefore\ \theta = \tan^{-1}(2)$ and $\tan^{-1}\left(-\dfrac{1}{2}\right)$.

Ex. 7: What does the equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ becomes when referred to rectangular axes through (2, 3), the new axes of x making angle 45° with the old.

Solution:

Here, the equation is

$$3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0 \quad ......(1)$$

Put $x = x\cos 45° - y\sin 45° + 2$

$$= x\frac{1}{\sqrt{2}} - y\frac{1}{\sqrt{2}} + 2 = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + 2$$

$y = y\cos 45° + x\sin 45° + 3$

$$= y.\frac{1}{\sqrt{2}} + x\frac{1}{\sqrt{2}} + 3 = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} + 3$$ in (1), we get

$$3\left[\frac{1}{\sqrt{2}}(x-y)+2\right]^2+2\left[\frac{1}{\sqrt{2}}(x-y)+2\right]\left[\frac{1}{\sqrt{2}}(x+y)+3\right]$$
$$+3\left[\frac{1}{\sqrt{2}}(x+y)+3\right]^2-18\left[\frac{1}{\sqrt{2}}(x-y)+2\right]$$
$$-22\left[\frac{1}{\sqrt{2}}(x+y)+3\right]+50=0$$

or $\frac{3}{2}(x-y)^2+\frac{12}{\sqrt{2}}(x-y)+12+x^2-y^2+\frac{6}{\sqrt{2}}(x-y)$
$$+12+\frac{4}{\sqrt{2}}(x+y)+\frac{3}{2}(x+y)^2+\frac{18}{\sqrt{2}}(x+y)$$
$$+27-\frac{18}{\sqrt{2}}(x-y)-36-\frac{22}{\sqrt{2}}(x+y)-66+50=0$$

or $\frac{3}{2}(x-y)^2+\frac{3}{2}(x+y)^2+x^2-y^2-1=0$

or $3[x^2-2xy+y^2+x^2+2xy+y^2]+2x^2-2y^2-2=0$

or $8x^2+4y^2-2=0$

$\therefore$ $4x^2+2y^2=1$

Hence the transferred equation is

$4x^2+2y^2=1.$

Ex. 8: Transform the equation $3x^2 - 2xy + 4y^2 + 8x - 10y + 8 = 0$ by translating the axes into an equation with linear term missing.

Solution:

Here, the equation of a curve is

$3x^2-2xy+4y^2+8x-10y+8=0$ .....(1)

Let the origin be transferred to (h, k).

So put $x = x + h$, $y = y + k$ in (1), we get,

$3(x+h)^2-2(x+h)(y+k)+4(y+k)^2+8(x+h)-10(y+k)+8=0$

or $3x^2-2xy+4y^2+2(3h-k+4)x+2(4k-h-5)y$
$$+(3h^2-2hk+4k^2+8h-10k+8)=0$$ .....(2)

To transform the equation containing only terms of the second degree i.e. missing of linear terms, we have

$3h-k+4=0$ .....(3)

and $4k-h-5=0$ .....(4)

Solving (2) and (3) $h=-1$, $k=1$.

Putting the value of h and k in (2), we get

$3x^2-2xy+4y^2=1$

This is the required transformed equation.

Exercise - 32

1. Transform the equation $x^2-3y^2+4x+6y=0$ by transferring the origin to the point (- 2, 1) and the coordinate axes remaining parallel.
2. Transform the equation $x^2+3y^2+3x-40=0$ to parallel axes through (4, - 1)
3. Transform the equation $x^2+y^2-5xy+4=0$ to parallel axes through the point (1, 3). [2057 Chaitra, B.E.]
4. Transform the equation $2x^2+4xy+5y^2-4x-22y+7=0$ to parallel axes through (- 2, 3).
5. What does the equation $x^2+2\sqrt{3}\,xy-y^2=2a^2$ become when the axes are turned through an angle 30° to the original axes.
6. What does the equation $2x+3y=\sqrt{2}$ become when the axes are turned through an angle 45° to the original axes.
7. What does the equation $3x^2+3y^2+2xy=2$ become when the axes are turned through an angle 45° to the original axes. [2057/060/062, B.E.]
8. Through what angle should the axes be rotated so that the equation. $9x^2-2\sqrt{3}\,xy+7y^2=10$ may be changed to $3x^2+5y^2=5$.
9. Find the angle through which the axes may be turned so that the equation $x+2y+5=0$ may be reduced to the form $x=c$ and determine the value of c.
10. What does the equation of the straight lines $7x^2+4xy+4y^2=0$ become when the axes are the bisectors of the angles between them.
11. Through what angle must the axes be rotated to remove the term containing xy in $3x^2+2xy+3y^2-\sqrt{2}\,x=0$. Also find the transformed equation.

12. Transform the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$ by translating the axes into an equation with linear term missing.

13. The equation $x^2 - y^2 = a^2$ is transferred to $xy = c^2$ by a change of rectangular axes. Find the inclination of new axes to the original axes and the value of c^2.

14. If (x, y) and (x_1, y_1) be the coordinates of the same point referred to two sets of rectangular axes with same origin and if $ux + vy$ where u and v are independent x, y, become $u_1x_1 + v_1y_1$. Show that $u^2 + v^2 = u_1^2 + v_1^2$.

15. Transform the equation $x^2 - 2xy + y^2 + x - 3y = 0$ to axes through the point (-1, 0) parallel to the lines bisecting the angles between original axes. [2059, B.E.]

Answers

1.	$x^2 - 3y^2 = 1$	2.	$x^2 + 3y^2 + 11x - 6y - 9 = 0$
3.	$x^2 + y^2 - 5xy - 13x + y = 1$	4.	$2x^2 + 4xy + 5y^2 - 22 = 0$
5.	$x^2 - y^2 = a^2$	6.	$5x + y = 2$
7.	$2x^2 + y^2 = 1$	8.	$60°$
9.	$\tan^{-1}(2); c = -\sqrt{5}$	10.	$8x^2 + 3y^2 = 0$
11.	$45°, 4x^2 + 2y^2 - x + y = 0$	12.	$6x^2 - 5xy + y^2 = 0$
13.	$45°, c^2 = -\frac{a^2}{2}$		

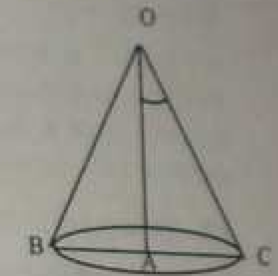
✧✧✧

Chapter - 20

The Ellipse

20.1 Conic Section

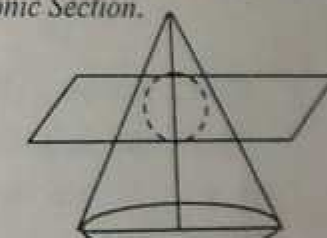
A cone is a surface generated by rotating a line about fixed point and fixed straight line making constant angle with fixed straight line. In the figure, O be the fixed point, A the vertex, OA the fixed straight line and OA the Axis,



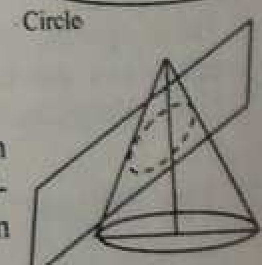
∠AOC the constant angle is semi-vertical angle of the cone.

If a plane in intersects a cone various ways, then the open or closed curve obtain from it, is called *Conic Section*.

If a plane intersects a cone perpendicular to the axis, then the section is a *Circle*.


Circle

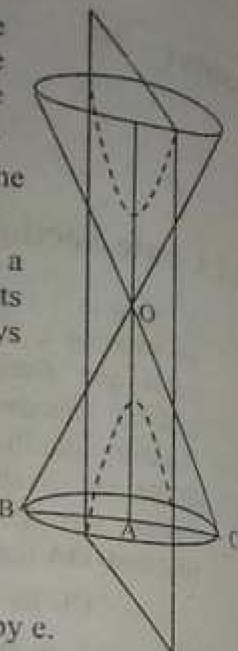
If a plane intersects a cone at a given angle with the axis greater than semi-vertical angle, then the section is an *Ellipse*.


Ellipse

If a plane which is parallel to the generator and does not pass through the vertex of a cone, intersects the cone, then the conic section is a *Parabola*.


Parabola

If a plane intersects a double right cone which makes the angle with axis of cone less than that of semi vertical angle of the cone, then the section is called a *Hyperbola*.



Hyperbola

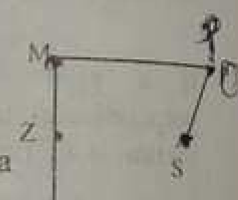
A conic section is also defined as the following ways:

A locus of a point which moves in such a way that its distance from fixed point and its distance from a fixed straight line is always constant ratio is called *Conic Section*.

The fixed point is *Focus*,

Fixed straight line is *Directrix*

Constant ratio $\frac{SP}{PM}$ is *Eccentricity* of the conic sectionand the eccentricity is denoted by e.



If e = 1, then the conic section is *Parabola*.

If e < 1, then the conic section is *Ellipse*.

If e > 1, then the conic section is a *Hyperbola*.

The straight line through focus and perpendicular to the directrix is called *Axis* and the intersection of the axis, the curve is called *Vertex* of the conic section.

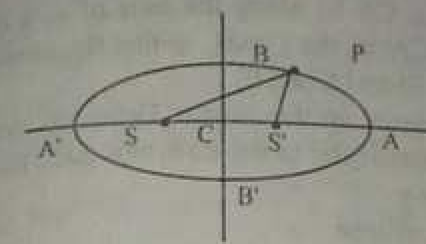
20.2 The Ellipse

An *ellipse* is a locus of a point in a plane, which moves in such a way that the ratio of its distance from the fixed point to its perpendicular distances from the fixed straight line is less than unity. It is also defined as a conic section in which the eccentricity is less than unity.

An *ellipse* is the locus of a point in a plane such that the sum of whose distances from two fixed points (the foci) in the plane is constant.

The fixed points S and S' are called *foci* and the midpoint of SS' is called *center* of the ellipse.

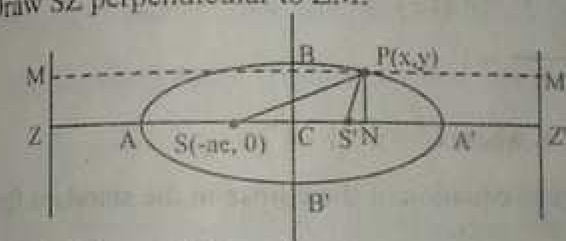
The line through the foci is called *major axis* of the ellipse. The line passes through the center and perpendicular to the major axis is called *minor axis*. The intersection of the ellipse with the major axis at two points A and A' are called *vertices* of the ellipse.



20.3 Standard form of the Equation of an Ellipse

Obtain the equation of ellipse in the standard form [2058 B E]

Let S be fixed point and ZM be the fixed straight line. Draw SZ perpendicular to ZM.



By definition of ellipse,

Let A be a point on the locus such that

$$SA = eAZ \quad \text{.........(1)}$$

Also the point A' be on the locus such that

$$SA' = eA'Z \quad \text{.........(2)}$$

Let C be the middle point of AA' and AA' = 2a then

$$CA = CA' = a$$

On addition (1) and (2),

$$SA + SA' = e(AZ + A'Z)$$

or $AA' = [e[(CZ - CA) + CA' + CZ]$

or $2a = e[(CZ - a + a + CZ]$

or $2a = 2e\ CZ$

$\therefore \quad CZ = \frac{a}{e}$

On subtraction (1) and (2),

$SA - SA' = e(AZ - A'Z)$

or $(CS + CA) - (CA' - CS) = eAA'$

or $2CS = 2ae$

$\therefore CS = ae$

Let C be the origin, CA be along the axis of x, a line through C perpendicular to ACA' be the y-axis, so that the coordinates of foci S and S' are S(- ae, 0) and S'(ae, 0).

Let P(x, y) be any point on the ellipse. Draw PN perpendicular to x-axis and PM perpendicular to the directrix such that

$CN = x,\ PN = y$

By definition of the ellipse

$SP = ePM.$

or $SP^2 = e^2 PM^2 = e^2(ZN)^2 = e^2(CZ + CN)^2$

or $(x + ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} + x\right)^2 = (a + ex)^2$

or $x^2 + 2aex + a^2e^2 + y^2 = a^2 + 2aex + e^2x^2$

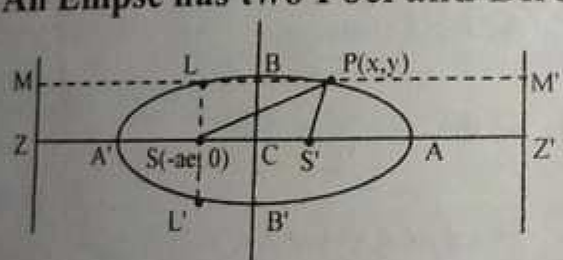
or $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

or $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$.

This is the required equation of the ellipse in the standard form.

20.4 An Ellipse has two Foci and Directrices



Let C be the center such that

$CZ' = CZ = \frac{a}{e}$

Thus the equation of the line Z'M' is $x = \frac{a}{e}$ and CS = CS' so that coordinate of S' is S'(ae, 0).

From the definition of an ellipse

$SP = ePM$

or $SP^2 = e^2 PM^2$

or $(x + ae)^2 + (y - 0)^2 = e^2 (ZN)^2 = e^2 (CZ + CN)^2 = e^2\left(\frac{a}{e} + x\right)^2$

or $x^2 + 2aex + a^2e^2 + y^2 = (a + ex)^2$

or $x^2 + 2aex + a^2e^2 + y^2 = a^2 + 2aex + e^2x^2$

or $x^2 + y^2 + a^2e^2 = a^2 + e^2x^2$

or $x^2 - 2aex + a^2e^2 + y^2 = a^2 + e^2x^2 - 2aex$

or $(x - ae)^2 + (y - 0)^2 = (a - ex)^2$

or $(x - ae)^2 + (y - 0)^2 = (a - ex)^2$

or $(x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2 = e^2\left(x - \frac{a}{e}\right)^2$

or $S'P^2 = e^2 PM'^2$

$\therefore S'P = ePM'$

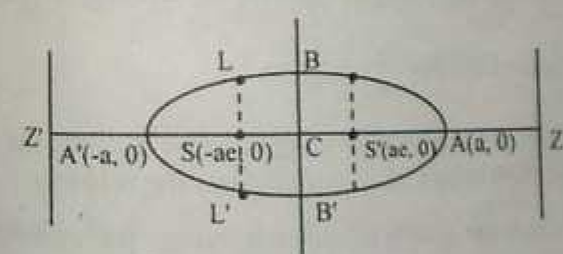
Hence an ellipse has two foci S and S' and two directrices ZM and Z'M'.

20.5 Some Important Definitions on an Ellipse

The standard form of the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Some definitions related to an ellipse are defined as follows:



Major axis:

The line segment AA' is called the *major axis* of the ellipse.

The length of major axis is AA' = 2a.

Coordinates of ends of major axis are A(- a, 0) and A'(a, 0)

Minor axis:

The line segment BB' is called the *minor axis* of the ellipse.

When $x = 0$, $y^2 = b^2$

$\therefore y = \pm b$

The length of minor axis is BB' = 2b.

Coordinates of ends of the minor axis are B(0, b) and B'(0, - b)

Center:

The intersection point of major and minor axis of the ellipse is called *center* of the ellipse.

Vertices:

The vertices of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for a > b, is the point of intersection of ellipse and major axis so that the vertices of the ellipse are A(a, 0) and A'(-a, 0).

Latus rectum:

The double ordinate passing through the focus and perpendicular to the major axis is called *latus rectum* of the ellipse.

For the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the ends of latus rectum L and L' are $\left(-ae, \frac{b^2}{a}\right), \left(-ae, -\frac{b^2}{a}\right)$ respectively.

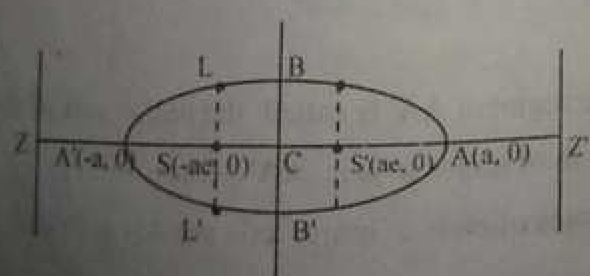
Length of the latus rectum = LL' = $\frac{2b^2}{a}$

20.6 An Ellipse whose Major Axis is Along x-axis

For the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

if a > b > 0, then the major axis of the ellipse is along the axis of x. In this case, eccentricity e is obtained from the relation

$b^2 = a^2(1 - e^2)$ and the graph of curve is symmetric on both axes.



Center is (0, 0)

Foci are S(-ae, 0), S'(ae, 0)

Vertices are A(a, 0), A'(- a, 0)

Directrices are x = - a/e, x = a/e

Major axis is AA' = 2a

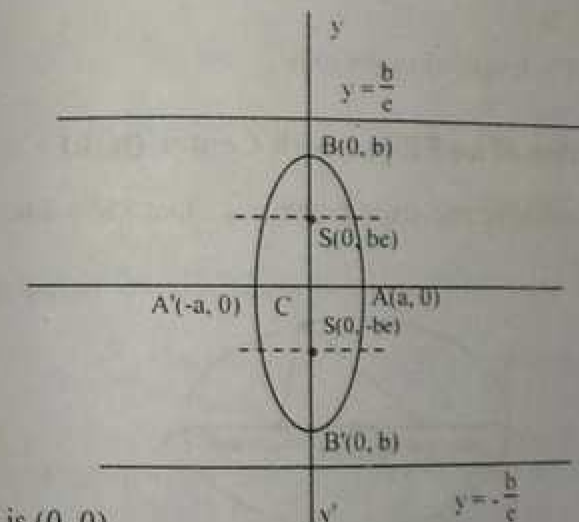
Minor axis is BB' = 2b

Latus rectum = $\frac{2b^2}{a}$.

20.7 An Ellipse whose Major Axis is Along y-axis

For the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if b > a > 0, then the major axis of the ellipse is along axis of y. In this case eccentricity e is obtained from the relation

$a^2 = b^2(1 - e^2)$ and the graph of this curve is symmetric on both axes.



Center is (0, 0)

Foci are S'(0, be), S(0, - be)

Vertices are B(0, b), B'(0, - b)

Directrices are $y = -\frac{b}{e}$, $y = \frac{b}{e}$

Major axis is (BB') = 2b

Minor axis is (AA') = 2a

Latus rectum = $\frac{2a^2}{b}$.

20.8 Sum of the Focal Chords of any Point of the Ellipse is Constant and Equal to the Major Axis

Let S and S' be foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and P(x, y) be any point on the ellipse.

Sum of focal distances of P

$= SP + S'P$

$= ePM + ePM'$

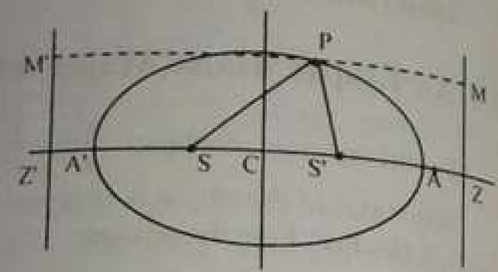
$= e(PM + PM')$

$= eZZ'$

$= e.2CZ$

$= e.2.\frac{a}{e}$

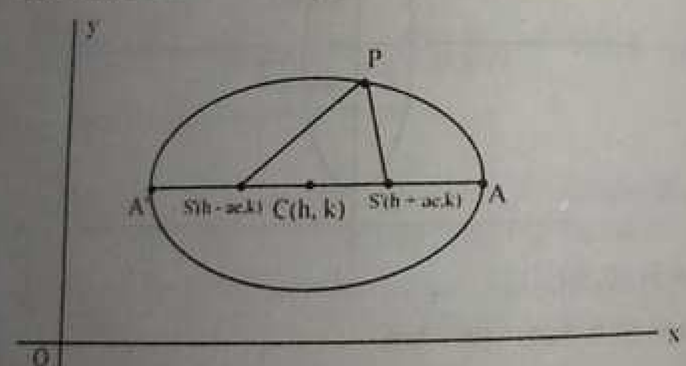
$= 2a$



$SP + S'P =$ length of major axis

20.8.1 Equation of an Ellipse with Center (h, k)

Let S and S' be two fixed points such that SS' = 2ae.



Take the line SS' as parallel to the x-axis and C be the middle point of SS' with center (h, k) then the coordinates of S and S' are (h + ae, k) and (h - ae, k) respectively.

Let P(x, y) be any point on the ellipse. By definition, we have

$SP + S'P = 2a$

or $\sqrt{(x - h - ae)^2 + (y - k)^2} + \sqrt{(x - h + ae)^2 + (y - k)^2} = 2a$

or $\sqrt{(x - h - ae)^2 + (y - k)^2} = 2a - \sqrt{(x - h + ae)^2 + (y - k)^2}$

Squaring,

$$(x - h - ae)^2 + (y - k)^2 = 4a^2 - 4a\sqrt{(x - h + ae)^2 + (y - k)^2} + (x - h + ae)^2 + (y - k)^2$$

or $(x - h)^2 - 2ae(x - h) + a^2e^2 + (y - k)^2$

$$= 4a^2 - 4a\sqrt{(x - h + ae)^2 + (y - k)^2} + (x - h)^2 + 2ae(x - h) + a^2e^2 + (y - k)^2$$

or $4a\sqrt{(x - h + ae)^2 + (y - k)^2} = 4ae(x - h) + 4a^2$

or $\sqrt{(x - h + ae)^2 + (y - k)^2} = e(x - h) + a$

Squaring

$(x - h + ae)^2 + (y - k)^2 = \{e(x - h) + a\}^2$

or $(x - h)^2 + 2ae(x - h) + a^2e^2 + (y - k)^2$

$$= e^2(x - h)^2 + 2ae(x - h) + a^2$$

or $(x - h)^2(1 - e^2) + (y - k)^2 = a^2(1 - e^2)$

or $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2(1 - e^2)} = 1$

$\therefore \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$.

is the required equation of the ellipse.

	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b>0$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, b>a>0$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $a>b>0$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ $b>a>0$
Center	(0, 0)	(0, 0)	(h, k)	(h, k)
Vertices	$(\pm a, 0)$	$(0, \pm b)$	$(h \pm a, k)$	$(h, k \pm b)$
Eccentricity	$b^2 = a^2(1-e^2)$ $e=\sqrt{1-\frac{b^2}{a^2}}$	$a^2 = b^2(1-e^2)$ $e=\sqrt{1-\frac{a^2}{b^2}}$	$b^2 = a^2(1-e^2)$ $e=\sqrt{1-\frac{b^2}{a^2}}$	$a^2 = b^2(1-e^2)$ $e=\sqrt{1-\frac{a^2}{b^2}}$
Foci	$(\pm ae, 0)$	$(0, \pm be)$	$(h \pm ae, k)$	$(h, k \pm be)$
Directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$
Length of Major axis	2a	2b	2a	2b
Minor axis	2b	2a	2b	2a
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$

Worked out Examples

Ex. 1: Find the equation of the ellipse

i. whose vertices at (± 4, 0) and foci (± 3, 0)

ii. whose focus is (0, ± 8), eccentricity $\frac{2}{3}$

iii. whose focus is (2, 5), eccentricity $\frac{2}{3}$, directrix x + y = 1

iv. whose foci (± 2, 0) and eccentricity $\frac{1}{3}$

i. **Vertices at (± 4, 0) and foci (± 3, 0)**

Solution:

Here, the major axis is along the x-axis.

$2a = 8 \quad \therefore a = 4.$

So $4e = 3 \quad \therefore e = \frac{3}{4}$

and $b^2 = a^2(1 - e^2)$ for a > b,

or $b^2 = 16\left(1 - \frac{9}{16}\right)$

or $b^2 = 16 \cdot \frac{7}{16}$

$\therefore b = \pm\sqrt{7}$

The equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or $\frac{x^2}{16} + \frac{y^2}{7} = 1.$

$\therefore 7x^2 + 16y^2 = 112$

ii. **Focus is (0, ± 8), eccentricity $\frac{2}{3}$**

Solution:

Here, the major axis is along the y-axis.

Given that $be = 8, \quad e = \frac{2}{3}$

So, $b \cdot \frac{2}{3} = 8 \quad \therefore b = 12$

Now,

$a^2 = b^2(1 - e^2)$, for b > a,

or $a^2 = 144\left(1 - \frac{4}{9}\right)$

or $a^2 = 144 \times \frac{5}{9} = 80$

or $a^2 = 80$

$\therefore a = \pm\sqrt{80}$

The equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or $\frac{x^2}{80} + \frac{y^2}{144} = 1$

$\therefore 9x^2 + 5y^2 = 720$

iii. **Focus is (2, 5), eccentricity $\frac{2}{3}$, directrix x + y = 1**

Solution:

Let P(x, y) be any point on the ellipse and given that

focus: S(2, 5), directrix: x + y = 1, eccentricity, $e = \frac{2}{3}$

By definition of ellipse,

$$(x - 2)^2 + (y - 5)^2 = \frac{4}{9}\frac{(x + y - 1)^2}{(1 + 1)}$$

or $(x - 2)^2 + (y - 5)^2 = \frac{2}{9}(x + y - 1)^2$

or $9(x^2 - 4x + 4 + y^2 - 10y + 25)$

$= 2(x^2 + y^2 + 1 + 2xy - 2y - 2x)$

or $9x^2 + 9y^2 - 36x - 90y + 261$

$= 2x^2 + 2y^2 + 4xy - 4x - 4y + 2$

$\therefore 7x^2 + 7y^2 - 4xy - 32x - 86y + 259 = 0$

is the required equation of the ellipse.

iv. **Foci (± 2, 0) and eccentricity $\frac{1}{3}$**

Solution:

Here, the major axis is along x-axis

Given that $ae = 2$, $e = \frac{1}{3}$

So, $a.\frac{1}{3} = 2$ $\quad \therefore a = 6$

Now, $b^2 = a^2 - a^2e^2$ for $a > b$,

or $b^2 = 36 - 4 = 32$

$\therefore b = \sqrt{32}$

The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or $\frac{x^2}{36} + \frac{y^2}{32} = 1$

$\therefore$ $8x^2 + 9y^2 = 288$ is the required equation of ellipse.

Ex. 2: Find the equation of ellipse whose center is origin, whose axes are the axes of co-ordinates and passes through (1, 6) and (2, - 5).

Solution:

Here, the standard equation of ellipse with center origin and axes are axes of coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{.........(1)}$$

Since it passes through (1, 6) and (2, - 5)

So, $\frac{1}{a^2} + \frac{36}{b^2} = 1$ ....(2)

and $\frac{4}{a^2} + \frac{25}{b^2} = 1$ ....(3)

Solving these

$b^2 = \frac{119}{3}$ $\quad \therefore b = \sqrt{\frac{119}{3}}$

and $a^2 = \frac{119}{11}$ $\quad \therefore a = \sqrt{\frac{119}{11}}$

So, the equation (1) becomes

$\frac{11.x^2}{119} + \frac{3.y^2}{119} = 1$

$\therefore$ $11x^2 + 3y^2 = 119$ is the required equation of ellipse.

Ex. 3: Find the eccentricity, latus rectum and foci of the ellipse

i. $4x^2 + 9y^2 = 36$

ii. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

iii. $9x^2 + 5y^2 - 30y = 0$

iv. $8x^2 + 6y^2 - 16x + 12y + 13 = 0$

i. $4x^2 + 9y^2 = 36$

Solution:

Here, the equation is

$4x^2 + 9y^2 = 36$

or $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Comparing it with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 9$, $\quad b^2 = 4$

$\therefore a = \pm 3$, $\quad b = \pm 2$

Clearly $a > b$,

So, $b^2 = a^2 - a^2e^2$

or $4 = 9 - 9e^2$

or $9e^2 = 5$

or $e^2 = \frac{5}{9}$ $\quad \therefore e = \frac{\sqrt{5}}{3}$

Foci $(\pm ae, 0) = (\pm\sqrt{5}, 0)$

Latus rectum $= \frac{2b^2}{a} = 2 \times \frac{4}{3} = \frac{8}{3}$

ii. $\frac{x^2}{4}+\frac{y^2}{9}=1$

Solution:

Here, the equation of ellipse is

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Comparing it with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a^2=4,\quad b^2=9$

$\therefore\ a=\pm 2,\quad b=\pm 3$

Clearly b > a. Thus the major axis is along the y-axis.

$a^2=b^2-b^2e^2$

or $4=9-9e^2$

or $9e^2=5$

or $e^2=\frac{5}{9}$ $\qquad \therefore e=\frac{\sqrt{5}}{3}$

Latus rectum $=\frac{2a^2}{b}=2\times\frac{4}{3}=\frac{8}{3}$

Foci $(0,\pm be)=(0,\pm\sqrt{5})$

iii. **$9x^2+5y^2-30y=0$**

Solution:

Here, the equation of the ellipse is

$9x^2+5y^2-30y=0$

or $9x^2+5(y^2-6y)=0$

or $9x^2+5\{(y-3)^2-9\}=0$

or $9x^2+5(y-3)^2-45=0$

or $9x^2+5(y-3)^2=45$

or $\frac{(x-0)^2}{5}+\frac{(y-3)^2}{9}=1$

Comparing it with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we get

Center $(h,k)=(0,3)$, $a^2=5$, $b^2=9$

Clearly b > a, the major axis of the ellipse is along the y-axis.

So, $a^2=b^2-b^2e^2$

or $5=9-9e^2$

or $9e^2=4$

or $e^2=\frac{4}{9}$ $\qquad \therefore e=\frac{2}{3}$

Latus rectum $=\frac{2a^2}{b}=\frac{2\times 5}{3}=\frac{10}{3}$

Foci $=(0,k\pm be)$

$=(0,3+2)$ and $(0,3-2)$

$=(0,5)$ and $(0,1)$

iv. **$8x^2+6y^2-16x+12y+13=0$**

Solution:

Here, the equation of the ellipse is

$8x^2+6y^2-16x+12y+13=0$

or $8(x^2-2x)+6(y^2+2y)+13=0$

or $8\{(x-1)^2-1\}+6\{(y+1)^2-1\}+13=0$

or $8(x-1)^2-8+6(y+1)^2-6+13=0$

or $8(x-1)^2+6(y+1)^2=1$

$\frac{(x-1)^2}{1/8}+\frac{(y+1)^2}{1/6}=1$

Comparing it with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we get

Center $(h,k)=(1,-1)$, $a^2=\frac{1}{8}$, $\quad b^2=\frac{1}{6}$

Clearly, b > a, the major axis is parallel to y-axis,

So, $a^2=b^2-b^2e^2$

or $\frac{1}{8}=\frac{1}{6}-\frac{1}{6}e^2$

or $\frac{e^2}{6}=\frac{1}{6}-\frac{1}{8}$

or $\frac{e^2}{6}=\frac{4-3}{24}$

or $e^2 = \frac{1}{4}$ $\quad\therefore\quad e = \frac{1}{2}$

Latus rectum $= \frac{2a^2}{b} = 2 \times \frac{1 \times \sqrt{6}}{8 \times 1} = \frac{\sqrt{6}}{4}$

Foci $= (h, k \pm be) = \left(1, -1 \pm \frac{1}{2\sqrt{6}}\right) = \left(1, -1 \pm \frac{\sqrt{6}}{12}\right)$

Ex. 4: The line x + y = 0 is a directrix of an ellipse, the point (2, 2) is the corresponding focus if the eccentricity be $\frac{1}{3}$, find the equation of other directrix. [2057 Chaitra B.E.]

Solution:

The equation of directrix is

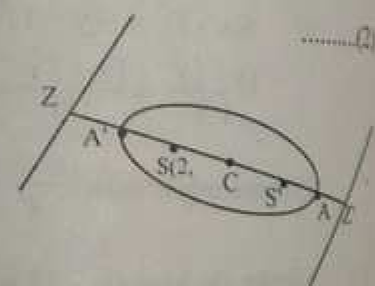
$$x + y = 0 \qquad \text{.........(1)}$$

Its slope is - 1 so that the slope of the major axis is 1.

The equation of major axis passes through focus (2, 2) and perpendicular to the directrix is

$$y - 2 = 1(x - 2)$$

or $\quad x - y = 0 \qquad$ ....(2)



Solving the equations (1) and (2)

We get Z(0, 0)

We have SA = eAZ

$$\frac{SA}{AZ} = \frac{1}{3}$$

The vertices A and A' divide the line SZ internally and externally in the ratio 1:3.Hence the coordinates of A and A' are

$$A = \left(\frac{3}{2}, \frac{1}{2}\right) \text{ and } A' = (3, 3) \text{ respectively.}$$

Let C be the center of AA'. So its coordinates is $\left(\frac{9}{4}, \frac{9}{4}\right)$

Also C be the middle point of ZZ'

If Z' = (h, k), then we have

$$\frac{9}{4} = \frac{0 + h}{2} \qquad \therefore\ h = \frac{9}{2}$$

and $\frac{9}{4} = \frac{0 + k}{2} \qquad \therefore\ k = \frac{9}{2}$

The coordinate of Z' is $\left(\frac{9}{2}, \frac{9}{2}\right)$.

The second directrix is parallel to the given directrix x + y = 0 so that the equation of second directrix is

$$x + y + \lambda = 0 \qquad \text{.........(1)}$$

Since the equation (1) passes through the point $Z'\left(\frac{9}{2}, \frac{9}{2}\right)$.

$\therefore\ \frac{9}{2} + \frac{9}{2} + \lambda = 0 \qquad$ i.e. $\lambda = -9$

Thus the equation (1) becomes

x + y = 9 is the required equation of other directrix.

Ex. 5: Prove that the sum of the square of the reciprocals of two perpendicular radius vectors of an ellipse is constant.

Solution:

Here, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{.........(1)}$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$ in (1),we get

$$r^2\frac{\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

or $\quad \frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$

If the radius vector CP being length r_1, makes an angle θ with x-axis, then $\quad \frac{1}{r_1^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \qquad$ ....(1)

and the radius vector CQ being length r_2 perpendicular to CP, then

$$\frac{1}{r_2^2} = \frac{\left\{\cos\left(\frac{\pi}{2} + \theta\right)\right\}^2}{a^2} + \frac{\left\{\sin\left(\frac{\pi}{2} + \theta\right)\right\}^2}{b^2}$$

$$\frac{1}{r_2^2} = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} \qquad \text{.........(2)}$$

Now, from (1) and (2)

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}+\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

$$\therefore \frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

Ex. 6: Find the eccentricity of an ellipse if its latus rectum is half of its minor axis.

Solution:

Here, the equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Given that, $\frac{2b^2}{a}=\frac{2b}{2}$

or $2b = a$

Squaring, $4b^2 = a^2$

or $4a^2 - 4a^2e^2 = a^2$

or $3a^2 = 4a^2e^2$

or $e^2 = \frac{3}{4}$ $\therefore e = \frac{\sqrt{3}}{2}$

Exercise - 33

1. Find the equation of ellipse
   i. whose foci $(\pm 2, 0)$ and eccentricity $\frac{1}{2}$.
   ii. whose vertices $(\pm 5, 0)$ and foci $(\pm 4, 0)$
   iii. focus (2, 4) eccentricity $\frac{1}{\sqrt{13}}$ and directrix $x - y + 13 = 0$
   iv. focus (-1, 1) eccentricity $\frac{1}{2}$ and directrix. $x - y + 3 = 0$
2. Find the equation of the ellipse whose center is the origin, whose axes are the axes of coordinates and which passes through
   i. the points (1, 4) and (- 3, 2)
   ii. the points (2, 1) and (1, - 3)
3. Find the equation of ellipse whose latus rectum 5 eccentricity $\frac{2}{3}$ and its axes as the axes of coordinates.
4. Find the eccentricity, latus rectum, directrix and foci of the ellipse
   i. $3x^2 + 4y^2 = 12$
   ii. $\frac{x^2}{9}+\frac{y^2}{16}=1$
   iii. $4x^2 + y^2 - 8x + 2y + 1 = 0$
   iv. $7x^2 + 6y^2 - 42x - 24y + 86 = 0$
   v. $8(x-1)^2 + 6(y+1)^2 = 1$
5. Find the center, length of axes, eccentricity and directrix of the ellipse
   i. $2x^2 + 3y^2 - 4x + 5y + 4 = 0$
   ii. $2x^2 + 3y^2 - 4x - 12y + 13 = 0$
6. Find the coordinates of the second focus and the equation of second directrix of the ellipse whose one focus is S(1, 2) and the corresponding directrix is the line $x - y = 5$ and $e = \frac{1}{2}$.
7. By transferring the origin to the point (2, 3) and turning the axes through an angle $\frac{\pi}{4}$ prove that $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ represents an ellipse.
8. Show that the point $x = a.\frac{1-t^2}{1+t^2}$ and $y = b.\frac{2t}{1+t^2}$ where t is a variable parameter, lies on an ellipse.
9. A point P moves in such a way that the sum of its distances from S and S_1 is 10 and $SS_1 = 8$. Show that the locus of P is an ellipse. Find its equation, its eccentricity and length of latus rectum.
10. Find the equation of the ellipse whose foci are at (-2, 4) and (4, 4) and major axis is 10. Also find eccentricity of the ellipse.
11. Find the eccentric angle of a point on the ellipse $\frac{x^2}{4}+\frac{y^2}{5}=2$ whose distance from the center is $\frac{\sqrt{34}}{2}$.

Answers

1. i. $3x^2 + 4y^2 = 48$ ii. $9x^2 + 25y^2 = 225$
   iii. $25x^2 + 25y^2 + 2xy - 130x - 134y + 169 = 0$
   iv. $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$
2. i. $3x^2 + 2y^2 = 35$ ii. $8x^2 + 3y^2 = 35$
3. $20x^2 + 36y^2 = 405$

4. i. $e = \frac{1}{2}, 3, x = \pm 4, (\pm 1, 0)$

ii. $\frac{\sqrt{7}}{4}, \frac{9}{2}, \quad y = \pm\frac{16}{\sqrt{7}}, \quad (0, \pm\sqrt{7})$

iii. $e = \frac{\sqrt{3}}{2}, 1, y = -1 \pm \frac{4\sqrt{3}}{3}, (1, -1 \pm \sqrt{3})$

iv. $e = \frac{1}{\sqrt{7}}, \frac{2\sqrt{6}}{7}, y = 2 \pm \frac{\sqrt{7}}{\sqrt{6}}, \left(3, 2 \pm \frac{1}{\sqrt{42}}\right)$

v. $e = \frac{1}{2}, \frac{\sqrt{6}}{4}, y = -1 \pm \sqrt{\frac{2}{3}}, \left(1, -1 \pm \frac{1}{2\sqrt{6}}\right)$

5. i. $\left(1, -\frac{5}{6}\right); \frac{1}{\sqrt{6}}, \frac{1}{3}, e = \frac{1}{\sqrt{3}}, x = 1 \pm \frac{1}{2\sqrt{2}}$

ii. $(1, 2), \sqrt{2}, \frac{2}{\sqrt{3}}, e = \frac{1}{\sqrt{3}}, x = 1 \pm \sqrt{\frac{3}{2}}$

6. $(-1, 4), x - y + 11 = 0$ 9. $9x^2 + 25y^2 = 225, e = \frac{4}{5}, \frac{18}{5}$

10. $16x^2 + 25y^2 - 32x - 200y + 16 = 0, e = \frac{3}{5}$ 11. $\pm\frac{\pi}{3}, \pm\frac{4\pi}{3}$

20.9 Tangent at a Given Point

To find the equation of the tangent at the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $P(x_1, y_1)$ be given point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $Q(x_2, y_2)$ be another point on the ellipse very close to P.

So $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ ....(1)

$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1$ ....(2)

On subtraction (1) and (2)

$$\frac{x_2^2 - x_1^2}{a^2} + \frac{y_2^2 - y_1^2}{b^2} = 0$$

or $\frac{(x_2 - x_1)(x_2 + x_1)}{a^2} = -\frac{(y_2 - y_1)(y_2 + y_1)}{b^2}$

or $\frac{y_2 - y_1}{x_2 - x_1} = -\frac{b^2}{a^2}\frac{(x_2 + x_1)}{(y_2 + y_1)}$ ....(3)

The equation of the chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $y - y_1 = -\frac{b^2}{a^2}\frac{(x_2 + x_1)}{y_2 + y_1}(x - x_1)$

When $Q \to P$, $x_2 \to x_1$, $y_2 \to y_1$, then

$$y - y_1 = -\frac{2b^2x_1}{2a^2y_1}(x - x_1)$$

or $\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2}$

or $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

Using (1)

$$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

is the required equation of tangent at (x_1, y_1) on the ellipse.

Note: In the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Any point of the ellipse is $(a\cos\phi, b\sin\phi)$, the equation of tangent at $(a\cos\phi, b\sin\phi)$ is $\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1$

20.10 Condition of Tangency

1. *To find the condition that the line $y = mx + c$ will touch the ellipse*

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1)

Let $y = mx + c$ ....(2)

be the straight line meet the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solving (1) and (2),

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

or $\frac{x^2}{a^2} + \frac{m^2x^2}{b^2} + \frac{2mcx}{b^2} + \frac{c^2}{b^2} = 1$

or $\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) x^2 + \frac{2mc}{b^2} x + \frac{c^2 - b^2}{b^2} = 0$

or $(b^2 + a^2m^2) x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$

This is quadratic equation in x, so x has two roots. The line (1) touches the one point if x has two equal roots. The two roots of x are equal if

$$(2mca^2)^2 - 4(b^2 + a^2m^2) a^2(c^2 - b^2) = 0$$

or $4m^2c^2a^4 - 4a^2(b^2 + a^2m^2)(c^2 - b^2) = 0$

or $m^2c^2a^2 - (b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2) = 0$

or $m^2c^2a^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$

or $-c^2 + b^2 + a^2m^2 = 0$

or $c^2 = a^2m^2 + b^2$

$\therefore$ $c = \pm\sqrt{a^2m^2 + b^2}$ is the required condition of tangency.

Note 1: Putting $c = \pm\sqrt{a^2m^2 + b^2}$ in (1),

We get $y = mx \pm \sqrt{a^2m^2 + b^2}$ is always tangent on the ellipse.

Note 2: Putting $c = \sqrt{a^2m^2 + b^2}$ in quadratic equation in x, we get,

$$(a^2m^2 + b^2) x^2 + 2m\sqrt{a^2m^2 + b^2}\ a^2x + a^2 (a^2m^2 + b^2 - b^2) = 0$$

or $(\sqrt{a^2m^2 + b^2}\, x)^2 + 2\sqrt{a^2m^2 + b^2}\, x.\ a^2m + (a^2m)^2 = 0$

or $(\sqrt{a^2m^2 + b^2}\, x + a^2m)^2 = 0$

$$\therefore x = -\frac{a^2m}{\sqrt{a^2m^2 + b^2}}$$

and $y = mx + \sqrt{a^2m^2 + b^2}$

$$= -m\frac{a^2m}{\sqrt{a^2m^2 + b^2}} + \sqrt{a^2m^2 + b^2} = \frac{-a^2m^2 + a^2m^2 + b^2}{\sqrt{a^2m^2 + b^2}}$$

$$\therefore y = \frac{b^2}{\sqrt{a^2m^2 + b^2}}$$

The point of contact is $\left(-\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$

To find the condition that the line $lx + my + n = 0$ will touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the point of contact. [2059/062 B.E.]

Let $P(x_1, y_1)$ be the coordinates of the point of contact, then the equation of tangent at $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \text{..........(1)}$$

Since the line $lx + my + n = 0$ ....(2)

is at tangent to the ellipse, then the equation (1)and (2) should be identical. Comparing their coefficients

$$\frac{l}{x_1/a^2} = \frac{m}{y_1/b^2} = \frac{-n}{1}$$

$$\therefore x_1 = -\frac{a^2l}{n}, \qquad y_1 = -\frac{b^2m}{n}$$

The point (x_1, y_1) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

or $\frac{a^4l^2}{a^2n^2} + \frac{b^4m^2}{b^2n^2} = 1$

$\therefore a^2l^2 + b^2m^2 = n^2$

is the required condition that the line (2) should be a tangent to the given ellipse (1).

20.11 Normal at the Point (x_1, y_1)

To find the equation of the normal at the point (x_1, y_1) of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation of tangent at the point (x_1, y_1) on the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

or $\frac{yy_1}{b^2} = -\frac{xx_1}{a^2} + 1$

or $y = -\frac{b^2x_1}{a^2y_1}x + \frac{b^2}{y_1}$

Slope of the tangent $= -\frac{b^2x_1}{a^2y_1}$

Slope of the normal $= \frac{a^2}{b^2}\frac{y_1}{x_1}$

Hence the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1)$$

$\therefore$ $\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}$ is the required equation of normal.

20.12 Normal at the Point ϕ

To find the equation of the normal at the point (a cosϕ, bsinϕ) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The equation of tangent at the point (acosϕ, bsinϕ) of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x\, a\cos\phi}{a^2} + \frac{y\, b\sin\phi}{b^2} = 1$$

or $\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1$ ....(I)

Slope of the tangent $= -\frac{b}{a}\frac{\cos\phi}{\sin\phi}$

Slope of the normal $= \frac{a}{b}\frac{\sin\phi}{\cos\phi}$

Hence the equation of normal at (acosϕ, bsinϕ) to the ellipse is

$$y - b\sin\phi = \frac{a}{b}\frac{\sin\phi}{\cos\phi}(x - a\cos\phi)$$

or $\frac{by - b^2\sin\phi}{\sin\phi} = \frac{ax - a^2\cos\phi}{\cos\phi}$

or $by\operatorname{cosec}\phi - b^2 = ax\sec\phi - a^2$

$\therefore$ $ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2$

is the equation of normal at the point (acosϕ, bsinϕ) to the ellipse

20.13 Normal in the Form y = mx + c

To find the equation of the normal in the form y = mx + c to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The equation of the normal at the point (x_1, y_1) is

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}$$

or $\frac{a^2(x - x_1)}{x_1} = \frac{b^2(y - y_1)}{y_1}$

or $\frac{a^2x}{x_1} - a^2 = \frac{b^2y}{y_1} - b^2$

or $y = \frac{a^2y_1}{b^2x_1}x - \frac{(a^2 - b^2)y_1}{b^2}$ ....(1)

Slope of the normal $m = \frac{a^2y_1}{b^2x_1}$

So that $x_1 = \frac{a^2y_1}{b^2m}$

Since (x_1, y_1) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

or $\frac{1}{a^2}\frac{a^4y_1^2}{b^4m^2} + \frac{y_1^2}{b^2} = 1$

or $\frac{a^2y_1^2}{b^4m^2} + \frac{y_1^2}{b^2} = 1$

or $\frac{(a^2 + b^2m^2)y_1^2}{b^4m^2} = 1$

$\therefore$ $y_1 = \frac{b^2m}{\sqrt{a^2 + b^2m^2}}$

So, (1) becomes

$$y = mx - \frac{(a^2 - b^2)}{b^2}\frac{(b^2m)}{\sqrt{a^2 + b^2m^2}}$$

$$y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$$

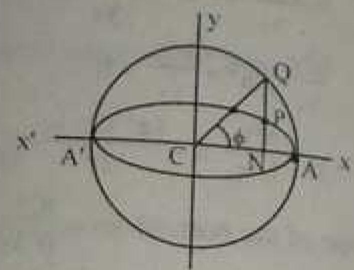
is the required equation of normal in slope form.

20.14 Auxiliary Circle, Eccentric Angle

The circle described on the major axis of an ellipse as diameters is called *Auxiliary circle* of the ellipse.

Draw a circle with the major axis AA' as the diameter so that the auxiliary circle is formed with center C and radius 'a'.

Let P be any point on the ellipse, draw the ordinate PN and produce NP to meet the circle at Q. Joint CQ and let $\angle QCN = \phi$, the angle ϕ is called eccentric angle of the point P on the ellipse.



We have $\cos\phi = \dfrac{CN}{CQ}$

or $CN = CQ\cos\phi$

$\therefore$ $CN = a\cos\phi$

Since P lies on the ellipse, we have

$$\frac{CN^2}{a^2} + \frac{PN^2}{b^2} = 1$$

or $\dfrac{a^2\cos^2\phi}{a^2} + \dfrac{PN^2}{b^2} = 1$

or $\dfrac{PN^2}{b^2} = \sin^2\phi$

$\therefore$ $PN = b\sin\phi$

Hence the co-ordinates of P is $P(a\cos\phi, b\sin\phi)$

Also, $QN = CQ\sin\phi$

$QN = a\sin\phi$

The co-ordinates of Q is

$Q(a\cos\phi, a\sin\phi)$

and the equation of auxiliary circle is $x^2 + y^2 = a^2$

20.15 Equation of Chord Joining two Points whose Eccentric Angles are θ and φ.

Let θ and ϕ be eccentric angles of the point P and Q on the ellipse, so the coordinates of P and Q are $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ respectively.

The equation of chord PQ is

$$y - b\sin\theta = \frac{b\sin\phi - b\sin\theta}{a\cos\phi - a\cos\theta}(x - a\cos\theta)$$

or $\dfrac{x - a\cos\theta}{a(\cos\phi - \cos\theta)} = \dfrac{y - b\sin\theta}{b(\sin\phi - \sin\theta)}$

or $\dfrac{\frac{x}{a} - \cos\theta}{-2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)} = \dfrac{\frac{y}{b} - \sin\theta}{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)}$

or $\dfrac{\frac{x}{a} - \cos\theta}{-\sin\left(\frac{\theta+\phi}{2}\right)} = \dfrac{\frac{y}{b} - \sin\theta}{\cos\left(\frac{\theta+\phi}{2}\right)}$

or $\dfrac{x}{a}\cos\left(\dfrac{\theta+\phi}{2}\right) - \cos\theta\cos\left(\dfrac{\theta+\phi}{2}\right)$

$$= -\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) + \sin\theta\sin\left(\frac{\theta+\phi}{2}\right)$$

or $\dfrac{x}{a}\cos\left(\dfrac{\theta+\phi}{2}\right) + \dfrac{y}{b}\sin\left(\dfrac{\theta+\phi}{2}\right)$

$$= \cos\theta\cos\left(\frac{\theta+\phi}{2}\right) + \sin\theta\sin\left(\frac{\theta+\phi}{2}\right)$$

or $\dfrac{x}{a}\cos\left(\dfrac{\theta+\phi}{2}\right) + \dfrac{y}{b}\sin\left(\dfrac{\theta+\phi}{2}\right) = \cos\left(\theta - \dfrac{\theta+\phi}{2}\right)$

$\therefore$ $\dfrac{x}{a}\cos\left(\dfrac{\theta+\phi}{2}\right) + \dfrac{y}{b}\sin\left(\dfrac{\theta+\phi}{2}\right) = \cos\left(\dfrac{\theta-\phi}{2}\right)$

Worked out Examples

Ex. 1: Find the equation of the tangent and normal at the point $(\sqrt{3}, 2)$ of the ellipse $4x^2 + 3y^2 = 24$

Solution:

Here, the equation of ellipse is

$$4x^2 + 3y^2 = 24$$

or $\frac{x^2}{6} + \frac{y^2}{8} = 1$ ....(1)

The equation of tangent at $(\sqrt{3}, 2)$ of the ellipse (1) is

$$\frac{x \times \sqrt{3}}{6} + \frac{y \times 2}{8} = 1$$

or $\frac{x \times \sqrt{3}}{6} + \frac{y}{4} = 1$

or $2\sqrt{3}\,x + 3y = 12$

The slope of the tangent $= -\frac{2\sqrt{3}}{3}$

The slope of the normal $= \frac{3}{2\sqrt{3}}$

Thus the equation of normal at $(\sqrt{3}, 2)$ of the ellipse is

$$y - 2 = \frac{3}{2\sqrt{3}}(x - \sqrt{3})$$

or $2y - 4 = \sqrt{3}\,x - 3$

or $\sqrt{3}\,x - 2y + 1 = 0$

$\therefore$ $3x - 2\sqrt{3}\,y + \sqrt{3} = 0$

Ex. 2: Show that the line $3x + 4y^2 = 1$ touches the ellipse $3x^2 + 4y^2 + \sqrt{7} = 0$, also find the point of contact.

Solution:

Here, the given line is $3x + 4y + \sqrt{7} = 0$ ....(1)

and the ellipse is $3x^2 + 4y^2 = 1$ ....(2)

Solving these two equations (1) and (2)

$$3x^2 + 4\left(\frac{-\sqrt{7} - 3x}{4}\right)^2 = 1$$

or $3x^2 + \frac{4}{16}(7 + 6\sqrt{7}\,x + 9x^2) = 1$

or $12x^2 + 7 + 6\sqrt{7}\,x + 9x^2 = 4$

or $21x^2 + 6\sqrt{7}\,x + 3 = 0$

or $7x^2 + 2\sqrt{7}\,x + 1 = 0$

or $(\sqrt{7}x + 1)^2 = 0$

or $\therefore x = -\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}$

From (1), $y = \frac{1}{4}\left(-\sqrt{7} + 3 \times \frac{1}{\sqrt{7}}\right)$

or $y = \frac{-7 + 3}{4\sqrt{7}} = -\frac{4}{4\sqrt{7}} = -\frac{1}{\sqrt{7}}$

$\therefore y = -\frac{1}{\sqrt{7}}$

Thus the given line (1) is tangent to the given ellipse and the point of contact is $\left(-\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$.

Ex. 3: Find the condition that the line $2x - 3y = c$ may normal to the ellipse $9x^2 + 16y^2 = 144$

Solution:

Here, the equation of line is

$2x - 3y = c$ ....(1)

and the equation of the ellipse is

$$9x^2 + 16y^2 = 144$$

or $\frac{x^2}{16} + \frac{y^2}{9} = 1$ ....(2)

Any point of this ellipse is $(4\cos\phi, 3\sin\phi)$.

The equation of normal at the point $(4\cos\phi, 3\sin\phi)$ to the ellipse is

$$\frac{16(x - 4\cos\phi)}{4\cos\phi} = \frac{9(y - 3\sin\phi)}{3\sin\phi}$$

or $4x\sec\phi - 16 = 3y\,\mathrm{cosec}\phi - 9$

or $4x\sec - 3y\,\mathrm{cosec}\phi = 7$ ...(3)

The equation of the line (1) is normal to the ellipse if it is identical with the equation (3).

Then

$$\frac{4\sec\phi}{2} = \frac{-3\ \text{cosec}\phi}{-3} = \frac{7}{c}$$

From first and third ratios

$$2\sec\phi = \frac{7}{c}$$

or $\sec\phi = \frac{7}{2c}$ $\quad \therefore \cos\phi = \frac{2c}{7}$

From second and third ratios

$$\text{cosec}\phi = \frac{7}{c} \qquad \therefore \sin\phi = \frac{c}{7}$$

Squaring and adding,

$$1 = \frac{4c^2}{49} + \frac{c^2}{49}$$

or $1 = \frac{5c^2}{49}$

or $c^2 = \frac{49}{5}$ $\quad \therefore c = \pm\frac{7}{\sqrt{5}}$

Ex. 4: Find the locus of the point of intersection of a pair of perpendicular tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution:

Let the straight lines

$$y = mx \pm \sqrt{a^2m^2+b^2} \qquad \text{...(1)}$$

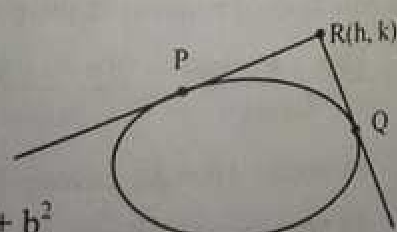
be the tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{....(2)}$$

Suppose the lines (1) meet at R(h, k),

So, $k = mh \pm \sqrt{a^2m^2+b^2}$

or $k - mh = \pm\sqrt{a^2m^2+b^2}$



Squaring

$$(k-mh)^2 = a^2m^2+b^2$$

or $k^2 - 2kmh + m^2h^2 = a^2m^2 + b^2$

or $m^2(h^2-a^2) - 2mkh + k^2 - b^2 = 0$

Which is a quadratic equation in m and thus m has two values, say m_1 and m_2.

Since the tangents are perpendicular

We have $m_1m_2 = -1$

or $\frac{k^2-b^2}{h^2-a^2} = -1$

or $k^2 - b^2 = -h^2 + a^2$

or $h^2 + k^2 = a^2 + b^2$

Hence the locus of the point of intersection R(h, k) is

$$x^2 + y^2 = a^2 + b^2$$

Clearly, it represents the equation of circle with center origin and radius $\sqrt{a^2+b^2}$. This circle is called *Director Circle*.

Ex. 5: Find the locus of a point from which two tangents can be drawn to an ellipse making angles θ_1 and θ_2 with the major axis such that $\tan\theta_1 + \tan\theta_2 = 2c$(constant)

Solution:

Here, the equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{..............(1)}$$

The equation of tangents to the ellipse (1) in slope form is

$$y = mx \pm \sqrt{a^2m^2+b^2} \qquad \text{..............(2)}$$

The tangents meet at a point (h, k), then

$$k = mh \pm \sqrt{a^2m^2+b^2}$$

or $k - mh = \sqrt{a^2m^2+b^2}$

Squaring

$$(k-mh)^2 = a^2m^2+b^2$$

or $m^2(h^2-a^2) - 2mkh + k^2 - b^2 = 0$

This is quadratic equation in m. So,m has two roots say, m_1 and m_2

Then $m_1 + m_2 = \frac{2kh}{h^2-a^2}$

$$\tan\theta_1 + \tan\theta_2 = \frac{2kh}{h^2-a^2}$$

But given that

$\tan\theta_1 + \tan\theta_2 = 2c$(constant)

or $\frac{2kh}{h^2 - a^2} = 2c$

or $kh = c(h^2 - a^2)$

Thus the locus of the point (h, k) is

$c(x^2 - a^2) - xy = 0$

Ex. 6: Find the condition that the straight line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

Let $(a\cos\phi, b\sin\phi)$ be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation of tangent at $(a\cos\phi, b\sin\phi)$ to the ellipse is

or $\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1$ ....(1)

If the line $x\cos\alpha + y\sin\alpha = p$ ....(2)

touches the ellipse, then it must be identical with (1).

Hence $\frac{\cos\phi}{a\cos\alpha} = \frac{\sin\phi}{b\sin\alpha} = \frac{1}{p}$

$\therefore \cos\phi = \frac{a}{p}\cos\alpha, \quad \sin\phi = \frac{b}{p}\sin\alpha$

Squaring and adding,

$$1 = \frac{a^2\cos^2\alpha}{p^2} + \frac{b^2\sin^2\alpha}{p^2}$$

$\therefore p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$ is the required condition.

Ex. 7: If the line the line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then, show that $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$ [2061 B.E.]

Solution:

Let $(a\cos\phi, b\sin\phi)$ be any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then the equation of normal at the point $(a\cos\phi, b\sin\phi)$ to the ellipse is

$$\frac{a^2(x - a\cos\phi)}{a\cos\phi} = \frac{b^2(y - b\sin\phi)}{b\sin\phi}$$

or $\frac{a(x - a\cos\phi)}{\cos\phi} = \frac{b(y - b\sin\phi)}{\sin\phi}$

or $ax\sec\phi - by\,\mathrm{cosec}\phi = a^2 - b^2$ ....(1)

If the line $lx + my + n = 0$ ....(2)

be the normal to the ellipse, then it must be identical with the equation (1)

So $\frac{a\sec\phi}{l} = \frac{-b\,\mathrm{cosec}\phi}{m} = \frac{a^2 - b^2}{-n}$

From first and last ratios

$$\frac{a\sec\phi}{l} = \frac{a^2 - b^2}{-n}$$

or $\sec\phi = \frac{l(a^2 - b^2)}{-an}$

$\therefore \cos\phi = -\frac{an}{l(a^2 - b^2)}$ ....(3)

From second and last ratios

$$\frac{b\,\mathrm{cosec}\phi}{m} = \frac{(a^2 - b^2)}{n}$$

or $\mathrm{cosec}\phi = \frac{m(a^2 - b^2)}{bn}$

$\therefore \sin\phi = \frac{bn}{m(a^2 - b^2)}$ ....(4)

Squaring and adding (3) and (4)

$$1 = \frac{a^2n^2}{l^2(a^2 - b^2)^2} + \frac{b^2n^2}{m^2(a^2 - b^2)^2}$$

or $1 = \frac{n^2}{(a^2 - b^2)^2}\left(\frac{a^2}{l^2} + \frac{b^2}{m^2}\right)$

$\therefore \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{b^2}$ is the required condition.

Ex. 8: Find the equation to the tangents to the ellipse $4x^2 + 3y^2 = 1$ which are perpendicular to the line $3x - 4y + 5 = 0$.

Solution:

Here, the equation of ellipse is

$4x^2 + 3y^2 = 1$

or $\dfrac{x^2}{1/4} + \dfrac{y^2}{1/3} = 1$ ......(1)

So, $a^2 = \dfrac{1}{4}$, $b^2 = \dfrac{1}{3}$.

Any line perpendicular to the line

$3x - 4y + 5 = 0$

is $4x + 3y + c = 0$ ......(2)

If this equation of line is tangent to the ellipse (1) then the condition of tangency is ......(3)

$a^2l^2 + b^2m^2 = n^2$

Here, $l = 4$, $m = 3$, $n = c$

So $\dfrac{1}{4} \times 4^2 + \dfrac{1}{3} \times 3^2 = c^2$

or $4 + 3 = c^2$

or $c^2 = 7$, $\therefore c = \pm\sqrt{7}$

Therefore the required equation of tangents are

$4x + 3y \pm \sqrt{7} = 0$

Exercise - 34

1. Find the equations of tangent and normal
   i. at the point (3, -2) of the ellipse $4x^2 + 9y^2 = 36$
   ii. at the point $\left(1, \dfrac{4}{3}\right)$ of the ellipse $4x^2 + 9y^2 = 20$
   iii. at the ends of the latus rectum of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
2. Find the equation of the tangents to the ellipse $9x^2 + 16y^2 = 144$ and passes through the point (2, 3).
3. Show that $4x + y + 7 = 0$ is a tangent in the ellipse $x^2 + 3y^2 = 3$ also find the point of contact.
4. Show that $x + y = \sqrt{a^2 + b^2}$ is a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and find the coordinates of the point of contact.
5. Find the equation of the tangents to the ellipse $3x^2 + 2y^2 = 1$ perpendicular to the line $x + 3y = 1$.
6. Find the equation of tangent to the ellipse $x^2 + 3y^2 = 3$ which are perpendicular to the straight line $x - 4y - 5 = 0$

7. Find the equation is to the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the straight line $3x - y + 7 = 0$, also find the point of contact.
8. Find the equations of the tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ which make equal intercepts on the both axes.
9. Find the points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ such that the tangents at each of them makes equal angles with the axes. Show that the length of the perpendicular from the center on either of these tangents is $\sqrt{\dfrac{a^2 + b^2}{2}}$.
10. If the normal at the end of a latus rectum of an ellipse passes through one extremity of the minor axis, then show that the eccentricity of the curve is given by the equation $e^4 + e^2 - 1 = 0$
11. Prove that locus of the foot of the perpendicular drawn from the center on any tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.
12. Find the locus of the feet of the perpendicular drawn upon any tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ from either focus.
13. If Q be the point on the auxiliary circle corresponding to a point P on an ellipse, then show that the normals at P and Q meet on a fixed circle $x^2 + y^2 = (a + b)^2$.
14. Find the locus of a point from which two tangents can be drawn to an ellipse making angles θ_1 and θ_2 with the major axis such that
    i. $\tan\theta_1 - \tan\theta_2 = d$(constant) ii. $\cot\theta_1 - \cot\theta_2 = 2d$ (constant)

Answers

1. i. $2x - 3y = 6, 3x + 2y = 5$ ii. $x + 3y = 5, 9x - 3y = 5$

   iii. $ex + y = a, x - ey = ae^3$

2. $x + y = 5, x + y = -5$ 3. $\left(-\frac{12}{7}, -\frac{1}{7}\right)$ 4. $\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right)$

5. $3x - y \pm \sqrt{\frac{7}{2}} = 0$ 6. $4x + y \pm 7 = 0$

7. $y = 3x \pm \frac{1}{2}\sqrt{\frac{155}{3}}$ $\left(\pm\frac{15}{2}\sqrt{\frac{3}{155}}, \frac{10}{3}\sqrt{\frac{3}{55}}\right)$

8. $x + y = \pm\sqrt{a^2+b^2}$ 9. $\left(\frac{a^2}{\sqrt{a^2+b^2}}, \frac{b^2}{\sqrt{a^2+b^2}}\right)$

12. $x^2 + y^2 = a^2$

❖❖❖

Chapter - 21

The Hyperbola

20.1 Introduction

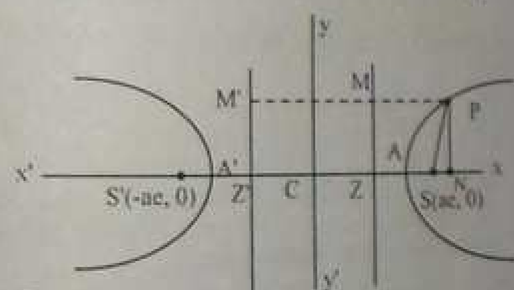
The locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point to its distance from a fixed straight line is greater than unity, is called a *Hyperbola*. The fixed point is called the *Focus* and the fixed straight line is called the *Directrix* of the hyperbola.

A hyperbola is also defined as the locus of a point in a plane such that the difference of the distances from two fixed points is a positive constant.

21.2 Equation of Hyperbola in Standard Form

Let S be the focus and ZM be the dirictrix of a hyperbola.

Draw ZS⊥ZM. Let A and A' be the points on the hyperbola such that



$$SA = eAZ \quad \text{.....(1)}$$

and $SA' = eA'Z$ .....(2)

Let C be the middle point of AA' such that $AA' = 2a$.

On subtraction (1) and (2),

$SA' - SA = e(A'Z - AZ)$

or $AA' = e[(CA' + CZ) - (CA - CZ)]$

or $2a = e.2CZ$

$\therefore CZ = \frac{a}{e}$

On addition (1) and (2),

$SA + SA' = e(AZ + A'Z)$

or $(CS - CA) + (CS + CA') = e[(CA - CZ) + (CZ + CA')]$

or $2CS = e.2a$

$\therefore CS = ae.$

Let C be the origin and CAX as the axis of x and a line through C perpendicular to CAX as the axis of y.

Therefore the coordinate of focus is S(ae, 0) and the equation of directrix is $x = a/e$.

If P(x, y) be any point on the hyperbola, then

$CN = x,\ PN = y$

By definition of the hyperbola

$SP = ePM$

or $SP^2 = e^2 PM^2 = e^2 ZN^2$

or $(x - ae)^2 + (y - 0)^2 = e^2 (CN - CZ)^2$

or $x^2 - 2aex + a^2e^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$

or $x^2 - 2aex + a^2e^2 + y^2 = e^2 x^2 - 2aex + a^2$

or $a^2(e^2 - 1) = x^2 (e^2 - 1) - y^2$

or $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$

$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$ is the equation of the hyperbola in standard form.

21.3 A Hyperbola has two Foci and Directrices

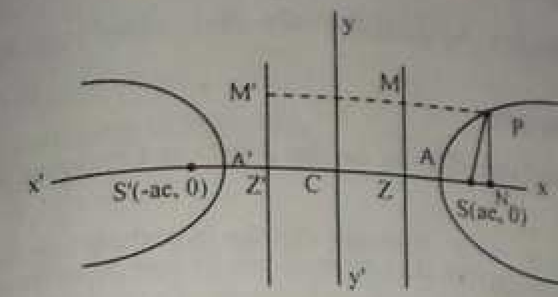
Let S' be a point on the x-axis left of the origin C such that

$S'C = SC = ae$

and another point Z' such that $CZ' = CZ = a/e$.

Therefore the coordinate of other focus is S'(- ae, 0) and the equation of the directrix Z'M' is

$x = -\frac{a}{e}$ $\qquad \therefore x + \frac{a}{e} = 0$



From the definition

$SP = ePM$

or $SP^2 = e^2 PM^2$

or $SP^2 = e^2ZN^2 = e^2(CN - CZ)^2$

or $(x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$

or $x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$

or $x^2 + a^2e^2 + y^2 = e^2x^2 + a^2$

or $(x + ae)^2 + (y - 0)^2 = e^2 (x + a/e)^2$

or $S'P^2 = e^2 (CN + CZ)^2$

$= e^2(Z'N)^2$

or $S'P^2 = e^2PM'^2$

or $S'P = e\ PM'$

$\therefore S'P = e\ PM'$

It shows that for any point P on the hyperbola there exist a second focus and second directrix.

21.4 Some Important Definition on a Hyperbola

The standard form of the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Some definitions related to it are mentioned below.

Transverse axis

The line segment AA' is the *Transverse Axis* of the hyperbola .Length of transverse axis,

$AA' = 2a$

Co-ordinates are A(- a, 0) and A'(a, 0).

Conjugate axis

The line segment BB' lie on the y-axis is *Conjugate Axis* of the hyperbola.

when $x = 0$, $y^2 = -b^2$. It shows that the hyperbola cuts the y-axis in imaginary points.

Since the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ contains only even powers of x and y, it is symmetrical on both transverse and conjugate axes.

Center

The intersection of transverse and conjugate axis of the hyperbola is called *Center* of the ellipse.

Vertices

The *Vertices* of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for $a > b$ is the point of intersection of the hyperbola and transverse axis. So the vertices of the hyperbola are A(a, 0) and A'(- a, 0).

Latus rectum

The double ordinate passing through the focus and perpendicular to the transverse axis is called *Latus rectum* of the hyperbola.

For the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Putting $x = ae$, $y = \pm \frac{b^2}{a}$

The ends of the latus rectum L and L' are

$$L\left(ae, \frac{b^2}{a}\right) \text{ and } L\left(ae, -\frac{b^2}{a}\right)$$

Length of latus rectum = Distance between L and L' = $\frac{2b^2}{a}$

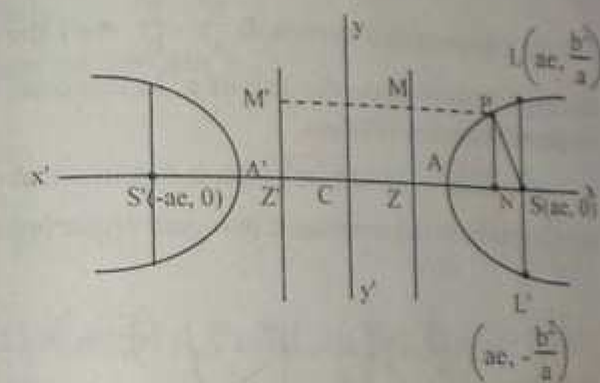
1.5 A Hyperbola whose Transverse Axis is Along x-axis

For the equation of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The transverse axis of the hyperbola is along the axis of x. In this case eccentricity e is obtained from the relation

$b^2 = a^2 (e^2 - 1)$ and the graph of the hyperbola is symmetrical on both axes.



Center is

C(0, 0)

Foci are

S'(- ae, 0), S(ae, 0)

Vertices are

A(a, 0), A' (- a, 0)

Directrices are

$$x = -\frac{a}{e},\ x = \frac{a}{e}$$

Transverse axis

AA' = 2a

Conjugate axis

BB' = Imaginary length

= 2b

Latus rectum

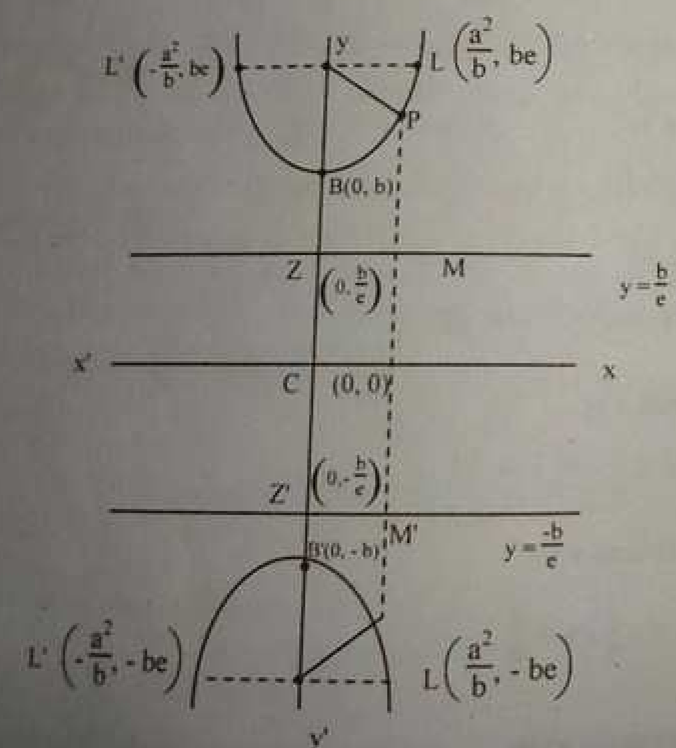
$LL' = \frac{2b^2}{a}$

21.6 A Hyperbola whose Transverse axis is Along the y-axis.

For the equation of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ the transverse axes of the hyperbola is along the axis of y. In this case, the eccentricity is obtained from the relation

$a^2 = b^2(e^2 - 1)$, but its graph is symmetrical on both axes.

This hyperbola is known as *Conjugate Hyperbola*.



Center is

C(0, 0).

Foci are

S(0, be) and S'(0, - be)

Vertices are

B(0, b), B'(0, - b)

Transverse axis

BB' = 2b.

Conjugate axis

AA' = imaginary length = 2a

Latus rectum

$LSL' = \frac{2a^2}{b}$

21.7 Focal Distances of a Point on the Hyperbola

In the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the coordinates of the foci S and S' are S(ae, 0) and S'(- ae, 0) respectively.

Let P(x, y) be any point on the hyperbola, then by the definition of a hyperbola

$$SP = ePM = e(CN - CZ)$$

$$= e\left(x - \frac{a}{e}\right) = ex - a$$

$\therefore$ $SP = ex - a$

and $S'P = ePM'$

$$= e(CZ + CN)$$

$$= e\left(\frac{a}{e} + x\right) = a + ex$$

Now, $S'P - SP = a + ex - ax + a$

$$= 2a = AA'$$

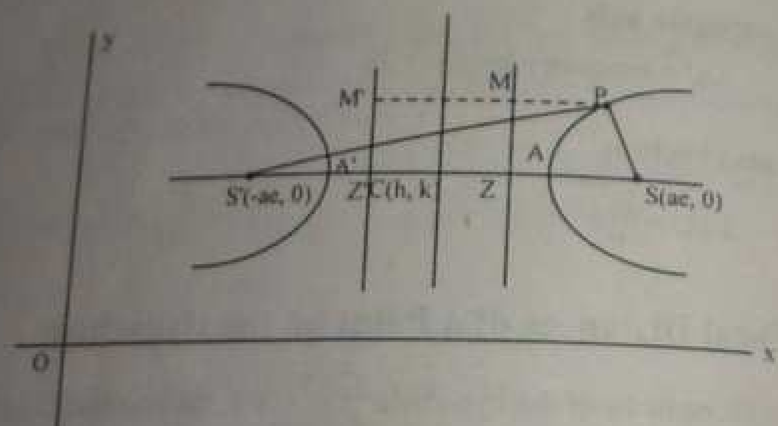
= length of transverse axis.

It shows that the difference of focal distance of any point on the hyperbola is equal to length of its transverse axis.

21.8 Equation of a Hyperbola with Center (h, k)

Let S and S' be two fixed point such that SS' = 2ae.

Take the line SS' as the parallel to the x-axis and C be the middle point of SS' with center C(h, k), then the coordinates of S and S' are S(h + ae, k) and S' (h - ae, k) respectively.



The equation directrix ZM is

$$x = h + \frac{a}{e}$$

Let P(x, y) be any point on the hyperbola, by definition of the hyperbola, we have,

$$S'P - SP = 2a$$

$$\sqrt{(x-h+ae)^2+(y-k)^2} - \sqrt{(x-h-ae)^2+(y-k)^2} = 2a$$

or $\sqrt{(x-h+ae)^2+(y-k)^2} = 2a + \sqrt{(x-h-ae)^2+(y-k)^2}$

Squaring,

$$(x+h+ae)^2+(y-k)^2 = 4a^2 + 4a\sqrt{(x-h-ae)^2+(y-k)^2} + (x-h-ae)^2+(y-k)^2$$

or $(x-h)^2 + 2ae(x-h) + a^2e^2 = 4a^2 + 4a\sqrt{(x-h-ae)^2+(y-k)^2} + (x-h)^2 - 2ae(x-h) + a^2e^2$

or $4ae(x-h) = 4a\left(a + \sqrt{(x-h-ae)^2+(y-k)^2}\right)$

or $e(x-h) = a + \sqrt{(x-h-ae)^2+(y-k)^2}$

or $e(x-h) - a = \sqrt{(x-h-ae)^2+(y-k)^2}$

Squaring,

$$e^2(x-h)^2 - 2ae(x-h) + a^2 = (x-h)^2 - 2ae(x-h) + a^2e^2 + (y-k)^2$$

or $(e^2-1)(x-h)^2 - (y-k)^2 = a^2(e^2-1)$

or $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{a^2(e^2-1)} = 1$

or $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $b^2 = a^2(e^2-1)$

is the required equation of hyperbola.

21.9 Parametric Equation of Hyperbola

In the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$x = a\sec\theta$, and $y = b\tan\theta$ is always satisfied in the equation.

Then the equation of the hyperbola in parametric form is

$x = a\sec\theta$, $y = b\tan\theta$ for all values of θ.

Therefore, any point on the hyperbola is $(a\sec\theta, b\,\mathrm{cosec}\theta)$.

	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
Center	(0, 0)	(0, 0)	(h, k)
Vertices	$(\pm a, 0)$	$(0, \pm b)$	$(h \pm a, k)$
Eccentricity	$b^2 = a^2(e^2-1)$ $e = \sqrt{1+\frac{b^2}{a^2}}$	$a^2 = b^2(e^2-1)$ $e = \sqrt{1+\frac{a^2}{b^2}}$	$b^2 = a^2(e^2-1)$ $e = \sqrt{1+\frac{b^2}{a^2}}$
Foci	$(\pm ae, 0)$	$(0, \pm be)$	$(h \pm ae, k)$
Directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$	$x = h \pm \frac{a}{e}$
Transverse axis	2a	2b	2a
Conjugate axis	2b	2a	2b
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$	$\frac{2b^2}{a}$

Worked out Examples

Ex. 1: Find the center, vertices, foci, eccentricity, directrices and latus rectum of the hyperbola.

i. $3x^2 - 4y^2 = 36$

ii. $x^2 - 4y^2 - 2x + 24y - 37 = 0$

iii. $\frac{x^2}{9} - \frac{y^2}{16} = -1$

i. $3x^2 - 4y^2 = 36$

Solution:

Here, the equation of the hyperbola is

$$3x^2 - 4y^2 = 36$$

or $\frac{x^2}{12} - \frac{y^2}{9} = 1$

Comparing it with the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We find $a^2 = 12$, $b^2 = 9$

$a = 2\sqrt{3}$, $b = 3$

Center (0, 0)

Clearly, the transverse axis is along the x-axis, vertices are $(-2\sqrt{3}, 0), (2\sqrt{3}, 0)$

We have,

$$b^2 = a^2(e^2 - 1)$$

or $9 = 12(e^2 - 1)$

or $9 + 12 = 12e^2$

or $21 = 12e^2$

or $e^2 = \frac{21}{12}$ $\therefore e = \frac{\sqrt{7}}{2}$

Foci = $(\pm ae, 0) = (\pm\sqrt{21}, 0)$

Directrices $x = \pm\frac{a}{e}$

$x = \pm 4\sqrt{\frac{3}{7}}$

Latus rectum $= \frac{2b^2}{a} = 2 \times \frac{9}{2\sqrt{3}} = 3\sqrt{3}$

ii. $x^2 - 4y^2 - 2x + 24y - 37 = 0$

Solution:

Here, the equation of the hyperbola is

$$x^2 - 4y^2 - 2x + 24y - 37 = 0$$

$$x^2 - 2x - 4(y^2 - 6y) - 37 = 0$$

or $(x-1)^2 - 1 - 4(y-3)^2 + 36 - 37 = 0$

or $(x-1)^2 - 4(y-3)^2 = 2$

or $\frac{(x-1)^2}{2} - \frac{(y-3)^2}{1/2} = 1$

Comparing it with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

We find $a^2 = 2$, $b^2 = \frac{1}{2}$

$\therefore a = \sqrt{2}$, $b = \frac{1}{\sqrt{2}}$

Center (h, k) = (1, 3),

Vertices = $(h \pm a, k) = (1 \pm \sqrt{2}, 3)$

We have,

$$b^2 = a^2(e^2 - 1)$$

or $\frac{1}{2} = 2(e^2 - 1)$

or $\frac{1}{2} + 2 = 2e^2$

or $e^2 = \frac{5}{4}$

or $e = \frac{\sqrt{5}}{2}$

$\therefore$ eccentricity $= \frac{\sqrt{5}}{2}$

Foci $= (h \pm ae, k) = \left(1 \pm \sqrt{\frac{5}{2}}, 3\right)$

Directrices, $x = h \pm \frac{a}{e}$

$\therefore \quad x = 1 \pm \frac{2\sqrt{2}}{5}$

Latus rectum $= \frac{2b^2}{a} = 2.\frac{1}{2 \times \sqrt{2}} = \frac{1}{\sqrt{2}}$

iii. $\frac{x^2}{9} - \frac{y^2}{16} = -1$

Solution:

Here, the equation of the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{16} = -1$$

Comparing it with the conjugate hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

We find

$a^2 = 9, \quad b^2 = 16,$

$\therefore \quad a = 3, \quad b = 4$

Center is $(0, 0)$

Vertices are $(0, \pm b) = (0, \pm 4)$

We have

$a^2 = b^2 (e^2 - 1)$

or $9 = 16 (e^2 - 1)$

or $9 + 16 = 16e^2$

or $e^2 = \frac{25}{16}$

or $e = \frac{5}{4}$

$\therefore$ eccentricity $= \frac{5}{4}$

Foci $= (0, \pm be) = (0, \pm 5)$

Directrices, $y = \pm \frac{b}{e} = \pm \frac{16}{5}$

Ex. 2: Find the equation of the hyperbola referred to its axes as axes of coordinates

i. **with foci (6, 4) and (- 4, 4) eccentricity 2.**

ii. **with length of conjugate axes is 5 and distance between foci is 13.**

iii. **with focus (2, 2) eccentricity 2 and directrix $x + y = 9$**

i. **Foci (6, 4) and (- 4, 4) eccentricity 2.**

Solution:

Given eccentricity, $e = 2$ and foci S(6, 4) and S'(- 4, 4) then the distances between them,

$2ae = \sqrt{(6+4)^2 + (4-4)^2}$

or $2ae = \sqrt{(10)^2}$

or $2ae = 10$

or $2a.2 = 10$

$\therefore a = \frac{5}{2}$

We have,

$b^2 = a^2 (e^2 - 1)$

or $b^2 = \frac{25}{4}(4 - 1) = \frac{25}{4} \times 3$

or $b^2 = \frac{75}{4}$

Hence the equation of the hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or $\frac{4x^2}{25} - \frac{4y^2}{75} = 1$

$\therefore \quad 12x^2 - 4y^2 = 75$

ii. Length of conjugate axes is 5, distance between foci is 13.

Solution:

Length of conjugate axis

$2b = 5 \qquad \therefore \quad b = \frac{5}{2}$

and $\quad 2ae = 13 \qquad \therefore \quad ae = \frac{13}{2}$

We have

$b^2 = a^2e^2 - a^2$

or $\frac{25}{4} = \frac{169}{4} - a^2$

or $a^2 = \frac{169}{4} - \frac{25}{4}$

or $a^2 = \frac{144}{4}$

$\therefore \quad a^2 = 36$

Hence the equation of hyperbola in standard form is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or $\frac{x^2}{36} - \frac{4y^2}{25} = 1$

$\therefore \quad 25x^2 - 144y = 900$

iii. Focus (2, 2) ,eccentricity 2 and directrix $x + y = 9$

Solution:

Given, focus (2, 2), eccentricity $e = 2$, directrix $x + y = 9$

Let P(x, y) be any point on the hyperbola.

So, $SP = \sqrt{(x-2)^2 + (y-2)^2}$

Let ZM be the line of the equation $x + y = 9$

$PM = \frac{x+y-9}{\sqrt{1+1}} = \frac{x+y-9}{\sqrt{2}}$

By definition

$SP = e\,PM$

or $SP^2 = e^2\,PM^2$

or $(x-2)^2 + (y-2)^2 = 4\,\frac{(x+y-9)^2}{2}$

or $x^2 - 4x + 4 + y^2 - 4y + 4 = 2(x^2 + y^2 + 81 + 2xy - 18y - 18x)$

or $x^2 + y^2 - 4x - 4y + 8 = 2x^2 + 2y^2 + 4xy - 36y - 36x + 162$

$\therefore \quad x^2 + y^2 + 4xy - 32x - 32y + 154 = 0$

is the required equation of the hyperbola.

Ex. 3: Show that the locus of a point which moves in such a way that the differences of its distances from two fixed points is constant is a hyperbola. [2060 B.E.]

Solution:

Let S and S' be two fixed points with coordinates are S(c, d) and S'(- c, d). Let P(x, y) be any point

Given that

$SP - S'P = \text{constant } (2a)$

or $\sqrt{(x-c)^2 + (y-d)^2} - \sqrt{(x+c)^2 + (y-d)^2} = 2a$

or $\sqrt{(x-c)^2 + (y-d)^2} = 2a - \sqrt{(x+c)^2 + (y-d)^2}$

Squaring,

$(x-c)^2 + (y-d)^2 = 4a^2 - 4a\sqrt{(x+c)^2 + (y-d)^2} + (x+c)^2 + (y-d)^2$

or $x^2 - 2xc + c^2 = 4a^2 - 4a\sqrt{(x+c)^2 + (y-d)^2} + x^2 + 2xc + c^2$

or $4a\sqrt{(x+c)^2 + (y-d)^2} = 4xc + 4a^2$

Squaring,

$a^2(x+c)^2 + a^2(y-d)^2 = (xc + a^2)^2$

or $a^2(x + c^2) + a^2(y-d)^2 = x^2c^2 + 2a^2xc + a^4$

$a^2x^2 + 2a^2xc + a^2c^2 + a^2(y-d)^2 = x^2c^2 + 2a^2xc + a^4$

or $(a^2 - c^2)x^2 + a^2(y-d)^2 = a^2(a^2 - c^2)$

or $\frac{x^2}{a^2} + \frac{(y-d)^2}{(a^2 - c^2)} = 1$

or $\frac{x^2}{a^2} - \frac{(y-d)^2}{(c^2 - a^2)} = 1$

$\therefore \frac{(x-0)^2}{a^2} - \frac{(y-d)^2}{b^2} = 1$ where $b^2 = (c^2 - a^2)$

which represents the equation of a hyperbola.

Ex. 4: Show that $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ and $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$ where 't' is a variable parameter represents a hyperbola.

Solution:

Here, $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ ....(1)

or $\frac{x}{a} = \frac{1}{2}\left(t+\frac{1}{t}\right)$

and $\frac{y}{b} = \frac{1}{2}\left(t-\frac{1}{t}\right)$ ....(2)

Eliminating 't' from (1) and (2),

Squaring and subtracting (1) from (2), we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{4}\left[\left(t+\frac{1}{t}\right)^2 - \left(t-\frac{1}{t}\right)^2\right]$$

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{4}\left[\left(t^2+2+\frac{1}{t^2}\right) - \left(t^2-2+\frac{1}{t^2}\right)\right]$

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{4}\left[t^2+2+\frac{1}{t^2} - t^2+2-\frac{1}{t^2}\right]$

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{4}\times 4$

$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a hyperbola.

Ex. 5: If e_1 and e_2 be the eccentricities of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ then show that $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ [2059/061/062 B.E.]

Solution:

Here, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If e_1 be the eccentricity of this hyperbola, then

$b^2 = a^2(e_1^2 - 1)$

or $\frac{b^2}{a^2} = e_1^2 - 1$

or $1 + \frac{b^2}{a^2} = e_1^2$

or $e_1^2 = \frac{a^2+b^2}{a^2}$

$\therefore \frac{1}{e_1^2} = \frac{a^2}{a^2+b^2}$ ....(1)

and the equation of conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

If e_2 be the eccentricity of this hyperbola, then

$a^2 = b^2(e_2^2 - 1)$

or $\frac{a^2}{b^2} = e_2^2 - 1$

or $\frac{a^2}{b^2} + 1 = e_2^2$

or $\frac{a^2+b^2}{b^2} = e_2^2$

$\therefore \frac{1}{e_2^2} = \frac{b^2}{a^2+b^2}$ ....(2)

On addition (1) and (2)

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$$

$\therefore \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$

Exercise - 35

1. Find the center, eccentricity, vertices, foci, directrices, length of axes and latus rectum of the hyperbola

   i. $4x^2 - 9y^2 = 36$

   ii. $16x^2 - 9y^2 = -144$

iii. $16x^2 - 25y^2 = 400$

iv. $9x^2 - 16y^2 + 72x - 32y - 16 = 0$

2. Find the equation to the hyperbola referred to its axes as axes of coordinates.

   i. with distances between the foci is 9, eccentricity is $\sqrt{3}$.

   ii. with distances between the foci is 16, eccentricity is $\sqrt{2}$.

   iii. with length of transverse and conjugate axes are 3 and 4 respectively.

   iv. with length of conjugate axis is 10, passes through the point (2, - 1).

   v. with length of conjugate axis 7 and passes through the (3, - 2).

3. Find the equation of the hyperbola whose directrix is $2x + y = 1$ and focus (1, 2) and eccentricity $\sqrt{3}$.

4. Find the equation of the hyperbola with vertices (± 5, 0) and foci and (± 7, 0).

5. Prove that the locus of a point which moves in such a way that the difference of its distances from the points (5, 0) and (- 5, 0) is 2 is a hyperbola.

6. Prove that the lines $\frac{x}{a} + \frac{y}{b} = k$ and $\frac{x}{a} - \frac{y}{b} = \frac{1}{k}$ where k is variable parameter always intersect on a hyperbola.

7. Show that the eccentricity of a hyperbola with transverse axis is 2a and axes are the axes of coordinates which passes through the point (h, k) is given by $\left(\frac{h^2 + k^2 - a^2}{h^2 - a^2}\right)^{1/2}$ [2058 B.E.]

Answers

1. i. $(0, 0), e = \frac{\sqrt{13}}{3}, (\pm 3, 0), (\pm\sqrt{13}, 0), x = \pm\frac{9}{\sqrt{13}}, 6, 4, \frac{8}{3}$

   ii. $(0, 0), e = \frac{5}{4}, (0, \pm 4), (0, \pm 5), 8, 6, \frac{9}{2}$

   iii. $(0, 0), e = \frac{\sqrt{41}}{5}, (\pm 5, 0), (\pm\sqrt{41}, 0), x = \pm\frac{25}{\sqrt{41}}, 10, 8, \frac{32}{5}$

iv. $(-4, -1), \frac{5}{4}, (-8, -1), (0, -1), (1, -1), (-9, -1), x = -\frac{4}{5}, x = -\frac{36}{5}, 8, 6, \frac{9}{2}$

2. i. $4x^2 - 2y^2 = 27$ ii. $x^2 - y^2 = 32$

   iii. $16x^2 - 9y^2 = 36$ iv $13x^2 - 2y^2 = 50$

   v. $65x^2 - 36y^2 = 441$

3. $7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

4. $24x^2 - 25y^2 = 600$

21.10 Tangent at a Given Point

To find the equation of tangent at the point (x_1, y_1) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let $P(x_1, y_1)$ be the given point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $Q(x_2, y_2)$ be another point on the hyperbola very close to P.

So $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ ....(1)

and $\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1$ ....(2)

On subtraction (1) and (2)

$$\frac{x_2^2 - x_1^2}{a^2} - \frac{(y_2^2 - y_1^2)}{b^2} = 0$$

or $$\frac{(x_2 + x_1)(x_2 - x_1)}{a^2} = \frac{(y_2 + y_1)(y_2 - y_1)}{b^2}$$

or $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{b^2 (x_2 + x_1)}{a^2 (y_2 + y_1)} \quad \text{............(3)}$$

The equation of the chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $$y - y_1 = \frac{b^2}{a^2}\frac{(x_2 + x_1)}{(y_2 + y_1)}(x - x_1)$$

When $Q \to P$, $x_2 \to y_1$, then

$$y - y_1 = \frac{b^2}{a^2}\frac{2x_1}{2y_1}(x - x_1)$$

or $\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$

or $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

Since (x_1, y_1) lies on the hyperbola,

Using (1)

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ is the equation of tangent at the point (x_1, y_1) on the hyperbola.

21.11 Condition of Tangency

To find the condition that the line $y = mx + c$ will touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let $y = mx + c$ ....(1)

be the straight line meet the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ....(2)

Solving (1) and (2)

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

or $\frac{x^2}{a^2} - \frac{m^2x^2}{b^2} - \frac{2mcx}{b^2} - \frac{c^2}{b^2} = 1$

or $(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$

which is quadratic equation in x, so x has two roots. The line (1) touches at the point if x has two equal roots. The two roots of x are equal if

$$(-2a^2mc)^2 - 4(b^2 - a^2m^2) \times \{-a^2(c^2 + b^2)\} = 0$$

or $4a^4m^2c^2 + 4a^2(b^2 - a^2m^2)(c^2 + b^2) = 0$

or $a^2m^2c^2 + b^2c^2 + b^4 - a^2m^2c^2 - b^2m^2b^2 = 0$

or $b^2c^2 + b^4 - a^2m^2b^2 = 0$

or $c^2 = a^2m^2 - b^2$

$\therefore$ $c = \pm\sqrt{a^2m^2 - b^2}$ is the required condition of tangency.

Note 1:

Putting $c = \pm\sqrt{a^2m^2 - b^2}$ in (1), we get

$y = mx \pm \sqrt{a^2m^2 - b^2}$ is always tangents to the hyperbola.

Note 2:

The equation of tangent at (x_1, y_1) to the hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

It must be identical with

$$mx - y \pm \sqrt{a^2m^2 - b^2} = 0$$

$$\frac{x_1/a^2}{m} = \frac{y_1/b^2}{1} = \frac{1}{\pm\sqrt{a^2m^2 - b^2}}$$

From first and last ratios

$$x_1 = \pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}$$

From second and last ratios

$$y_1 = \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}$$

Therefore the point of contact $\left(\pm\frac{a^2m}{a^2m^2 - b^2}, \pm\frac{b^2}{a^2m^2 - b^2}\right)$

21.12 Normal at a Given Point

To find the equation of normal at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let the equation of tangent at (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Slope of the tangent $= -\frac{x_1/a^2}{-y_1/b^2} = \frac{b^2}{a^2}\frac{x_1}{y_1}$

Slope of the normal $= -\frac{a^2}{b^2}\frac{y_1}{x_1}$

Any equation of line through (x_1, y_1) with given slope is

$$y - y_1 = -\frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1)$$

or $$\frac{x - x_1}{x_1/a^2} = -\frac{y - y_1}{y_1/b^2}$$

Which is the required equation of normal at the given point to the hyperbola.

The proofs of the above results can be obtained from the corresponding proofs in the case of ellipse by replacing b^2 to $-b^2$.

1. Condition of tangency:

I. The condition that the line $y = mx + c$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c = \pm\sqrt{a^2m^2 - b^2}$

The equation of the line

$y = mx \pm \sqrt{a^2m^2 - b^2}$ are tangents to the hyperbola, for all values of m.

II. The condition that the line $lx + my + n = 0$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a^2l^2 - b^2m^2 = n^2$

III. The condition that the line $x\cos\alpha + y\sin\alpha = p$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$

2. Equation of tangent

I. The equation of tangent at the point (x_1, y_1) to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

II. The equation of tangent at the point $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x\sec\phi}{a} - \frac{y\tan\phi}{b} = 1$

3. Equation of normal

I. The equation of normal at the point (x_1, y_1) to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{x - x_1}{x_1/a^2} = -\frac{y - y_1}{y_1/b^2}$$

II. The equation normal at the point $(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $ax\cos\phi + by\cot\phi = a^2 + b^2$.

4. Director circle

The equation of the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$x^2 + y^2 = a^2 - b^2$$

The director circle is clearly

i. imaginary circle when $a < b$

ii. reduced to a point circle when $a = b$.

iii. real circle when $a > b$.

5. Auxiliary circle

The circle describes on transverse axis as diameter is called *Auxiliary circle* and equation of auxiliary circle is

$$x^2 + y^2 = a^2$$

21.13 Asymptote

A straight line which touches the curve at the point at infinity but itself does not be wholly at infinity is called an asymptote.

Alternative definition : An asymptote is a straight line if the perpendicular distance from the point on the curve to the line tends to zero as the point moves to infinity.

21.14 Asymptotes of a Hyperbola

To find the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Let $y = mx + c$ ....(1)

be an asymptote of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{......(2)}$$

Solving (1) and (2)

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

or $b^2x^2 - a^2(m^2x^2 + 2mxcx + c^2) = a^2b^2$

or $(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2 + b^2) = 0$ ....(3)

This is quadratic equation in x, so x has two roots. The line (1) to be an asymptote of the hyperbola (2) if the two roots of the equation (3) are at infinity, we must have

Coeff. of $x^2 = 0$ and coeff. of $x = 0$

or $b^2 - a^2m^2 = 0$ and $a^2mc = 0$

$\therefore$ $m = \pm\frac{b}{a}$ and $c = 0$

Putting the value of m and c in the equation (1), we find the two asymptotes are

$$y = \pm\frac{b}{a}x.$$

Note 1

The asymptotes to the hyperbola is

$$y = \pm\frac{b}{a}x$$

or $ay = \pm bx$

Squaring,

$$a^2y^2 = b^2x^2$$

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

This is the joint equation of the asymptotes to the hyperbola.

It shows that the equation of the asymptotes differs from the equation of the hyperbola only by a constant term.

If the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then its asymptote are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

and its conjugate hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

Note 2

The equation of conjugate hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

and its asymptotes are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

Thus

If the asymptotes are

$lx + my + n = 0$ and $l_1x + m_1y + n_1 = 0$, then the equation of the hyperbola will be of the form

$$(lx + my + n)(l_1x + m_1y + n_1) + k = 0$$

And conversely,

if $(lx + my + n)(l_1x + m_1y + n_1) + k = 0$

be the equation of the hyperbola then its asymptotes will be

$lx + my + n = 0$ and $l_1x + m_1y + n_1 = 0$

21.15 Rectangular Hyperblola

In the equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if the transverse axes and conjugate axes of the hyperbola are equal in length, then the hyperbola is called *Rectangular hyperbola.*

The equation of the hyperbola in standard form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Writing a = b,

$x^2 - y^2 = a^2$ is the equation of *Rectangular hyperbola.*

21.16 Equation of Rectangular Hyperbola Refereed to its Asymptotes as Axes of Coordinates.

To find the equation of rectangular hyperbola whose axes are the asymptotes of the rectangular hyperbola $x^2 - y^2 = a^2$

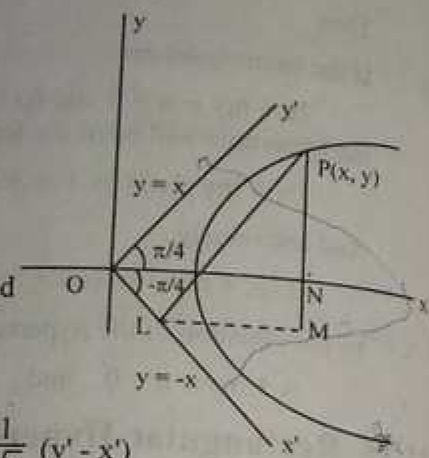
Let P(x, y) be any point on the hyperbola

$x^2 - y^2 = a^2$ ....(1)

with respect to original axes Ox and Oy and (x', y') be the coordinates of the point P with respect to asymptotes Ox' and Oy' as axes of coordinates and since the two asymptotes y = x and y = -x are right angles. Thus the equation of the rectangular hyperbola is obtained by rotating through an angle $\frac{\pi}{4}$.

We have $\angle y'Ox = \frac{\pi}{4}$,

$$x = x\cos\theta - y\sin\theta \text{ and}$$
$$y = x\sin\theta + y\cos\theta$$



Put $\theta = -\frac{\pi}{4}$

$$x = x'\cos\frac{\pi}{4} + y'\sin\frac{\pi}{4} \text{ and}$$

$$y = -x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4}$$

$$x = \frac{1}{\sqrt{2}}(x' + y') \text{ and } y = \frac{1}{\sqrt{2}}(y' - x')$$

Putting these in (1) we get

$$\left(\frac{1}{\sqrt{2}}(x' + y')\right)^2 - \left(\frac{y' - x'}{\sqrt{2}}\right)^2 = a^2$$

or $\frac{1}{2}(x' + y')^2 - \frac{1}{2}(y' - x')^2 = a^2$

or $(x'^2 + 2x'y' + y'^2) - (y'^2 - 2x'y' + x'^2) = 2a^2$

or $4x'y' = 2a^2$

or $x'y' = \frac{1}{2}a^2$

The locus of the point (x', y') is

$$xy = c^2 \text{ where } c^2 = \frac{1}{2}a^2$$

is the required equation of rectangular hyperbola referred to its asymptotes as axes of coordinates.

Note

In the equation of the rectangular hyperbola, $xy = c^2$,

Clearly, $x = ct$, $y = \frac{c}{t}$ is satisfied for all values of t, hence the point $\left(ct, \frac{c}{t}\right)$, $t \neq 0$, lie on the rectangular hyperbola.

21.17 Equation of Tangent

To find the equation of tangent at (x_1, y_1) to the rectangular hyperbola $xy = c^2$

Here, the equation of the rectangular hyperbola is

$$xy = c^2$$

Differentiating,

$$x\frac{dy}{dx} + y = 0$$

or $\frac{dy}{dx} = -\frac{y}{x}$

or $\left(\frac{dy}{dx}\right)_{at\,(x_1, y_1)} = -\frac{y_1}{x_1}$

Slope of the tangent $= -\frac{y_1}{x_1}$

The equation of the tangent at the point (x_1, y_1) to the hyperbola is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

or $\frac{y - y_1}{y_1} = -\frac{(x - x_1)}{x_1}$

or $\frac{y}{y_1} - 1 = -\frac{x}{x_1} + 1$

$\therefore$ $\frac{x}{x_1} + \frac{y}{y_1} = 2$ is the required equation of tangent.

Note

The equation of tangent at $\left(ct, \frac{c}{t}\right)$ to the curve $xy = c^2$ is

$$\frac{x}{ct} + \frac{ty}{c} = 2$$

$\therefore$ $x + yt^2 = 2\,ct$

21.18 Equation of Normal

To find the equation of normal at the point (x_1, y_1) to the curve $xy = c^2$

The equation of the tangent at the point (x_1, y_1) to the curve $xy = c^2$ is

$$\frac{x}{x_1} + \frac{y}{y_1} = 2$$

Slope of the tangent $= -\frac{y_1}{x_1}$

Slope of the normal $= \frac{x_1}{y_1}$

The equation of the normal at the point (x_1, y_1) to the curve is

$$y - y_1 = \frac{x_1}{y_1}(x - x_1)$$

or $yy_1 - y_1^2 = xx_1 - x_1^2$

$\therefore$ $xx_1 - yy_1 = x_1^2 - y_1^2$

is the required equation of normal

Note

The equation of the normal at the point $\left(ct, \frac{c}{t}\right)$ to the hyperbola

$xy = c^2$ is

$$x \times ct - y \times \frac{c}{t} = c^2t^2 - \frac{c^2}{t^2}$$

$\therefore$ $xt^2 - y = ct^3 - \frac{c}{t}$ is the required equation of normal.

Worked out Examples

Ex. 1: Find the equation of tangent and normal at the point $\left(5, \frac{8}{3}\right)$ to the hyperbola $16x^2 - 9y^2 = 144$.

Solution:

Here, the equation of the hyperbola is $16x^2 - 9y^2 = 144$

or $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we find

$a^2 = 9, b^2 = 16$

The equation of tangent at $\left(5, \frac{8}{3}\right)$ to the given hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

or $\frac{x \times 5}{9} - \frac{y \times 8/3}{16} = 1$

or $\frac{5x}{9} - \frac{y}{6} = 1$

or $10x - 3y = 18$

This is the required equation of tangent.

Slope of the tangent $= \frac{-10}{-3} = \frac{10}{3}$.

Slope of the normal $= -\frac{3}{10}$

The equation of normal at the point $\left(5, \frac{8}{3}\right)$ to the hyperbola is

$$y - \frac{8}{3} = -\frac{3}{10}(x - 5)$$

or $10y - \frac{80}{3} = -3x + 15$

or $3x + 10y = \frac{80}{3} + 15$

or $3x + 10y = \frac{80 + 45}{3}$

$\therefore$ $9x + 30y = 125$

Ex. 2 Show that the line $x - y + 2 = 0$ touches the hyperbola $5x^2 - 9y^2 = 45$. Also find the point of contact.

Solution:

Here, the equation of the line is

$x - y + 2 = 0$ ....(1)

and the hyperbola is

$5x^2 - 9y^2 = 45$ ....(2)

Solving (1) and (2)

$$5x^2 - 9(x+2)^2 = 45$$

or $5x^2 - 9(x^2 + 4x + 4) = 45$

or $5x^2 - 9x^2 - 36x - 36 - 45 = 0$

or $-4x^2 - 36x - 81 = 0$

or $4x^2 + 36x + 81 = 0$

or $(2x+9)^2 = 0$

$\therefore$ $x = -9/2$ and $y = -5/2$

Hence the given line is tangent to the given hyperbola and the point of contact is $\left(-\frac{9}{2}, -\frac{5}{2}\right)$.

Ex. 3 Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$ which are parallel to the line $4y = 5x + 7$.

Solution:

Here, the equation of the hyperbola is

$$4x^2 - 9y^2 = 1$$

or $\frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$ ....(1)

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We find, $a^2 = \frac{1}{4}$, $b^2 = \frac{1}{9}$

The equations of the tangents in slope form are

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

or $y = mx \pm \sqrt{\frac{1}{4}m^2 - \frac{1}{9}}$ ....(2)

Since the tangent line (2) is parallel to the line

$$4y = 5x + 7$$

or $y = \frac{5}{4}x + \frac{7}{4}$ ....(3)

Slope of this line $= \frac{5}{4}$

$\therefore m = \frac{5}{4}$

Putting the value of m in (2)

$$y = \frac{5}{4}x \pm \sqrt{\frac{1}{4} \times \frac{25}{16} - \frac{1}{9}}$$

or $y = \frac{5}{4}x \pm \sqrt{\frac{225 - 64}{64 \times 9}}$

or $y = \frac{5}{4}x \pm \frac{\sqrt{161}}{24}$

$\therefore$ $24y - 30x \pm \sqrt{161} = 0$ is the required equation of tangent.

Ex. 4 Find the equation of the tangents to the hyperbola $3x^2 - 4y^2 - 12 = 0$ which are perpendicular to the line $x - y + 2 = 0$. Also fin the point of contact.

Solution:

The equation of the hyperbola is

$$3x^2 - 4y^2 - 12 = 0$$

or $3x^2 - 4y^2 = 12$

or $\frac{x^2}{4} - \frac{y^2}{3} = 1$ ....(1)

Comparing it with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We find $a^2 = 4$, $b^2 = 3$,

The equation of tangent in slope from is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$ ......(2)

Since the tangent line (2) is perpendicular to the line

$$x - y + 2 = 0$$

or $y = x + 2$ ....(3)

Its slope = 1,

So, $m \times 1 = -1$.

$\therefore$ $m = -1$

So (2) becomes

$$y = -x \pm \sqrt{4 \times 1 - 3}$$

or $y = -x \pm \sqrt{4 - 3}$

or $y = -x \pm 1$

Therefore the equations of the tangents are

$x + y = 1, \quad x + y = -1$

Let (x_1, y_1) be the point of contact. The equation of tangent at (x_1, y_1) to the hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

or $\frac{xx_1}{4} - \frac{yy_1}{3} = 1$ ....(4)

Comparing it with $x + y = 1$

$$\frac{x_1/4}{1} = \frac{-y_1/3}{1} = \frac{1}{1}$$

From first and last ratios $\frac{x_1}{4} = 1$ $\quad \therefore x_1 = 4$

From second and last ratios $\frac{-y_1}{3} = 1$ $\quad \therefore y_1 = -3$

The point of contact is (4, - 3)

Also, comparing (4) with $x + y = -1$

$$\frac{x_1/4}{1} = \frac{-y_1/3}{1} = \frac{1}{-1}$$

From first and last ratios $\frac{x_1}{4} = -1$ $\quad \therefore x_1 = -4$

From second and last ratios $\frac{-y_1}{3} = -1$ $\quad \therefore y_1 = 3$

The point of contact is (- 4, 3).

Ex. 5 Find the equation of hyperbola whose center is (1, 2) and its asymptotes are parallel to $2x + 3y = 0$ and $3x + 2y = 0$ and passes through (5, 3).

Solution:

Here, the equation of an asymptote parallel to $2x + 3y = 0$ is

$2x + 3y = k$

Since it passes through the center (1, 2)

So $2 \times 1 + 3 \times 2 = k$

or $2 + 6 = k$

$\therefore k = 8$

Thus the line $2x + 3y - 8 = 0$ is the equation of an asymptote.

The equation of other asymptote parallel to $3x + 2y = 0$ is

$3x + 2y = c$

Since it passes through the center (1, 2) of the hyperbola,

So $3 \times 1 + 2 \times 2 = c$

or $3 + 4 = c$

$\therefore c = 7$

Thus the equation of other asymptote is

$3x + 2y = 7$

The combined equation of the asymptotes is

$(2x + 3y - 8)(3x + 2y - 7) = 0$

As the equation of the hyperbola will differ from its asymptotes only by a constant, it may be given by

$(2x + 3y - 8)(3x + 2y - 7) = \lambda$ where λ is constant

Since it passes through the point (5, 3)

So $(2 \times 5 + 3 \times 3 - 8)(3 \times 5 + 2 \times 3 - 7) = \lambda$

or $(10 + 9 - 8)(15 + 6 - 7) = \lambda$

or $11 \times 14 = \lambda$

$\therefore \lambda = 154$

Thus the equation of the hyperbola is

$(2x + 3y - 8)(3x + 2y - 7) - 154 = 0.$

Ex. 6: Show that the pair of tangents drawn from the center of a hyperbola are its asymptotes.

Solution:

Let the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{.........(1)}$$

The equation of tangents to the hyperbola is ....(2)

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

Since it meets the center (0, 0) of the hyperbola

$\therefore 0 = m \times 0 \pm \sqrt{a^2m^2 - b^2}$

or $0 = \pm\sqrt{a^2m^2 - b^2}$

or $a^2m^2 - b^2 = 0$

$$\therefore\ m = \pm\frac{b}{a}$$

So the equation of tangents are

$$y = \pm\frac{b}{a}x.$$

This is also the equation of asymptotes of the hyperbola.

Exercise - 36

1. Find the equation of tangent, and normal at the point $(5\sqrt{2}, 4)$ to the hyperbola $16x^2 - 25y^2 = 400$.
2. Find the equation of the tangent and normal to the hyperbola $2x^2 - 3y^2 = 6$ at the end of its latus rectum in the first quadrant.
3. Prove that the straight line $21x + 5y = 116$ is a tangent to the hyperbola $7x^2 - 5y^2 = 232$ and find its point of contact.
4. Find the equation of tangent to the hyperbola $3x^2 - 4y^2 = 1$ which is parallel to the line $2x - y + 4 = 0$.
5. Find the equation of tangent to the hyperbola $3x^2 - 4y^2 = 1$ and perpendicular to the line $2x + 3y = 5$.
6. Find the point of intersection of the tangents at t_1 and t_2 on the hyperbola $xy = c^2$.
7. Find the equation of a hyperbola whose asymptotes are $2x - y - 3 = 0$ and $3x + y - 7 = 0$ and passes through the point (1, 1).
8. Prove that the normal at a point 't' of the rectangular hyperbola $xy = c^2$ meets the curve again at a point 't_1' such that $t^3 t_1 = -1$.

Answers

1. $4\sqrt{2}x - 5y = 20,\ 5x + 4\sqrt{2}y = 41\sqrt{2}$
2. $\sqrt{5}x - \sqrt{3}y = 3,\ 3x + \sqrt{15}y = 5\sqrt{5}$
3. (6, - 2)
4. $2\sqrt{3}x - \sqrt{3}y \pm \sqrt{2} = 0$
5. $3x - 2y \pm \sqrt{2} = 0$
6. $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$
7. $6x^2 - xy - y^2 - 23x + 4y + 15 = 0$

❖❖❖

Chapter - 22

General Equation of Conic Section

22.1 General Equation of Second Degree

The equation of the curve in the plane is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c are not simultaneously zero, is known as *General equation of second degree in x and y.*

Prove that the general equation of the second degree in x and y represents a conic section.

Let the general equation of the second degree be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{.........(1)}$$

For the simplicity,

Let the axes be rotated through an angle θ. So the transformed equation is obtained by putting

$$x = x\cos\theta - y\sin\theta$$

and $y = x\sin\theta + y\cos\theta$ in (1), we get

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta) + 2f(x\sin\theta + y\cos\theta) + c = 0$$

or $$(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)x^2 + 2xy[h(\cos^2\theta - \sin^2\theta) + (b - a)\sin\theta\cos\theta] + (a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta)y^2 + 2x(g\cos\theta + f\sin\theta) + 2y(f\cos\theta - g\sin\theta) + c = 0 \quad ...(2)$$

Let the angle θ be so chosen that the coefficient of xy in (2) becomes zero

So $2(b - a)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta) = 0$

or $(b - a)\sin 2\theta + 2h\cos 2\theta = 0$

or $(a - b)\sin 2\theta = 2h\cos 2\theta$

or $\tan 2\theta = \frac{2h}{a-b}$

$$\therefore \theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right) \qquad ....(3)$$

It shows that the values of θ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$.

On substituting the values of h, a, b, and θ determined from (3) in (2), the equation takes

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0 \qquad .....(4)$$

The following different cases may arise:

Case I

When neither A nor B is zero, i.e. $A \neq 0,\ B \neq 0.$

Then the equation (4) can be written as

$$A\left(x^2 + \frac{2G}{A}x\right) + B\left(y^2 + \frac{2F}{B}\right) = -C$$

or $$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A^2} + \frac{F^2}{B^2} - C$$

Shifting the origin to $\left(-\frac{G}{A}, -\frac{F}{B}\right)$ the equation becomes

$$Ax^2 + By^2 = \frac{G^2}{A^2} + \frac{F^2}{B^2} - C$$

or $$Ax^2 + By^2 = k \qquad ....(5)$$

where $k = \frac{G^2}{A^2} + \frac{F^2}{B^2} - C$

Now the following sub cases are

i. If $k = 0$, then the equation (5) becomes

$Ax^2 + By^2 = 0$ and it represents *a pair of straight lines.*

ii. If $k \neq 0$, then the equation (5) becomes

$\frac{x^2}{k/A} + \frac{y^2}{k/B} = 1$ and it represents an *ellipse* when $\frac{k}{A}$ and $\frac{k}{B}$ are of same signs and a *hyperbola* when $\frac{k}{A}$ and $\frac{k}{B}$ are of different signs and when $\frac{k}{A}$ and $\frac{k}{B}$ are both negative it is an *imaginary ellipse.*

Case II

When either A or B = 0, i.e., $A = 0,\ B \neq 0$

Then the equation (4) can be written as

$$By^2 + 2Gx + 2Fy + C = 0$$

or $$B\left(y^2 + \frac{2F}{B}y\right) + 2Gx + C = 0$$

or $$B\left(y + \frac{F}{B}\right)^2 - \frac{F^2}{B} + 2Gx + C = 0$$

or $$B\left(y + \frac{F}{B}\right)^2 + 2G\left(x - \frac{F^2}{2BG} + \frac{C}{2G}\right) = 0$$

Shifting the origin to $\left(\frac{F^2}{2BG} - \frac{C}{2G}, -\frac{F}{B}\right)$ the equation becomes

$$By^2 + 2Gx = 0$$

or $y^2 = -\frac{2G}{B}x$ which represent a *parabola.*

Similarly, suppose $B = 0,\ A \neq 0$, we will also find a parabola.

Note :

We have by the theory of invariant

$$AB = ab - h^2$$

When either A or B is zero, $ab - h^2 = 0$

Thus the equation $ax^2 + 2hxy + by^2 + 2gx + 2fg + c = 0$ represent a *parabola* if $ab - h^2 = 0$

When neither A nor B is zero, $ab - h^2 \neq 0$

If A and B are both positive, then $ab - h^2 = AB$ is also positive and the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an *ellipse.*

If A and B are opposite sign, then $ab - h^2 = AB$ is negative and the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a *hyperbola.*

Thus the general equation of second degree

$ax^2 + 2hxy + by^2 + 2gx + c = 0$ represents

i. a pair of straight lines if

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

ii. a parabola if $\Delta \neq 0$, $ab - h^2 = 0$

iii. an ellipse if $\Delta \neq 0$, $ab - h^2 > 0$ or $h^2 - ab < 0$

iv. a hyperbola if $\Delta \neq 0$, $ab - h^2 < 0$ or $h^2 - ab > 0$

v. a circle if $a = b$, $h = 0$

vi. two parallel straight lines if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

vii. a rectangular hyperbola if $a + b = 0$, $\Delta \neq 0$

22.2 Center of Conic

A point in the plane is called *center of conic* if every chord of the conic passes through the point and is bisected at that point.

Find the center of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let $P(\alpha, \beta)$ be the center of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.......(1)}$$

The equation of line through the point $P(\alpha, \beta)$ with slope

$m = \tan\theta$ is

$$y - \beta = \tan\theta\,(x - \alpha)$$

or $$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r(\text{say})$$

where r is the algebraic distance of the point on the conic from the point P.

Any point on the line that is also the point of conic is

$$x = \alpha + r\cos\theta, \quad y = \beta + r\sin\theta$$

If the point (x, y) lies on the conic (1), then we have

$$a(\alpha + r\cos\theta)^2 + 2h(\alpha + r\cos\theta)(\beta + r\sin\theta) + b(\beta + r\sin\theta)^2 + 2g(\alpha + r\cos\theta) + 2f(\beta + r\sin\theta) + c = 0$$

or $r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2r[(a\alpha + h\beta + g)\cos\theta + (h\alpha + b\beta + f)\sin\theta] + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$

Since (α, β) is the middle point of (1), therefore

coeff. of $r = 0$

$(a\alpha + h\beta + g)\cos\theta + (h\alpha + b\beta + f)\sin\theta = 0$, for all value of θ.

$\therefore$ $a\alpha + h\beta + g = 0$ ....(2)

and $h\alpha + b\beta + f = 0$ ....(3)

Solving (2) and (3),

α β 1

a h g a h

h b f h b

$$\frac{\alpha}{hf - bg} = \frac{\beta}{hg - af} = \frac{1}{ab - h^2}$$

$$\therefore \alpha = \frac{hf - bg}{ab - h^2}, \quad \beta = \frac{gh - af}{ab - h^2}, \quad ab - h^2 \neq 0$$

Thus the coordinates of the center of the conic is

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right).$$

Rules for finding the coordinate of center of conic

Let the conic be

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{......(1)}$$

Differentiate it partially with respect to x and y

$$\frac{\partial S}{\partial x} = 2ax + 2hy + 2g$$

and $$\frac{\partial S}{\partial y} = 2hx + 2by + 2f$$

Now, $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$ gives

$ax + hy + g = 0$ and

$hx + by + f = 0$

Solving these two equations

x y 1

$$\frac{x}{hf - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2}$$

$$\therefore \quad x = \frac{hf - bg}{ab - h^2}, \; y = \frac{gh - af}{ab - h^2}$$

The coordinate of center of conic is

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$$

22.3 Equation of a Conic Referred to the Center as the Origin.

Find the equation of conic referred to the center as the origin.

Let (α, β) be the center of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{.......(1)}$$

Shifting the origin to (α, β).

Put $x = X + \alpha$, $y = y + \beta$ in (1),

$$a(X + \alpha)^2 + 2h\,(X + \alpha)\,(y + \beta) + b(Y + \beta)^2 + 2g(X + \alpha) + 2f\,(Y + \beta) + c = 0$$

$$aX^2 + 2hXY + bY^2 + 2\,[X(a\alpha + h\beta + g) + Y(h\alpha + b\beta + f)] + a\alpha^2 + b\beta^2 + 2h\alpha\beta + 2g\alpha + 2f\beta + c = 0 \qquad \text{......(2)}$$

If the center of the conic is at the origin, then the coefficient of X and coefficient of Y in (2) should be zero.

i.e. $a\alpha + h\beta + g = 0$ ....(3)

and $h\alpha + b\beta + f = 0$ ....(4)

Solving (3) and (4)

$$\frac{\alpha}{hf - bg} = \frac{\beta}{gh - af} = \frac{1}{ab - h^2}$$

$$\alpha = \frac{hf - bg}{ab - h^2}, \; \beta = \frac{gh - af}{ab - h^2}$$

Using (3) and (4) the equation (2) becomes

$$aX^2 + 2hXY + bY^2 + a\alpha^2 + b\beta^2 + 2h\alpha\beta + 2g\alpha + 2f\beta + c = 0 \quad \text{...(5)}$$

Now,

$$a\alpha^2 + b\beta^2 + 2h\alpha\beta + 2g\alpha + 2f\beta + c = \alpha(a\alpha + h\beta + g) + \beta\,(h\alpha + b\beta + f) + g\alpha + f\beta + c$$

Using (3) and (4)

$$= \alpha.0 + \beta.0 + g\alpha + f\beta + c = g\alpha + f\beta + c$$

$$= g\,\frac{hf - bg}{ab - h^2} + f\,\frac{gh - af}{ab - h^2} + c$$

$$= \frac{fgh - bg^2 + fgh - af^2 + abc - ch^2}{ab - h^2}$$

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = \frac{\Delta}{ab - h^2}$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Thus the equation (5) becomes

$$aX^2 + 2hXY + bY^2 + \frac{\Delta}{ab - h^2} = 0$$

or $$aX^2 + 2hXY + bY^2 = \frac{\Delta}{h^2 - ab}$$

or $$\frac{a(h^2 - ab)}{\Delta}\,X^2 + \frac{2h(a^2 - ab)}{\Delta}\,XY + \frac{b(h^2 - ab)}{\Delta}\,Y^2 = 1$$

and this is of the form

$$AX^2 + 2HXY + BY^2 = 1$$

where $A = \frac{a(h^2 - ab)}{\Delta}$, $H = \frac{h(b^2 - ab)}{\Delta}$, $B = \frac{b(h^2 - ab)}{\Delta}$, $\Delta \neq 0$.

is the required *equation of central conic* referred as center as origin.

22.4 Axes and Latus Rectum of Parabola

Find the axis and latus rectum of the parabola whose equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Let the equation of the parabola is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{.......(1)}$$

Since it represents a parabola, $h^2 - ab = 0$ and the second degree terms of the equation form a perfect square.

Thus the equation (1) can be written as

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$$

where $a = \alpha^2$, $b = \beta^2$, $\alpha\beta = h$.

or $(\alpha x + \beta y)^2 = -(2gx + 2fy + c)$ ………(2)

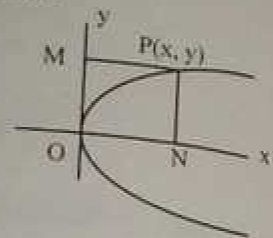
We have following two properties in a parabola.

i. Here, the equation of the parabola

$y^2 = 4ax$

or $(PN)^2 = 4a.ON$

or $(PN)^2 = 4a.PM$



(The distance of any point P from its axis)^2

= latus rectum × the distance of a point P from the tangent at the vertex.

ii. *The axis and the tangent at the vertex are perpendicular.*

So we shall write the equation (2) is such a form which express the above property in relation to two straight lines.

The equation (2) can be written as

$$(\alpha x + \beta y + \lambda)^2 = 2x(\lambda\alpha - g) + 2y(\lambda\beta - f) + \lambda^2 - c$$

where λ is constant, we choose λ so that two straight lines

$\alpha x + \beta y + \lambda = 0$ and ……………(3)

$2x(\lambda\alpha - g) + 2y(\lambda\beta - f) + \lambda^2 - c = 0$ ……………(4)

Slope of the line (3) $= -\dfrac{\alpha}{\beta}$

and slope of the line (4) $= -\dfrac{\lambda\alpha - g}{\lambda\beta - f}$

The condition for the line (3) and (4) are to be perpendicular is that

$$\left(-\frac{\alpha}{\beta}\right) \times \left(-\frac{\lambda\alpha - g}{\lambda\beta - f}\right) = -1$$

or $\alpha(\lambda\alpha - g) + \beta(\lambda\beta - f) = 0$

or $\lambda = \dfrac{g\alpha + f\beta}{\alpha^2 + \beta^2}$

With this value of λ, the line (3) represents the *equation of the axis* and the equation (4) the *tangent at the vertex* and the point of intersection of (3) and (4) is the *vertex* of the parabola.

To find the latus rectum, the equation can be written as in the following form:

$$\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2 (\alpha^2 + \beta^2)$$

$$= \left\{\frac{2(\lambda\alpha - g)x + 2(\lambda\beta - f)y + (\lambda^2 - c)}{2\sqrt{(\lambda\alpha - g)^2 + (\lambda\beta - f)^2}}\right\} \cdot 2\sqrt{(\lambda\alpha - g)^2 + (\lambda\beta - f)^2}$$

or $$\left(\frac{\alpha x + \beta y + \lambda}{\sqrt{\alpha^2 + \beta^2}}\right)^2 = 2.\frac{\sqrt{(\lambda\alpha - g)^2 + (\lambda\beta - f)^2}}{\alpha^2 + \beta^2} \left\{\frac{2(\lambda\alpha - g)x + 2(\lambda\beta - f)y + (\lambda^2 - c)}{2\sqrt{(\lambda\alpha - g)^2 + (\lambda\beta - f)^2}}\right\}$$

Which is of the form $Y^2 = 4AX$

Thus the length of latus rectum

$$= \frac{2\sqrt{(\lambda\alpha - g)^2 + (\lambda\beta - f)^2}}{\alpha^2 + \beta^2}$$

Substituting the value of λ

$$= \frac{2}{(\alpha^2 + \beta^2)} \sqrt{\left[\frac{\alpha(g\alpha + f\beta)}{\alpha^2 + \beta^2} - g\right]^2 + \left[\frac{\beta(g\alpha + f\beta)}{\alpha^2 + \beta^2} - f\right]^2}$$

$$= \frac{2}{(\alpha^2 + \beta^2)} \sqrt{\left(\frac{g\alpha^2 + \alpha\beta f - g\alpha^2 - g\beta^2}{\alpha^2 + \beta^2}\right)^2 + \left(\frac{\alpha\beta g + f\beta^2 - f\alpha^2 - f\beta^2}{\alpha^2 + \beta^2}\right)^2}$$

$$= \frac{2}{(\alpha^2 + \beta^2)} \sqrt{\frac{(\alpha\beta f - g.\beta^2)^2 + (\alpha\beta g - f\alpha^2)^2}{(\alpha^2 + \beta^2)^2}}$$

$$= \frac{2}{(\alpha^2 + \beta^2)^2} \sqrt{[\beta^2(\alpha f - g\beta)^2 + \alpha^2(\beta g - f\alpha)^2]}$$

$$= \frac{2}{(\alpha^2 + \beta^2)^2} \sqrt{(\alpha^2 + \beta^2)(\alpha f - \beta g)^2}$$

$$= \frac{2(\alpha f - \beta g)}{(\alpha^2 + \beta^2)^{3/2}}$$

Therefore length of latus rectum $= \dfrac{2(\alpha f - \beta g)}{(\alpha^2 + \beta^2)^{3/2}}$.

Note :

The focus of the parabola is obtained by solving the equations $X = A$ and $Y = 0$

22.5 Axes of Central Conic

Find the length and the equation of the axes to the central conic $ax^2 + 2hxy + by^2 = 1$

Let the equation of the central conic be

$$ax^2 + 2hxy + by^2 = 1 \qquad \text{.........(1)}$$

Let us consider the concentric circle

$$x^2 + y^2 = r^2 \qquad \text{.........(2)}$$

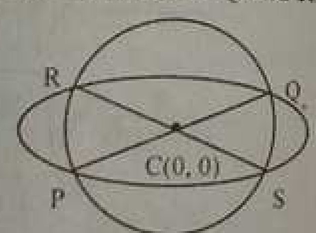
The equation of the lines joining the origin (center) to the points of intersection of (1) and (2) is obtained by making (1) homogeneous with the help of (2).

So $\quad ax^2 + 2hxy + by^2 = \dfrac{x^2 + y^2}{r^2}$

or $\quad \left(a - \dfrac{1}{r^2}\right)x^2 + 2hxy + \left(b - \dfrac{1}{r^2}\right)y^2 = 0 \qquad \text{.........(3)}$

It shows that the equation gives a pair of straight lines passing through origin as center and represents as the diameters PQ and RS

These two straight lines PQ and RS will coincide if the lines lie along either axes of conic.



The equation will represent a pair of coincide lines if

$$4h^2 - 4\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = 0$$

or $\quad h^2 = \left(a - \dfrac{1}{r^2}\right)\left(b - \dfrac{1}{r^2}\right) \qquad \text{.........(4)}$

or $\quad h^2 - ab + (a + b)\dfrac{1}{r^2} - \dfrac{1}{r^4} = 0$

or $\quad \dfrac{1}{r^4} - (a + b)\dfrac{1}{r^2} + ab - h^2 = 0$

which is quadratic equation in $\frac{1}{r^2}$. So it gives two values of r^2, say r_1^2 and r_2^2. Thus r_1 and r_2 are length of *semi axes* of the conic.

Case I

If r_1^2 and r_2^2 obtained from the above equation both positive, then the conic is an *ellipse*. If $r_1 > r_2$, then r_1 is the length of the *semi-major axis* and r_2 that of the *semi-minor axis*.

Case II

If r_1^2 is positive and r_2^2 is negative, then the conic is a *hyperbola* and r_1 is the length of the *semi-transverse axis* and $\sqrt{|r_2^2|}$ is the length of *semi-conjugate axis*.

For the equation of axis, multiplying the equation (3) by $\left(a - \frac{1}{r^2}\right)$

We get

$$\left(a - \frac{1}{r^2}\right)^2 x^2 + 2h\left(a - \frac{1}{r^2}\right)xy + \left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right)y^2 = 0$$

Using (4) $\quad \left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2$, we get

$$\left(a - \frac{1}{r^2}\right)^2 x^2 + 2h\left(a - \frac{1}{r^2}\right)xy + h^2y^2 = 0$$

or $\quad \left\{\left(a - \dfrac{1}{r^2}\right)x + hy\right\}^2 = 0$

Thus the equation of the axes corresponding to r_1^2 and r_2^2 are

$$\left(a - \frac{1}{r_1^2}\right)x + hy = 0$$

and $\quad \left(a - \dfrac{1}{r_2^2}\right)x + hy = 0$

where r_1^2 and r_2^2 are the roots of the equation (4).

22.6 Eccentricity of a Central Conic

I. If the central conic is an ellipse r_1^2 and r_2^2 be squares of the semi-major and minor axis then the eccentricity of the ellipse is obtained by the relation

$$r_2^2 = r_1^2(1 - e^2)$$

or $$\frac{r_2^2}{r_1^2} = 1 - e^2$$

or $$e^2 = 1 - \frac{r_2^2}{r_1^2}$$

$\therefore$ $$e = \frac{\sqrt{r_1^2 - r_2^2}}{r_1}$$

II. If the central conic is a hyperbola, $r_1^2 > 0$, $r_2^2 < 0$ be the squares of the semi transverse and conjugate axes, then the eccentricity of the hyperbola is obtained by the relation

$$r_2^2 = r_1^2(e^2 - 1)$$

or $$\frac{r_2^2}{r_1^2} = e^2 - 1$$

or $$1 + \frac{r_2^2}{r_1^2} = e^2$$

$\therefore$ $$e = \frac{\sqrt{r_1^2 + r_2^2}}{r_1}$$

22.7 Foci of a Conic

I. The conic is an ellipse. Let $C(\alpha, \beta)$ be the coordinates of the center and r_1 the length of semi-major axis. The foci are the points on the major axis at a distance r_1 from the center $C(\alpha, \beta)$ of the conic.

Let θ be the angle of inclination of major axis of the conic to the x-axis.

In the right angle triangle CPN,

$$\cos\theta = \frac{CP}{CS}$$

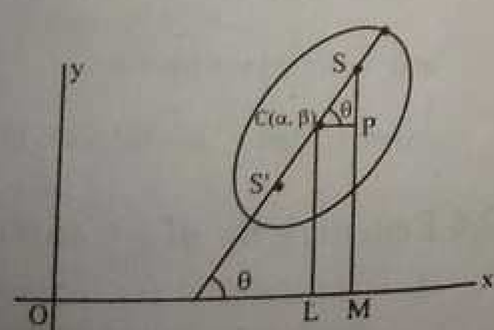
or $CP = CS\cos\theta$

$\therefore$ $CP = er_1\cos\theta$

and $$\sin\theta = \frac{SP}{CS}$$

or $SP = CS\sin\theta$

$\therefore$ $SP = er_1\sin\theta$



Then coordinate of S is

$$(OM, MS) = (OL + LM, SP + PM)$$
$$= (\alpha + er_1\cos\theta, \beta + er_1\sin\theta)$$

Similarly, the co-ordinates of S' is

$$S'(\alpha - er_1\cos\theta, \beta - er_1\sin\theta)$$

II. If the conic section is a hyperbola then the coordinates of the foci holds the same as in the case of the ellipse

$$S(\alpha + er_1\cos\theta, \beta + er_1\sin\theta) \text{ and}$$
$$S'(\alpha - er_1\cos\theta, \beta - er_1\sin\theta)$$

22.8 Working Rule in the Tracing of Conics

The equation of conic is

$$\phi = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........(1)}$$

We first observe that whether or not the second-degree terms in the general equation of conic form a perfect square. If the second degree terms form a perfect square, then the conic (1) represents a *parabola*

If the second-degree terms do not form a perfect square, then the conic will be either an *ellipse* or a *hyperbola.*

If $h^2 - ab < 0$, then the conic will be an *ellipse* and if $h^2 - ab > 0$,then the conic will be a *hyperbola*. When the conic is an ellipse or a hyperbola, then we find center, length of axes, the equation of axes, eccentricity, foci and directrices which are described as follows:

I. Center of conic

The equation of conic is

$$\phi = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\frac{\partial\phi}{\partial x} = 2ax + 2hy + 2g$$

and $$\frac{\partial\phi}{\partial y} = 2hx + 2by + 2f$$

Putting $\frac{\partial\phi}{\partial x} = 0$ and $\frac{\partial\phi}{\partial y} = 0$ gives

$$ax + hy + g = 0$$

and $$hx + by + f = 0$$

Let (α, β) be center of conic, then

$$a\alpha + h\beta + g = 0$$

$$h\alpha + b\beta + f = 0$$

Solving these

$$\frac{\alpha}{hf - bg} = \frac{\beta}{gh - af} = \frac{1}{ab - h^2}$$

$$\alpha = \frac{hf - bg}{ab - h^2}, \quad \beta = \frac{gh - bg}{ab - h^2}$$

Center of the conic is

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - bg}{ab - h^2}\right)$$

II. Length of axes

Let the general equation of conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

It can be reduced into the form

$$Ax^2 + 2Hxy + By^2 = 1$$

Where, $A = \frac{a(ab - h^2)}{\Delta}$, $H = \frac{h(ab - h^2)}{\Delta}$, $B = \frac{b(ab - h^2)}{\Delta}$, $\Delta \neq 0$

and the quadratic equations in $\frac{1}{r^2}$ is

$$\frac{1}{r^4} - (A + B)\frac{1}{r^2} + AB - H^2 = 0$$

Clearly, it gives two values of r^2. If both values of r^2 are positive, then the conic represents an *ellipse*. If r_1^2 is greater than r_2^2 then r_1 is *length of semi-major axis* and r_2 is *length of semi-minor axis*. If both values of r^2 have opposite signs, then the conic represents a *hyperbola*. If r_1^2 is positive and r_2^2 is negative, then r_1 is *length of semi-transverse axis* and $\sqrt{|r_2^2|}$ is *length of semi-conjugate axis* of the hyperbola.

III. Equation of axes

The equations of axes corresponding to the value of r_1^2 and r_2^2 are

$$\left(A - \frac{1}{r_1^2}\right) X + HY = 0 \text{ and}$$

$$\left(A - \frac{1}{r_2^2}\right) X + HY = 0$$

IV. Eccentricity

If r_1^2 and r_2^2 are both positive and r_1^2 is greater than r_2^2 then the conic represents *an ellipse* and the *eccentricity* of the ellipse is obtained by the relation

$$r_2^2 = r_1^2 (1 - e^2)$$

or $$\frac{r_2^2}{r_1^2} = 1 - e^2$$

or $$e^2 = 1 - \frac{r_2^2}{r_1^2}$$

$$\therefore \quad e = \frac{\sqrt{r_1^2 - r_2^2}}{r_1}$$

If r_1^2 and r_2^2 both are opposite sign, then the conic represents a *hyperbola*. If r_1^2 is positive and r_2^2 is negative ,then the *eccentricity* of the hyperbola is obtained by the relation

$$r_2^2 = r_1^2 (e^2 - 1)$$

or $$\frac{r_2^2}{r_1^2} = e^2 - 1$$

or $$e^2 = \frac{r_2^2}{r_1^2} + 1$$

$$\therefore \quad e = \frac{\sqrt{r_1^2 + r_2^2}}{r_1}$$

V. Foci

If the conic is an ellipse and r_1 be the length of semi-major axis then foci are the points on the major axis at a distance er_1 from the center (α, β).

So the foci are the points $(\alpha \pm er_1 \cos\theta, \beta \pm er_1 \sin\theta)$

$$e = \frac{\sqrt{r_1^2 - r_2^2}}{r_1}$$

where θ is the inclination of major axis on the x-axis. The equation of major axis

$$\left(A - \frac{1}{r_1^2}\right) x + Hy = 0 \text{ gives}$$

$$\tan\theta = -\frac{A - \frac{1}{r_1^2}}{H}$$

If the conic is a hyperbola then foci are the points

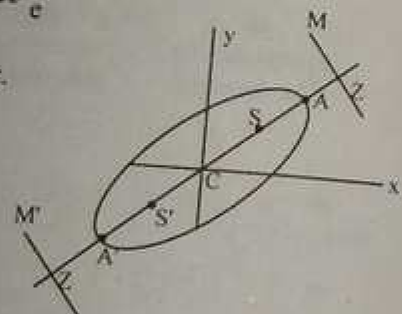
$$(\alpha \pm er_1 \cos\theta, \beta \pm er_1 \sin\theta), \text{ where } e = \frac{\sqrt{r_1^2 + r_2^2}}{r_1}$$

VI. Directrices

Let ZM and Z'M' be the directrices perpendicular to the major axis AA' of the ellipse and θ be the angle made by the major axis with x-axis at the center.

The directrices are at a distance $\frac{r_1}{e}$

i.e. $\frac{r_1 \times r_1}{\sqrt{r_1^2 - r_2^2}}$ from the center.



Thus the equation of the line ZM and Z'M' are as in the normal form

$$x\cos\theta + y\sin\theta = \pm \frac{r_1^2}{\sqrt{r_1^2 - r_2^2}}$$

which are the *equation of directrices* of the central ellipse.

On shifting the origin to the center (α, β), the *equation of directrices* of the general ellipse are

$$(x - \alpha) \cos\theta + (y - \beta) \sin\theta = \pm \frac{r_1^2}{\sqrt{r_1^2 - r_2^2}}$$

If r_1^2 is positive and r_2^2 is negative, then the conic represents a hyperbola. So, r_1^2 is the square of the semi-transverse axis. Let θ be the angle made by the transverse axis with x-axis. Thus the *equation of directrices* of central hyperbola are

$$x\cos\theta + y\sin\theta = \pm \frac{r_1^2}{\sqrt{r_1^2 - r_2^2}}$$

On shifting the origin to the center (α, β) the *equation of directrices* of general hyperbola are

$$(x - \alpha) \cos\theta + (y - \beta) \sin\theta = \pm \frac{r_1^2}{\sqrt{r_1^2 - r_2^2}}$$

2.9 Asymptotes

An asymptote is a straight line, which touches the curve at infinity but does not lie altogether at infinity.

Find the equation asymptote to the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Let the equation of conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........(1)}$$

Since, the equation of the asymptotes differs from the equation of the conic in constant term only. Thus the equations of the asymptotes will be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda = 0 \quad \text{.........(2)}$$

where λ is constant which will be determined by choosing that (2) represents a pair of straight lines.

The equation (2) gives a pair of straight lines if

$$ab(c + \lambda) + 2fgh - af^2 - bg^2 - (c + \lambda) h^2 = 0$$

or $(ab - h^2) \lambda + abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or $\lambda = -\frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}$

$$\therefore \lambda = -\frac{\Delta}{ab - h^2}$$

So the equation (2) becomes

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2} = 0$$

This is the required equation of the asymptotes to the conic.

Worked out Examples

Ex. 1: Find the latus rectum, equation of axis, vertex, equation tangent at the vertex, focus and directrix of the parabola. $16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0$

Solution:

Here, the second-degree terms

$$16x^2 - 24xy + 9y^2 = (4x - 3y)^2$$

form a perfect square, the conic represent a parabola and the given equation is

$$16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0$$

or $(4x - 3y)^2 + 77x - 64y + 95 = 0$

It can be written as

$$(4x - 3y + \lambda)^2 + 77x - 64y + 95 - 8\lambda x + 6\lambda y - \lambda^2 = 0$$

or $(4x - 3y + \lambda)^2 + (77 - 8\lambda)x - (64 - 6\lambda)y + 95 - \lambda^2 = 0$

or $(4x + 3y + \lambda)^2 = (8\lambda - 77)x + (64 - 6\lambda)y + \lambda^2 - 95$ ......(1)

Let us choose λ in such a way that the lines

$4x - 3y + \lambda = 0$ ......(2)

and $(8\lambda - 77)x + (64 - 6\lambda)y + \lambda^2 - 95 = 0$ ......(3)

are perpendicular to each other.

Slope of the line (2) $= -\frac{4}{-3} = \frac{4}{3}$

Slope of the line (3) $= -\frac{8\lambda - 77}{64 - 6\lambda}$

The product of slope of the lines (2) and (3) = - 1

Thus $\frac{4}{3} \times -\frac{(8\lambda - 77)}{64 - 6\lambda} = -1$

or $32\lambda - 308 - 192 + 18\lambda = 0$

or $50\lambda - 500 = 0$

$\therefore \lambda = 10$

Putting $\lambda = 10$ in (1), the equation (1) becomes

$$(4x - 3y + 10)^2 = 3x + 4y + 5$$

or $\left(\frac{4x - 3y + 10}{\sqrt{4^2 + (-3)^2}}\right)^2 \times [4^2 + (-3)^2] = \frac{3x + 4y + 5}{\sqrt{3^2 + 4^2}} \times \sqrt{3^2 + 4^2}$

or $\left(\frac{4x - 3y + 10}{\sqrt{4^2 + (-3)^2}}\right)^2 = \frac{\sqrt{25}}{25}\left(\frac{3x + 4y + 5}{5}\right)$

or $\left(\frac{4x - 3y + 10}{5}\right)^2 = \frac{1}{5}\left(\frac{3x + 4y + 5}{5}\right)$

Which is of the form, $Y^2 = 4AX$.

Latus rectum

Length of latus rectum $= 4A = \frac{1}{5}$

The equation of axis is $Y = 0$

or $\frac{4x - 3y + 10}{5} = 0$

$\therefore 4x - 3y + 10 = 0$ ....(4)

and the tangent at the vertex is $X = 0$

or $\frac{3x + 4y + 5}{5} = 0$

$\therefore 3x + 4y + 5 = 0$ ....(5)

Vertex

We know that the vertex is the intersection of axis and the tangent at the vertex to parabola. So solving (4) and (5), we get

	x		y		1	
4	-3	10		4		-3
3	4	5		3		4

$$\frac{x}{-15 - 40} = \frac{y}{30 - 20} = \frac{1}{16 + 9}$$

or $\frac{x}{-55} = \frac{y}{10} = \frac{1}{25}$

or $x = -\frac{55}{25}, y = \frac{10}{25}$

$\therefore x = -\frac{11}{5}, y = \frac{2}{5}$

The coordinates of the vertex is $\left(-\frac{11}{5}, \frac{2}{5}\right)$.

Focus

We know that the focus is the intersection of the equation of latus rectum and axis of the parabola. We first find the equation latus rectum, which is

$$X = A$$

or $\frac{3x + 4y + 5}{5} = \frac{1}{20}$

or $3x + 4y + 5 = \frac{1}{4}$

$\therefore \quad 12x + 16y + 19 = 0$ ....(6)

Solving (4) and (6)

$$\begin{array}{cccccc} & x & & y & & 1 \\ 4 & -3 & 10 & & 4 & -3 \\ 12 & 16 & 19 & & 12 & 16 \end{array}$$

$$\frac{x}{-57 - 160} = \frac{y}{120 - 76} = \frac{1}{64 + 36}$$

or $\frac{x}{-217} = \frac{y}{44} = \frac{1}{100}$

$x = -\frac{217}{100}, \quad y = \frac{44}{100} = \frac{11}{25}$

The coordinate of the focus is $\left(-\frac{217}{100}, \frac{11}{25}\right)$

Directrix

In the equation of the parabola, $Y^2 = 4AX$

The equation of the directrix is

$$X = -A$$

or $\frac{3x + 4y + 5}{5} = -\frac{1}{20}$

or $3x + 4y + 5 = -\frac{1}{4}$

$\therefore \quad 12x + 16y + 21 = 0.$

Ex. 2: Show that the conic represented by the equation $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ is an ellipse. Also find its

i. **center**
ii. **equation referred to the center**
iii. **equation of axes and length of axes**
iv. **eccentricity**
v. **latus rectum**
vi. **foci**
vii. **directrices**

Solution:

Here, the equation of conic is

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$$

Comparing it with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$a = 14, \quad h = -2, \quad b = 11, \quad g = -22, \quad f = -29, \quad c = 71$$

Now, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 14 \times 11 \times 71 + 2 \times (-29) \times (-22) \times (-2) - 14 \times (-29)^2 - 11 \times (-22)^2 - 71 \times (-2)^2$

$= 10934 - 2552 - 11774 - 5324 - 284$

$= -9000 \neq 0$

and $ab - h^2 = 14 \times 11 - (-2)^2 = 154 - 4 = 150 > 0$

Thus $\Delta \neq 0$, and $ab - h^2 > 0$,

Hence the conic represents an ellipse.

Center

Let $\phi = 14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$

$$\frac{\partial \phi}{\partial x} = 28x - 4y - 44$$

and $\frac{\partial \phi}{\partial y} = -4x + 22y - 58$

For the coordinates of the center,

Put $\frac{\partial \phi}{\partial x} = 0$

and $\frac{\partial\phi}{\partial y} = 0$

or $28x - 4y - 44 = 0$

and $-4x + 22y - 58 = 0$

or $7x - y - 11 = 0$ ....(1)

and $2x - 11y + 29 = 0$ ....(2)

Solving (1) and (2),

$$\begin{array}{cccccc} & & x & y & 1 & \\ 7 & -1 & -11 & 7 & -1 & \\ 2 & -11 & 29 & 2 & -11 & \end{array}$$

$$\frac{x}{-29-121} = \frac{y}{-22-203} = \frac{1}{-77+2}$$

or $\frac{x}{-150} = \frac{y}{-225} = \frac{1}{-75}$

or $x = \frac{-150}{-75} = 2$ and $y = \frac{-225}{-75} = 3$

$\therefore x = 2$ and $y = 3$

The center of the ellipse is (2, 3).

Equation referred to the center

The equation referred to the center is (2, 3) is

$$\frac{a(h^2-ab)}{\Delta}x^2 + \frac{2h(h^2-ab)}{\Delta}xy + \frac{b(h^2-ab)}{\Delta} = 1$$

or $\frac{14(4-154)}{9000}x^2 + \frac{2\times(-2)(4-154)}{-9000}xy + \frac{11(4-154)}{9000}y^2 = 1$

or $\frac{14(-150)}{-9000}x^2 + \frac{4\times 150}{-9000}xy + \frac{11(-150)}{-9000}y^2 = 1$

$$\frac{7}{30}x^2 - \frac{1}{15}xy + \frac{11}{60}y^2 = 1$$

This is the required equation of conic referred to center (2, 3).

Comparing it with $Ax^2 + 2Hxy + By^2 = 1$.

We find $A = \frac{7}{30}$, $H = -\frac{1}{30}$, $B = \frac{11}{60}$.

Length of axes

We have

$$\frac{1}{r^4} - \frac{1}{r^2}(A+B) + (AB - H^2) = 0$$

or $\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{7}{30} + \frac{11}{60}\right) + \left[\frac{7}{30}\times\frac{11}{60} - \left(-\frac{1}{30}\right)^2\right] = 0$

or $\frac{1}{r^4} - \frac{5}{12r^2} + \frac{1}{24} = 0$

or $r^4 - 10r^2 + 24 = 0$

or $(r^2 - 4)(r^2 - 6) = 0$

$\therefore r_1 = \sqrt{6}$ and $r_2 = 2$

Length of semi-major axis $= \sqrt{6}$

Length of semi- minor axis = 2.

Equation of axes

The equation of major axis with center as origin is

$$\left(A - \frac{1}{r_1^2}\right)X + HY = 0$$

or $\left(\frac{7}{30} - \frac{1}{6}\right)X + \frac{1}{30}Y = 0$

or $\frac{1}{15}X - \frac{1}{30}Y = 0$

$2X - Y = 0$

The equation of minor axis with centers as origin is

$$\left(A - \frac{1}{r_2^2}\right)X + HY = 0$$

or $\left(\frac{7}{30} - \frac{1}{4}\right)X - \frac{1}{30}Y = 0$

or $\frac{14-15}{60}X - \frac{1}{30}Y = 0$

or $-\frac{1}{60}X - \frac{1}{30}Y = 0$

$X + 2Y = 0$

Eccentricity

Here, $r_1 = \sqrt{6}, \quad r_2 = 2$

We have $r_2^2 = r_1^2(1 - e^2)$

or $4 = 6(1 - e^2)$

or $\frac{4}{6} = 1 - e^2$

or $e^2 = 1 - \frac{2}{3}$

or $e^2 = \frac{1}{3}$

$\therefore \quad e = \frac{1}{\sqrt{3}}$.

Latus rectum

$$\text{Latus rectum} = \frac{2r_2^2}{r_1} = \frac{2 \times 4}{\sqrt{6}} = 4\sqrt{\frac{2}{3}}$$

Foci

Slope of major axis $2X - Y = 0$ is 2

$\therefore \tan\theta = 2$

$$\cos\theta = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}$$

$$\sin\theta = \frac{\sin\theta}{\cos\theta} \times \cos\theta$$

$$= \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = \frac{2}{\sqrt{1+4}} = \frac{2}{\sqrt{5}}$$

Also, center $(\alpha, \beta) = (2, 3)$, $r_1 = \sqrt{6}$, $r_2 = 2$.

$e = \frac{1}{\sqrt{3}}$,

The coordinates of the foci of the ellipse are

$(\alpha \pm er_1 \cos\theta, \beta \pm er_1 \sin\theta)$

or $(\alpha \pm er_1 \cos\theta, \beta \pm er_1 \sin\theta)$

or $\left(2 \pm \frac{1}{\sqrt{3}}\sqrt{6}\frac{1}{\sqrt{5}}, 3 \pm \frac{1}{\sqrt{3}}\sqrt{6}\frac{2}{\sqrt{5}}\right)$

or $\left(2 \pm \sqrt{\frac{2}{5}}, 3 \pm \frac{2\sqrt{2}}{\sqrt{5}}\right)$

Equation of directrices

The equation of directrices of the ellipse with center (α, β) is

$$(x - \alpha)\cos\theta + (y - \beta)\sin\theta = \pm\frac{r_1}{e}$$

or $(x - 2)\frac{1}{\sqrt{5}} + (y - 3)\frac{2}{\sqrt{5}} = \pm\sqrt{6} \times \sqrt{3}$

or $(x - 2) + 2(y - 3) = \pm 3\sqrt{2} \times \sqrt{5}$

or $x - 2 + 2y - 6 = \pm 3\sqrt{10}$

$\therefore \quad x + 2y = 8 \pm 3\sqrt{10}$

Ex. 3: Show that the conic $3x^2 + 8xy - 3y^2 - 40x - 20y + 50 = 0$ represents a hyperbola and hence find its

i. center
ii. standard form referred to center
iii. length of semi-axes
iv. equation of transverse and conjugate axes
v. equation of asymptotes
vi. eccentricity
vii. length of latus rectum
viii. foci
ix, equation of directrices

Solution:

Here, the equation of conic is

$$3x^2 + 8xy - 3y^2 - 40x - 20y + 50 = 0$$

Comparing it with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$a = 3, \quad h = 4, \quad b = -3, \quad g = -20, \quad f = -10, \quad c = 50$

Now,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 3 \times -3 \times 50 + 2 \times (-10)(-20) \times 4 - 3 \times (-10)^2 - (-3) \times (-20)^2 - 50 \times (4)^2$

$= -450 + 1600 - 300 + 1200 - 800$

$= 1250 \neq 0$

Also, $ab - h^2 = 3 \times -3 - (4)^2$

$= -9 - 16 = -25 = -25 < 0$

Thus $\Delta \neq 0$ and $ab - h^2 < 0$,

The equation of conic represents a hyperbola.

Center

Let $\phi = 3x^2 + 8xy - 3y^2 - 40x - 20y + 50 = 0$

$$\frac{\partial\phi}{\partial x} = 6x + 8y - 40$$

and $\frac{\partial\phi}{\partial y} = 8x - 6y - 20$

For the coordinates of the center,

Put $\frac{\partial\phi}{\partial x} = 0$ and $\frac{\partial\phi}{\partial y} = 0$

or $6x + 8y - 40 = 0$ ....(1)

and $8x - 6y - 20 = 0$ ....(2)

Solving (1) and (2),

$$\frac{x}{-160 - 240} = \frac{y}{-320 + 120} = \frac{1}{-36 - 64}$$

or $\frac{x}{-400} = \frac{y}{-200} = \frac{1}{-100}$

$\therefore x = 4$, and $y = 2$

Center of the hyperbola is (4, 2).

Standard form referred to the center

We know that the equation of conic referred to the center $(\alpha, \beta) = (4, 2)$ is

$$\frac{a(h^2 - ab)}{\Delta}x^2 + \frac{2h(h^2 - ab)}{\Delta}xy + \frac{b(h^2 - ab)}{\Delta} = 1$$

or $\frac{3(16+9)}{1250}x^2 + \frac{2 \times 4(16+9)}{1250}xy + \frac{(-3)(16+9)}{1250}y^2 = 1$

or $\frac{75}{1250}x^2 + \frac{200}{1250}xy - \frac{75}{1250}y^2 = 1$

or $\frac{3}{50}x^2 + \frac{4}{25}xy - \frac{3}{50}y^2 = 1$

This is the required equation of conic in standard form referred to the center (4, 2).

Comparing it with $Ax^2 + 2Hxy + By^2 = 1$

$$A = \frac{3}{50},\ H = \frac{2}{25},\ B = -\frac{3}{50}$$

Length of axes

For the length of axes, we have the equation

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or $\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{3}{50} - \frac{3}{50}\right) + \left[\frac{3}{50} \times \frac{-3}{50} - \left(\frac{2}{25}\right)^2\right] = 0$

or $\frac{1}{r^4} + \left(-\frac{9}{2500} - \frac{4}{625}\right) = 0$

or $\frac{1}{r^4} - \frac{25}{2500} = 0$

or $\frac{1}{r^4} - \frac{1}{100} = 0$

or $100 - r^4 = 0$

or $(r^2 - 10)(r^2 + 10) = 0$

$\therefore r_1^2 = 10, \quad r_2^2 = -10$

The length of semi-transverse axis $= \sqrt{10}$

The length semi-conjugate axis $= \sqrt{10}$ (taking real)

Equation of axes

The equation of transverse axis is

$$\left(A - \frac{1}{r_1^2}\right) X + HY = 0$$

Putting $r_1^2 = 10$, $A = \frac{3}{50}$, $H = \frac{2}{25}$

or $\left(\frac{3}{50} - \frac{1}{10}\right) X + \frac{2}{25} Y = 0$

or $-\frac{2}{50} X + \frac{2}{25} Y = 0$

or $-X + 2Y = 0$

$\therefore$ $X - 2Y = 0$

Shifting the origin back from (4, 2)

Put $x = X + 4$, $y = Y + 2$

$\therefore$ $X = x - 4$, $Y = y - 2$

The equation of transverse axis is

$$x - 4 - 2(y - 2) = 0$$

or $x - 4 - 2y + 4 = 0$

or $x - 2y = 0$

Since, $r_2^2 = 10$ (Taking real)

Thus the equation of conjugate axis is

$$x - 2y = 0.$$

Equation of asymptotes

The equation of conic is

$$3x^2 + 8xy - 3y^2 - 40x - 20y + 50 = 0$$

Since the equation of asymptotes differ from the equation of a hyperbola by constant.

So, the equation of an asymptote is

$$3x^2 + 8xy - 3y^2 - 40x - 20y + 50 - \frac{\Delta}{ab - h^2} = 0$$

or $3x^2 + 8xy - 3y^2 - 40x - 20y + 50 - \frac{1250}{(-25)} = 0$

or $3x^2 + 8xy - 3y^2 - 40x - 20y + 50 + 50 = 0$

or $3x^2 + 8xy - 3y^2 - 40x - 20y + 100 = 0$

Eccentricity

Here, $r_1^2 = 10$

and $r_2^2 = 10$ (taking real)

We have

$$r_2^2 = r_1^2 (e^2 - 1)$$

or $10 = 10 (e^2 - 1)$

or $1 = e^2 - 1$

or $e^2 = 2$

$\therefore$ $e = \sqrt{2}$

Length of latus rectum

The length of latus rectum

$$= \frac{2|r_2^2|}{r_1} = \frac{2 \times 10}{\sqrt{10}} = 2\sqrt{10}$$

Slope of transverse axis $x - 2y = 0$ is $\frac{1}{2}$

So, $\tan\theta = \frac{1}{2}$, $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1 + \tan^2\theta}} = \frac{1}{\sqrt{1 + \frac{1}{4}}} = \frac{2}{\sqrt{5}}$

$$\sin\theta = \frac{\sin\theta}{\cos\theta} \times \cos\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = \frac{\frac{1}{2}}{\sqrt{1 + \frac{1}{4}}}$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

Center of conic is

$(\alpha, \beta) = (4, 2)$, $r_1 = \sqrt{10}$, $e = \sqrt{2}$

The coordinates of the foci of the hyperbola are

$(\alpha \pm er_1 \cos\theta, \beta \pm er_1 \sin\theta)$

or $\left(4 \pm \sqrt{2}\sqrt{10}\frac{2}{\sqrt{5}}, 2 \pm \sqrt{2}\sqrt{10}\frac{1}{\sqrt{5}}\right)$

or $(4 \pm 4, 2 \pm 2)$

The foci are (0, 0) and (8, 4).

Equation of directrices

The equations of directrices of the hyperbola are

$$(x - \alpha)\cos\theta + (y - \beta)\sin\theta = \pm\frac{r_1}{e}$$

or $(x - 4)\frac{2}{\sqrt{5}} + (y - 2)\frac{1}{\sqrt{5}} = \pm\frac{\sqrt{10}}{\sqrt{2}}$

or $2x - 8 + y - 2 = \pm 5$

or $2x + y - 10 \pm 5 = 0$

The equations of directrices are $2x + y - 15 = 0$, $2x + y - 5 = 0$.

Ex. 4: Prove that the product of the semi-axis of the conic is $5x^2 + 6xy + 5y^2 + 12x + 4y - 4 = 0$ is 3

Solution:

Here, the equation of conic is

$$5x^2 + 6xy + 5y^2 + 12x + 4y - 4 = 0$$

Comparing it with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$a = 5, h = 3, b = 5, g = 6, f = 2, c = -4$$

Now, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 5 \times 5 \times (-4) + 2 \times 2 \times 6 \times 3 - 5 \times 2^2 - 5 \times 6^2 - (-4) \times 3^2$

$= -100 + 72 - 20 - 180 + 36$

$= -300 + 108$

$= -192 \neq 0$

and $ab - h^2 = 5 \times 5 - (3)^2 = 25 - 9 = 16 > 0$

Thus $\Delta \neq 0$ and $ab - h^2 > 0$.

The conic represents an ellipse.

The equation of conic referred to its center is

$$\frac{a(h^2 - ab)}{\Delta}X^2 + \frac{2h(h^2 - ab)}{\Delta}XY + \frac{b(h^2 - ab)}{\Delta}Y^2 = 1$$

or $\frac{5(9 - 25)}{-192}X^2 + \frac{2\times3(9 - 25)}{-192}XY + \frac{5(9 - 25)}{-192}Y^2 = 1$

or $\frac{5(-16)}{-192}X^2 + \frac{6(-16)}{-192}XY + \frac{5(-16)}{-192}Y^2 = 1$

or $\frac{5}{12}X^2 + \frac{1}{2}XY + \frac{5}{12}Y^2 = 1$

Comparing it with $AX^2 + 2HXY + BY^2 = 1$

$$A = \frac{5}{12}, \quad H = \frac{1}{4}, \quad B = \frac{5}{12}$$

We have the equation

$$\frac{1}{r^4} - \frac{1}{r^2}(A + B) + (AB - H^2) = 0$$

or $\frac{1}{r^4} - \frac{1}{r^2}\left(\frac{5}{12} + \frac{5}{12}\right) + \left(\frac{5}{12} \times \frac{5}{12} - \left(\frac{1}{4}\right)^2\right) = 0$

or $\frac{1}{r^4} - \frac{5}{6r^2} + \frac{1}{9} = 0$

or $2r^4 - 15r^2 + 18 = 0$

or $(2r^2 - 3)(r^2 - 6) = 0$

or $r_1^2 = 6$ and $r_2^2 = \frac{3}{2}$

Length of semi-major axis $= \sqrt{6}$

Length of semi-minor axis $= \sqrt{\frac{3}{2}}$

The product of semi-axes $= \sqrt{6} \times \sqrt{\frac{3}{2}} = \sqrt{2} \times \sqrt{3} \times \frac{\sqrt{3}}{\sqrt{2}} = 3$

Exercise - 37

1. Show that the conic $9x^2 - 24xy + 16y^2 - 50x - 100y + 225 = 0$ represents a parabola. Find the latus rectum, vertex, focus and directrix of the parabola.
2. Show that the conic $x^2 - 2xy + y^2 - 2x - 2y + 3 = 0$ represents a parabola. Find latus rectum, vertex, focus and directrix of the parabola.
3. Find the positions and lengths of the axes of the conic $x^2 - 4xy - 2y^2 + 10x + 4y = 0$
4. Find the product of semi-axis of the conic $x^2 - 4xy + 5y^2 = 2$

5. Find the center, lengths of the axes and eccentricity of the conic.
   i. $9x^2 + 4xy + 6y^2 - 22x - 16y + 9 = 0$
   ii. $2x^2+3y^2-4x-12y+13=0$ [2060B.E.]
6. Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$
7. Find the foci and eccentricity of the conic $x^2+4xy+y^2-2x+2y-6=0$ [2058B.E.]

Answers

1. Latus rectum = 4, vertex=(1, 2), focus$\left(\frac{9}{5},\frac{13}{5}\right)$, directrix: $4x+3y-5=0$
2. Latus rectum = $\sqrt{2}$, vertex: $\left(\frac{2}{4},\frac{2}{4}\right)$, focus (1, 1) directrix: $x + y = 1$
3. Transverse axis, $x + 2y = 3$, length of transverse axis : $\sqrt{2}$ conjugate axis $y - 2x = 4$, length of conjugate axis $=\frac{2\sqrt{3}}{3}$
4. 2
5. i. $(1, 1); 1, \sqrt{2}; e=\frac{1}{\sqrt{2}}$ ii. $(1,2); \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
6. $2x + y + 2 = 0, x + 2y + 1 = 0$
7. $\left(-1\pm\frac{2\sqrt{2}}{3}, 1\pm\frac{2\sqrt{2}}{\sqrt{3}}\right)$, 2.

❖❖❖

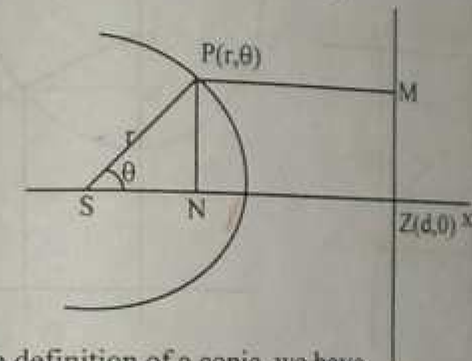
22.10 Equation of a Conic in Polar Coordinates

Let S x be the initial line, the focus S in the pole and (r, θ) the polar coordinates of a point P . Let ZM be the directrix and e be eccentricity of the conic. Let d be distance from fixed point to the directrix.

i.e. SZ = d .

Draw SZ perpendicular to ZM and PM perpendicular to ZM. If $P(r, \theta)$ be any point on the conic with respect to S as pole, then

$$SP = r \text{ and } \angle PSx = \theta.$$



By the definition of a conic, we have

$$SP = e\,PM$$

or $r = e\,ZN = e(SZ - SN) = e(d - SP\cos\theta)$

or $r = de - r\,e\cos\theta$

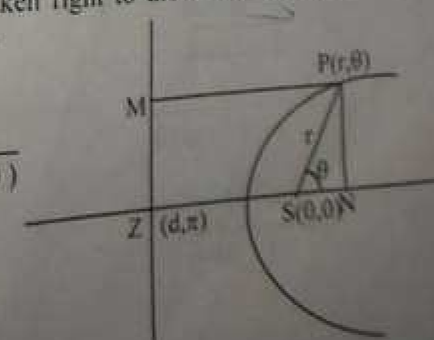
or $r + r\,e\cos\theta = de$

or $r(1 + e\cos\theta) = de$

$\therefore\ r = \frac{de}{1+e\cos\theta}$ is the *polar equation of the conic section.*

If the focus S to be taken right to the directrix with d >0, then the equation of the conic is

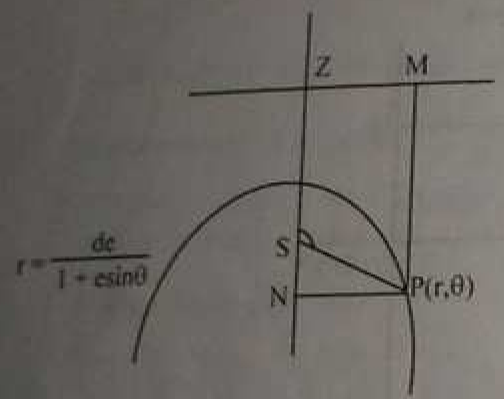
$$r = \frac{de}{1+e\cos(\pi-\theta)} = \frac{de}{1-e\cos\theta}$$

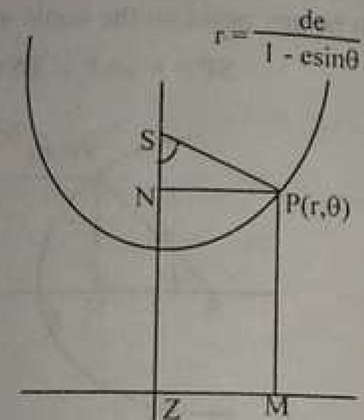


If the directrix to be taken perpendicular to the polar axis then the equation of the conic are

$$r = \frac{de}{1 + e\cos\left(\frac{\pi}{2} - \theta\right)} \quad \text{and} \quad r = \frac{de}{1 + e\cos\left(\frac{3\pi}{2} - \theta\right)}$$

i.e. $r = \dfrac{de}{1 + e\sin\theta}$ and $r = \dfrac{de}{1 - e\sin\theta}$ as shown in the figures.





The conic section is *a parabola* if $e = 1$, *an ellipse* if $e < 1$ and *a hyperbola* if $e > 1$.

Worked out Examples

Ex.1 :Describe and sketch the graph of the equation $r = \dfrac{3}{2 + 2\cos\theta}$.

Solution:

Here the equation of conic is

$$r = \frac{3}{2 + 2\cos\theta}$$

or $$r = \frac{3}{2(1 + \cos\theta)}$$

or $$r = \frac{\frac{3}{2}}{1 + \cos\theta}$$

Comparing it with $r = \dfrac{de}{1 + e\cos\theta}$

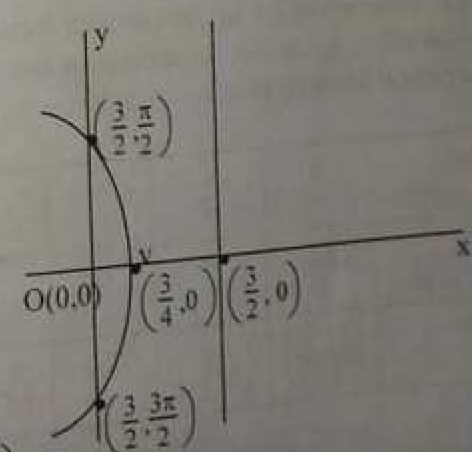
We find

$d = \dfrac{3}{2}$, $\quad e = 1$.

Therefore the conic represents a parabola with focus as the pole. The expression $\cos\theta$ tells us its axis is on the polar axis. The coordinates of $P(r,\theta)$ on the curve is following table.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{3}{2 + 2\cos\theta}$	$\frac{3}{4}$	$\frac{3}{2}$	∞	$\frac{3}{2}$
(r,θ)	$\left(\frac{3}{4}, 0\right)$	$\left(\frac{3}{2}, \frac{\pi}{2}\right)$		$\left(\frac{3}{2}, \frac{3\pi}{2}\right)$

The vertex of the parabola is $\left(\frac{3}{4}, 0\right)$. These points together type of conic lead to sketch the graph of the parabola which is shown as in the figure.



Description:

Vertex: $v\left(\frac{3}{4}, 0\right)$

Focus : O(0,0).

We have the equation of directrix is

$$x = r\cos\theta = \frac{3}{2}\cos 0 = \frac{3}{2} \times 1 = \frac{3}{2}.$$

Ex.2: Describe and sketch the graph of the equation $r = \dfrac{12}{3 + 2\cos\theta}$.

Solution:

Here the equation of conic is $r = \dfrac{12}{3 + 2\cos\theta}$

or $r = \dfrac{12}{3\left(1 + \frac{2}{3}\cos\theta\right)}$

or $r = \dfrac{4}{\left(1 + \frac{2}{3}\cos\theta\right)}$

or $r = \dfrac{6 \frac{2}{3}}{\left(1 + \frac{2}{3}\cos\theta\right)}$.

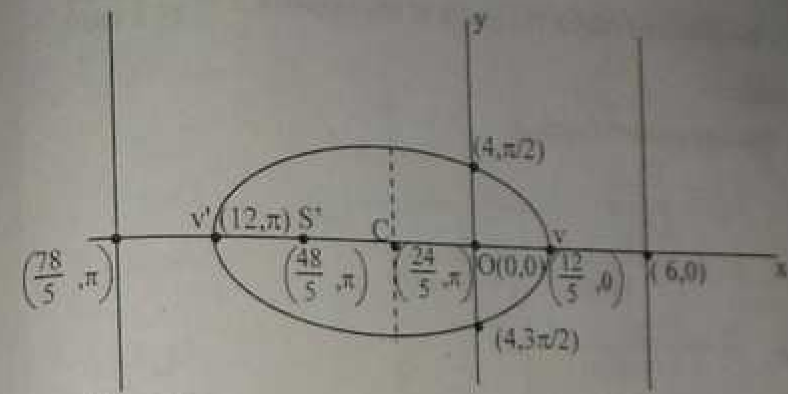
Comparing it with $r = \dfrac{de}{1 + e\cos\theta}$

We find $d = 6$, $e = \dfrac{2}{3} < 1$.

Therefore the conic represents an ellipse with focus as the pole. The expression $\cos\theta$ tells us that its axis is on the polar axis. The coordinates of $P(r, \theta)$ on the curve is following table.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{12}{3 + 2\cos\theta}$	$\frac{12}{5}$	4	12	4
(r, θ)	$\left(\frac{12}{5}, 0\right)$	$\left(4, \frac{\pi}{2}\right)$	$(12, \pi)$	$\left(4, \frac{3\pi}{2}\right)$

The vertices of the ellipse are $v\left(\frac{12}{5}, 0\right)$ and $v'(12, \pi)$. This is shown as in the following figure.



Description:

Vertices: $v\left(\frac{12}{5}, 0\right)$ and $v'(12, \pi)$

Here $2a = 12 + \frac{12}{5} = \frac{72}{5}$

$\therefore a = \frac{36}{5}$

Length of major axis: $2a = \frac{72}{5}$

Centre: $C\left(\frac{24}{5}, 0\right)$

We have $CS' = CO = \frac{24}{5}$

Foci: $O(0,0)$, $S'\left(\frac{48}{5}, \pi\right)$

We have the equation of directrices are,

$x = r\cos\theta = 6\cos 0 = 6 \times 1 = 6$ and

$x = \left(\frac{48}{5} + 6\right)\cos\pi = \frac{78}{5} \times (-1) = -\frac{78}{5}$

Also $b^2 = a^2(1 - e^2) = \left(\frac{36}{5}\right)^2\left(1 - \frac{4}{9}\right)$

$= \frac{1296}{25}\left(\frac{9-4}{9}\right) = \frac{1296}{25} \times \frac{5}{9} = \frac{144}{5}$

$\therefore b = \frac{12}{\sqrt{5}}$

Length of minor axis: $2b = \frac{24}{\sqrt{5}}$.

Ex.3: Describe and sketch the graph of the equation $r = \dfrac{12}{2 - 6\cos\theta}$.

Solution:

Here the equation of conic is

$$r = \frac{12}{2 - 6\cos\theta}$$

or $$r = \frac{12}{2(1 - 3\cos\theta)} = \frac{6}{(1 - 3\cos\theta)}$$

or $$r = \frac{2.3}{(1 - 3\cos\theta)}$$

Comparing it with

$$r = \frac{de}{1 - e\cos\theta}$$

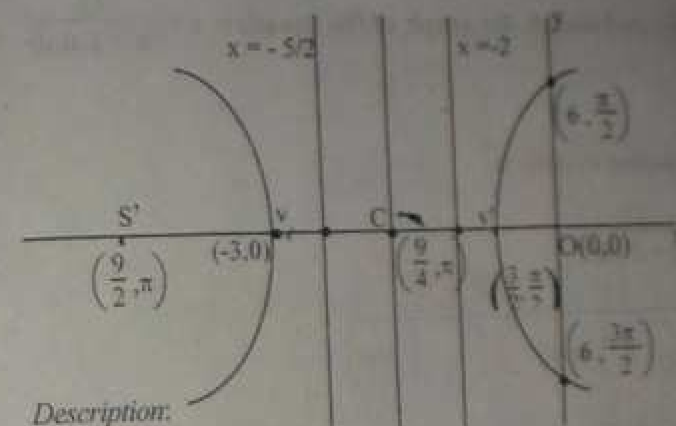
We find

$$d = 2, \qquad e = 3 > 1.$$

Therefore the conic represents a hyperbola with focus as the pole. The expression $\cos\theta$ tells us that its transverse axis is on the polar axis. The coordinates of $P(r,\theta)$ on the curve is following table.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{12}{2 - 6\cos\theta}$	-3	6	$\frac{3}{2}$	6
(r,θ)	(-3,0)	$\left(6, \frac{\pi}{2}\right)$	$\left(\frac{3}{2}, \pi\right)$	$\left(6, \frac{3\pi}{2}\right)$

The vertices of the hyperbola are $v(-3,0)$ and $v'\left(\frac{3}{2}, \pi\right)$. Thee points $\left(6, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$ together type of conic help us to sketch the branch of right side of the hyperbola and the other branch is obtained by symmetry which is shown as in the figure.



Description:

Vertices: $v(-3,0)$, $v'\left(\frac{3}{2}, \pi\right)$

Here $\quad 2a = 3 - \frac{3}{2} = \frac{3}{2}$

$$\therefore a = \frac{3}{4}$$

Length of transverse axis: $2a = \frac{3}{2}$.

Centre: $C\left(\frac{9}{4}, \pi\right)$

We have $CS' = CO = \frac{9}{4}$

Foci: $O(0,0)$, $S'\left(\frac{9}{2}, \pi\right)$

We have the equation of directrices are

$$x = 2\cos\theta = 2\cos\pi = 2 \times (-1) = -2 \text{ and}$$

$$x = \left(\frac{9}{2} - 2\right)\cos\theta = \frac{5}{2}\cos\pi = \frac{5}{2} \times (-1) = \frac{5}{2}$$

Also $$b^2 = a^2(e^2 - 1) = \left(\frac{3}{4}\right)^2 (9 - 1) = \frac{9}{16} \times 8 = \frac{9}{2}$$

$$\therefore b = \frac{3}{\sqrt{2}}$$

Length of Conjugate axis: $2b = 2\dfrac{3}{\sqrt{2}} = 3\sqrt{2}$.

Ex .4: Describe and sketch the graph of the equation $r = \dfrac{12}{6 + 2\sin\theta}$.

Solution:

Here the equation of conic is

$$r = \frac{12}{6 + 2\sin\theta}$$

$$\text{or} \quad r = \frac{12}{6\left(1 + \frac{1}{3}\sin\theta\right)} = \frac{2}{\left(1 + \frac{1}{3}\sin\theta\right)}$$

$$\text{or} \quad r = \frac{6 \cdot \frac{1}{3}}{\left(1 + \frac{1}{3}\sin\theta\right)}$$

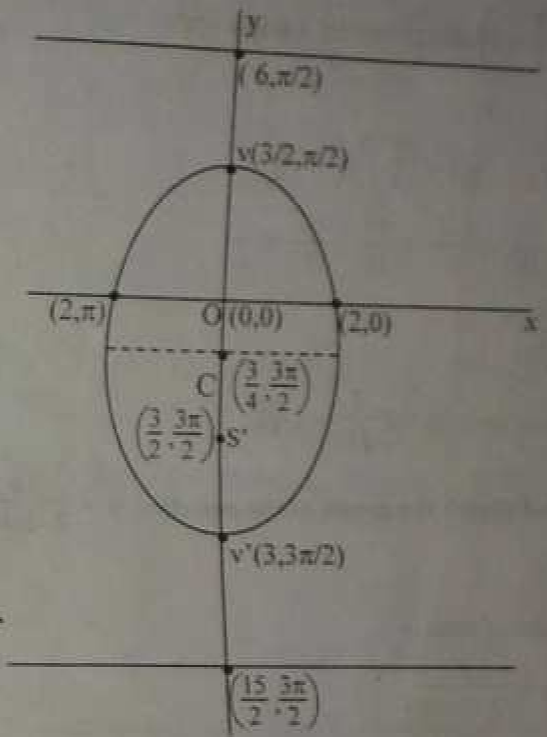
Comparing it with $r = \dfrac{de}{1 + e\sin\theta}$

We find $d = 6$, $e = \frac{1}{3} < 1$.

Therefore, the conic represents an ellipse with focus as the pole. The expression $\sin\theta$ tells us that the major axis of the ellipse is perpendicular to the polar axis. The coordinates of $P(r, \theta)$ on the curve is following table.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{12}{6 + 2\sin\theta}$	2	$\frac{3}{2}$	2	3
(r, θ)	(2,0)	$\left(\frac{3}{2}, \frac{\pi}{2}\right)$	$(2, \pi)$	$\left(3, \frac{3\pi}{2}\right)$

The vertices of the major axis of the ellipse are $v\left(\frac{3}{2}, \frac{\pi}{2}\right)$, $v'\left(3, \frac{3\pi}{2}\right)$. These points together type of conic lead to sketch the graph of the ellipse is shown as in the figure.



Description:

Vertices: $v\left(\frac{3}{2}, \frac{\pi}{2}\right)$, $v'\left(3, \frac{3\pi}{2}\right)$.

Here $2a = \frac{3}{2} + 3 = \frac{3+6}{2} = \frac{9}{2}$

$\therefore a = \frac{9}{4}$

Length of major axis: $2a = \frac{9}{2}$

Centre: $C\left(\frac{3}{4}, \frac{3\pi}{2}\right)$

We have $CS' = CO = \frac{3}{4}$

Foci: $O(0,0)$, $S'\left(\frac{3}{2}, \frac{\pi}{2}\right)$.

We have the equation of directrices are

$y = r\sin\theta = 6\sin\frac{\pi}{2} = 6 \times 1 = 6$ and

$$y=\left(\frac{3}{2}+6\right)\sin\frac{3\pi}{2} = \frac{15}{2}\times(-1) = -\frac{15}{2}$$

Also

$$b^2 = a^2(1-e^2) = \left(\frac{9}{4}\right)^2\left(1-\frac{1}{9}\right)$$

$$= \frac{81}{16}\cdot\frac{9-1}{9} = \frac{81}{16}\cdot\frac{8}{9} = \frac{9}{2}$$

$$\therefore\ b = \frac{3}{\sqrt{2}}$$

Length of minor axis: $2b = 2\frac{3}{\sqrt{2}} = 3\sqrt{2}$.

Ex.5: Describe and sketch the graph of the equation $r=\dfrac{10}{2-3\sin\theta}$.

Solution:

Here the equation of conic is

$$r=\frac{10}{2-3\sin\theta}$$

or $$r=\frac{10}{2\left(1-\frac{3}{2}\sin\theta\right)}$$

or $$r=\frac{\frac{10}{3}\cdot\frac{3}{2}}{\left(1-\frac{3}{2}\sin\theta\right)}$$

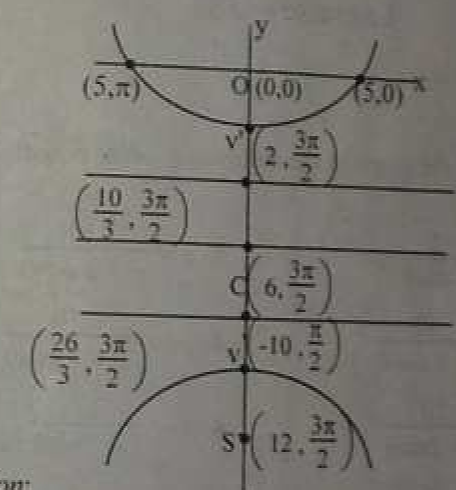
Comparing it with $r=\dfrac{de}{1-e\sin\theta}$

We find $d=\dfrac{10}{3}$, $e=\dfrac{3}{2}>1$.

Therefore the conic represents a hyperbola with focus as the pole. The expression $\sin\theta$ tells us that the transverse axis of the hyperbola is perpendicular to the polar axis. The coordinates of $P(r,\theta)$ on the curve is following table.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r=\frac{10}{2-3\sin\theta}$	5	-10	5	2
(r,θ)	(5,0	$\left(-10,\frac{\pi}{2}\right)$	$(5,\pi)$	$\left(2,\frac{3\pi}{2}\right)$

The points (5,0) and $(5,\pi)$ on the graph helps us to sketch lower branch of the hyperbola. The vertices of the hyperbola are $v\left(-10,\frac{\pi}{2}\right)$, $v'\left(2,\frac{3\pi}{2}\right)$. These points together type of conic help us to sketch the lower and upper branch of the hyperbola which is shown as in the figure.



Description:

Vertices: $v\left(2,\frac{\pi}{2}\right)$, $v'\left(-10,\frac{\pi}{2}\right)$

$2a = 10-2 = 8$ $\quad\therefore a=4$

Length of transverse axis: $2a=8$

Centre: $C\left(6,\frac{3\pi}{2}\right)$

We have $CS' = CO = 6$

Foci: $O(0,0)$, $S'\left(12,\frac{3\pi}{2}\right)$.

We have the equation of directrices are

$$y = r\sin\theta = \frac{10}{3}\sin\frac{3\pi}{2} = \frac{10}{3}\times(-1) = -\frac{10}{3}$$

and

$$y = \left(12 - \frac{10}{3}\right)\sin\frac{3\pi}{2} = \frac{26}{3}\times(-1) = -\frac{26}{3}$$

Also $\quad b^2 = a^2(e^2 - 1) = 4^2\left(\frac{9}{4} - 1\right)$

$$= 16 \cdot \frac{9-4}{4} = 16 \cdot \frac{5}{4} = 20$$

$\therefore \quad b = \sqrt{20} = 2\sqrt{5}$.

Length of transverse axis: $2b = 4\sqrt{5}$.

Exercise – 38

Describe and sketch the graph of the following polar equation of conic section.

1. $r = \dfrac{2}{1-\cos\theta}$
2. $r = \dfrac{10}{3 - 2\cos\theta}$
3. $r = \dfrac{12}{3+\cos\theta}$
4. $r = \dfrac{10}{3+2\cos\theta}$
5. $r = \dfrac{10}{3+2\sin\theta}$
6. $r = \dfrac{12}{3-2\sin\theta}$
7. $r = \dfrac{6}{2+3\sin\theta}$
8. $r = \dfrac{4}{1+3\cos\theta}$
9. $r = \dfrac{10\,\text{cosec}\theta}{2\,\text{cosec}\theta+3}$
10. $r = \dfrac{7}{2+5\sin\theta}$
11. $r = \dfrac{12}{2+4\sin\theta}$
12. $r = \dfrac{6\,\text{cosec}\theta}{2\,\text{cosec}\theta-3}$

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