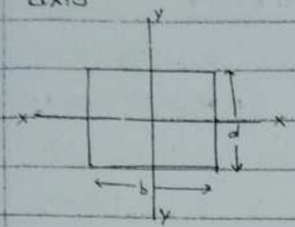
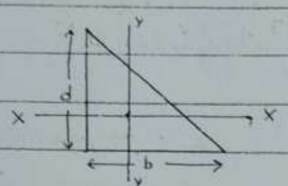
Moment of inertia of common plane areas about centroidal axis

1)



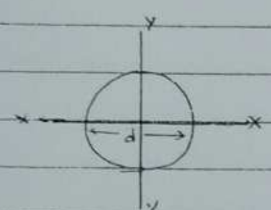
$$I_{xx} = \frac{bd^3}{12} \qquad I_{yy} = \frac{db^3}{12}$$

2)



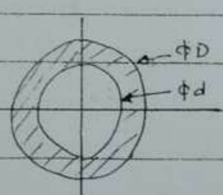
$$I_{xx} = \frac{bd^3}{36} \qquad I_{yy} = \frac{db^3}{36}$$

3)



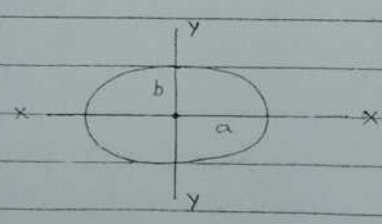
$$I_{xx} = I_{yy} = \frac{\pi d^4}{64}$$

4)



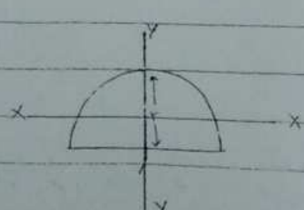
$$I_{xx} = I_{yy} = \frac{\pi (D^4 - d^4)}{64}$$

5)



$$I_{xx} = \frac{\pi ab^3}{4} \qquad I_{yy} = \frac{\pi ba^3}{4}$$

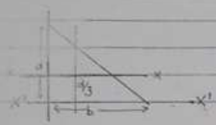
6)



$$I_{xx} = 0.11r^4 \qquad I_{yy} = \frac{\pi r^4}{8}$$

7 

$I_{xx} = I_{yy} = 0.055 r^4$

8 

$$I_{x'x'} = I_{xx} + A h^2$$
$$= \frac{bd^3}{36} + \frac{1}{2} bd \times \left(\frac{d}{3}\right)^2$$
$$= \frac{bd^3}{12}$$

Mechanics of structure / materials

Strength of material is the study of deformation and internal stresses induced in real bodies under the action of external forces. So, it is the science dealing with principles and methods of analysis and proper design of structural components and machine parts for strength, stiffness and stability.

Strength: It is the property of material to resist applied load without collapse or failure.

Stiffness: It is the property of structure or its components to resist excessive deformations such as deflection, elongation and settlement.

Stability: It is the property of structure or its components to return in its original position due to application of external loads.

Chapter 1 : Introduction

wt. density
$\gamma = \frac{W}{V}$

weight = $\gamma \cdot V$
= $\gamma \cdot bD$

Types of loads

a) According to design purpose,
i) Dead load / self weight
ii) Live load / imposed load
iii) Dynamic load (due to earthquake, vibration)
iv) Vibration load
v) Temperature stress
vi) Erection stress
vii) Snow load
viii) Wind load
ix) Impact load (body dropped from certain ht.)
x) Earthquake load.
xi) Settlement stress

b) According to intensity,
i) Point or concentrated load
ii) Distributed load — UDL / UVL

c) According to resisting mechanism,
i) Axial load
ii) Shear force (crossection [illegible] tangentially [illegible] load)
iii) Bending load
iv) Torsion load

Types of support

a) Roller support
b) Hinged / Pin support
c) Fixed support
d) Cable support (suspension bridge)

suspended bridge

The loads on structures can be broadly classified as:
a) Dead loads
b) Imposed loads
i) Operational loads
ii) Natural loads

a) Dead loads
They are fixed in magnitude and direction. Dead load is simply weight of the structure itself. It can be determined most accurately.

b) Imposed loads
They vary in both magnitude and position with time. They are further divided into operational loads and natural loads.

i) Operational loads are the one that result as a consequence of use of the structure. These include live load, vehicle load, machine load and vibration due to moving bodies.

ii) Natural loads are the loads caused by the nature. These include wind loads, earthquake loads, snow loads, earth pressure, hydro-static & hydro-dynamic loads and temperature loads.

According to intensity of loads,
a) Concentrated or point loads
b) Distributed loads
c) Couple or moments

a. Concentrated or point loads
A concentrated load is applied at a point and has dimensions of force (i.e. KN or tones). In reality, point loads are rarely applied but a load distributed over a relatively small area as compared to the dimension of loaded member can be considered as point load.

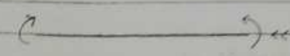
b) Distributed loads
They are applied over an area and have dimensions of force per unit area (KN/m^2 or N/m^2). Sometimes, the area may have a unit width in which the load has a unit of force per unit length (KN/m or N/m)

i) uniformly distributed load (UDL) is distributed along the length of the loaded member or over a defined area in a uniform manner.

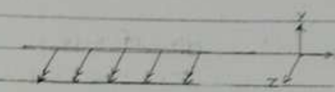
ii) Uniformly varying load (UVL) is distributed along the member but varying uniformly. It is also called triangular load.

c) Couple or moments
A couple or moment is usually applied at a point and has dimension of force times length (KNm or KNcm). If the

moment is distributed over a length, its unit will be KN m/m or simply KN. A moment is usually represented as a double arrow using the right hand thumb rule.



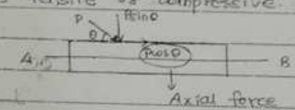
concentrated couples



Distributed couples

According to resisting mechanism,
a) Axial load
b) Shear load
c) Bending load
d) Torsion load

a) Axial load
- developed along the axis of resisting member.
- may be tensile or compressive.



b) Shear load
- these loads tend to shear off the member
- it is the summation of vertical loads
- also called transverse load

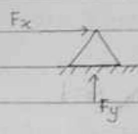
c) Bending load
- these loads create a bending effect in the member.
- these members are called flexural members.

3) Torsion load
- these loads tend to rotate the member
- these loads create twisting effect.

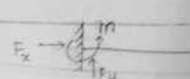
Types of support
There are 4 types of support as follows:
a) Roller support
b) Pin / hinge support
c) Fixed support
d) Cable support

a) Roller support
It is free to move in X-direction and rotate about Z-axis. It can't move in Y-direction so restrained is provided only in Y-axis and reaction is developed only in Y-direction.

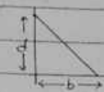
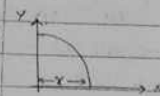
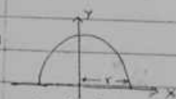
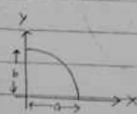
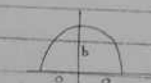
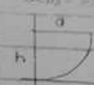
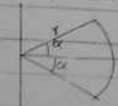
b) Hinge support
It can't move in X and Y-direction but is free to rotate about Z-axis. So, it has restrained in X and Y-axis. And reaction is developed in both X and Y-direction.



c) Fixed support
It restricts to move in X and Y-direction as well as to rotate along Z-axis. So, reactions are developed in X and Y-axis along with the moment.



Centroid of plane areas

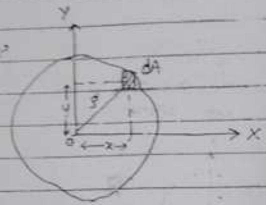
	Shape	Area (A)	$\bar{x}_1$	$\bar{y}_1$
1)		bd	$b/2$	$d/2$
2)		$\frac{1}{2} bd$	$b/3$	$d/3$
3)		$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
4)	 semi-circle	$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$
5)		$\frac{\pi ab}{4}$	$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$
6)	 semi-elliptical	$\frac{\pi ab}{2}$	0	$\frac{4b}{3\pi}$
7)	 semi-parabolic	$\frac{2ah}{3}$	$\frac{3a}{8}$	$\frac{3h}{5}$
8)	 circular sector	αr^2	$\frac{2r\sin\alpha}{\alpha}$	0

Polar moment of inertia (Perpendicular axis theorem)

The moment of inertia of plane area about an axis normal to the plane is equal to sum of moment of inertia about any two mutually perpendicular axis lying in the plane and passing through given axis.

The point of intersection of x- and Y-axis is called pole and axis passing through 'O' and perpendicular to area is called polar axis. The moment of inertia about this axis is called polar moment of inertia.

Take an elementary area 'dA' at a distance 'ρ' from origin having co-ordinate (x, y)



Polar moment of inertia is given by

$I_p = \Sigma \rho^2 dA$

but $\rho^2 = x^2 + y^2$

$I_p = \Sigma (x^2+y^2) dA$

$= \Sigma x^2 dA + \Sigma y^2 dA$

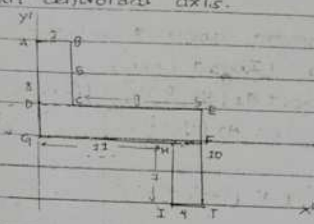
$\boxed{I_p = I_{yy} + I_{xx}}$

$I_{xx} = \int y^2 dA$

$I_{yy} = \int x^2 dA$

The polar moment of inertia of a beam's cross sectional area measures the beam's ability to resist torsion. The larger the polar moment of inertia, lesser will be the tendency to twist.

Q. Determine the polar moment of inertia of the following figure about centroidal axis.



shape	Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1) rect. ABCD	2×5 = 10	1	$10+\frac{5}{2}$ = 12.5	10	125
2) rect. DEFG	15×3 = 45	15/2 = 7.5	7 + 3/2 = 8.5	~~181.5~~ 337.5	~~280.5~~ 382.5
3) rect. HFJI	4×7 = 28	11+2 = 13	7/2 = 3.5	364 ~~252~~	98

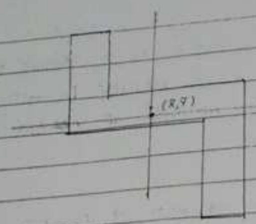
$\Sigma A_i = 83$

$\Sigma A_i x_i =$ ~~443.5~~ 711.5

$\Sigma A_i y_i =$ ~~503.5~~ 605.5

$\bar{x} =$ ~~6.25~~ 8.57

$\bar{Y} =$ ~~7.09~~ 7.3



Moment of inertia about centroidal XX axis

$I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3$

$= \left[(I_{xx})_{CG} + A_1(y_1-\bar{y})^2\right] + \left[(I_{xx})_{CG} + A_2(y_2-\bar{y})^2\right] +$
$\left[(I_{xx})_{CG} + A_3(y_3-\bar{y})^2\right]$

$= \left[\frac{2\times5^3}{12} + 10(12.5-7.3)^2\right] + \left[\frac{15\times3^3}{12} + 45(8.5-7.3)^2\right]$
$+ \left[\frac{4\times7^3}{12} + 28(3.5-7.3)^2\right]$

$= 908.43$

Moment of inertia about centroidal YY axis

$I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3$

$= \left[(I_{yy})_{CG} + A_1(x_1-\bar{x})^2\right] + \left[(I_{yy})_{CG} + A_2(x_2-\bar{x})^2\right] +$
$\left[(I_{yy})_{CG} + A_3(x_3-\bar{x})^2\right]$

$= \left[\frac{5\times2^3}{12} + 10(1-8.57)^2\right] + \left[\frac{3\times15^3}{12} + 45(7.5-8.57)^2\right]$
$+ \left[\frac{7\times4^3}{12} + 28(13-8.57)^2\right]$

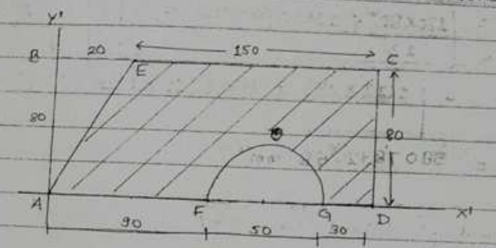
$= 2058.5$

Polar moment of inertia $I_p = I_{zz} = I_{xx} + I_{yy}$

$= 908.43 + 2058.5$

$= 2966.93$

Q. Find polar moment of inertia about centroidal axis.



shape	Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1) rect. ABCD	$170\times80 = 13600$	$170/2 = 85$	$80/2 = 40$	1156000	544000
2) triangle ABE	$\frac{1}{2}\times20\times80 = -800$	$20/3 = 6.67$	$\frac{2}{3}\times80 = 53.33$	-5336	-42664
3) semi-circle FOG	$\frac{\pi\times25^2}{2} = -981.7$	$90+25 = 115$	$\frac{4\times25}{3\pi} = 10.61$	-112895.5	-10415.837

$\Sigma A_i = 11818.3$

$\Sigma A_i x_i = 1037768.5$

$\Sigma A_i y_i = 490920.163$

$\bar{X} = 87.81$ mm

$\bar{Y} = 41.54$ mm

tia about xx-axis

$(I_{xx})_2 - (I_{xx})_3$

$= \left[(I_{xx})_{cg} + A_1(y_1-\bar{y})^2\right] - \left[(I_{xx})_{cg} + A_2(y_2-\bar{y})^2\right] -$

$\left[(I_{xx})_{cg} + A_3(y_3-\bar{y})^2\right]$

$\left[\;\; + 13600(40-41.54)^2\right] - \left[\frac{20\times 80^3}{36} + 800(53.33-41.54)^2\right]$

$\left[0.11\times 25^4 + 981.7(10.61-41.54)^2\right]$

$= 807812.68\ mm^4$

inertia about yy-axis

$(I_{yy})_1 - (I_{yy})_2 - (I_{yy})_3$

$\left[(I_{yy})_{cg} + A_1(x_1-\bar{x})^2\right] - \left[(I_{yy})_{cg} + A_2(x_2-\bar{x})^2\right] -$

$\left[(I_{yy})_{cg} + A_3(x_3-\bar{x})^2\right]$

$\left[\frac{\;\;0^3}{12} + 13600(85-87.81)^2\right] - \left[\frac{80\times 20^3}{36} + 800(6.67-87.81)^2\right]$

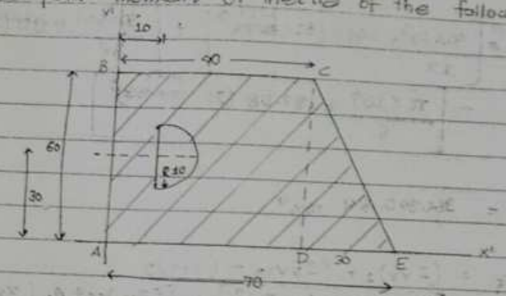
$\left[\frac{\;\;25^4}{8} + 981.7(115-87.81)^2\right]$

$= 96817.78\ mm^4$

moment of inertia $I_P = I_{xx} + I_{yy}$

$= 32604630.46\ mm^4$

8. Find the polar moment of inertia of the following shaded area.



	shape	Area(A_i)	x_i	y_i	A_ix_i	A_iy_i
1)	rect. ABCD	$40\times 60 = 2400$	$40/2 = 20$	$60/2 = 30$	48000	72000
2)	triangle CDE	$\frac{1}{2}\times 30\times 60 = 900$	$40+\frac{30}{3} = 50$	$\frac{60}{3} = 20$	45000	18000
3)	semi-circle	$\frac{\pi\times 10^2}{2} = -157.08$	$10+\frac{4\times 10}{3\pi} = 14.24$	30	-2236.82	-9712.4

$\Sigma A_i = 3142.92$

$\Sigma A_ix_i = 90763.18$

$\Sigma A_iy_i = 85287.6$

$\bar{x} = 28.88\ mm$

$\bar{y} = 27.14\ mm$

Moment of inertia

$I_{xx} = (I_{xx})_1 + (I_{xx})_2 - (I_{xx})_3$

$= \left[(I_{xx})_{cg} + A_1(y_1-\bar{y})^2\right] + \left[(I_{xx})_{cg} + A_2(y_2-\bar{y})^2\right] -$

$\left[(I_{xx})_{cg} + A_3(y_3-\bar{y})^2\right]$

$$= \left[\frac{90\times60^3}{12} + 2400\left(30-\cancel{28.88}^{\,27.74}\right)^2\right] + \left[\frac{90\times60^3}{36} + 900\left(20-\cancel{28.88}^{\,27.74}\right)^2\right]$$

$$- \left[\frac{\pi\times10^4}{8} + 157.08\left(30-\cancel{28.88}^{\,27.74}\right)^2\right]$$

$$= 960300.84 \text{ mm}^4$$

$$I_{yy} = (I_{yy})_1 + (I_{yy})_2 - (I_{yy})_3$$

$$= \left[(I_{yy})_{cg} + A_1(x_1-\bar{x})^2\right] + \left[(I_{yy})_{cg} + A_2(x_2-\bar{x})^2\right] - \left[(I_{yy})_{cg} + A_3(x_3-\bar{x})^2\right]$$

$$= \left[\frac{60\times40^3}{12} + 2400(20-28.88)^2\right] + \left[\frac{60\times30^3}{36} + 900(50-28.88)^2\right]$$

$$- \left[0.11\times10^4 + 157.08(14.24-28.88)^2\right]$$

$$= \cancel{824289.57 \text{ mm}^4}\ 820832.63 \text{ mm}^4$$

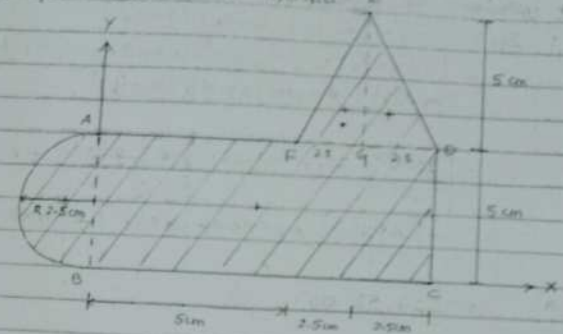
∴ Polar moment of inertia is

$$I_p = I_{xx} + I_{yy}$$

$$= \cancel{1884600.41 \text{ mm}^4}$$

$$= 1881233.47 \text{ mm}^4 \quad //$$

6. Find the polar moment of inertia



shape	Area (A_i)	x_i	y_i	A_ix_i	A_iy_i
1) rect. ABCD	10×5 = 50	$10/2$ = 5	$5/2$ = 2.5	250	125
2) triangle EFG	$\frac{1}{2}\times2.5\times5$ = 6.25	$5+\frac{2}{3}\times2.5$ = 6.67	$5+\frac{5}{3}$ = 6.67	41.68	41.68
3) triangle EGD	$\frac{1}{2}\times2.5\times5$ = 6.25	$7.5+\frac{2.5}{3}$ = 8.33	$5+5/3$ = 6.67	52.06	41.68
4) semicircle	$\frac{\pi\times2.5^2}{2}$ = 9.81	$-\frac{4\times2.5}{3\pi}$ = −1.06	2.5	−10.39	24.52

$\Sigma A_i = 72.31 \text{ cm}^2$

$\Sigma A_ix_i = 333.35$

$\Sigma A_iy_i = 232.88$

$\bar{X} = 4.61$ cm

$\bar{Y} = 3.22$ cm

Moment of inertia

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3 + (I_{xx})_4$$

$$= \left[(I_{xx})_{cg} + A_1 (y_1 - \bar{y})^2\right] + \left[(I_{xx})_{cg} + A_2 (y_2 - \bar{y})^2\right] + \left[(I_{xx})_{cg} + A_3 (y_3 - \bar{y})^2\right] + \left[(I_{xx})_{cg} + A_4 (y_4 - \bar{y})^2\right]$$

$$= \left[\frac{10 \times 5^3}{12} + 50 (2.5 - 3.22)^2\right] + \left[\frac{2.5 \times 5^3}{36} + 6.25 (6.67 - 3.22)^2\right]$$

$$+ \left[\frac{2.5 \times 5^3}{36} + 6.25 (6.67 - 3.22)^2\right] + \left[\frac{\pi \times 2.5^4}{8} + 9.81 (2.5 - 3.22)^2\right]$$

$$= 317.51 \text{ cm}^4$$

$$I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3 + (I_{yy})_4$$

$$= \left[\frac{5 \times 10^3}{12} + 50 (5 - 4.61)^2\right] + \left[\frac{5 \times 2.5^3}{36} + 6.25 (6.67 - 4.61)^2\right]$$

$$+ \left[\frac{5 \times 2.5^3}{36} + 6.25 (8.33 - 4.61)^2\right] + \left[0.11 \times 2.5^4 + 9.81 (-1.06 - 4.61)^2\right]$$

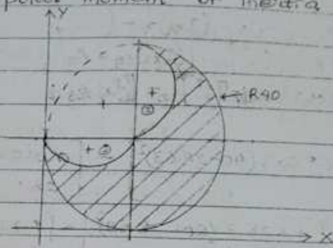
$$= 861.30 \text{ cm}^4$$

$\therefore$ Polar moment of inertia is

$$I_p = I_{xx} + I_{yy}$$

$$= 1178.81 \text{ cm}^4 \quad //$$

Q. Determine polar moment of inertia of following figure.



	Shape	Area(A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1)	big circle	$\pi \times 40^2$	40	40		
		$= 5026.5$			201060	201060
2)	quarter circle	$\frac{\pi \times 40^2}{4}$	$40 - \frac{4 \times 40}{3\pi}$	$40 + \frac{4 \times 40}{3\pi}$		
		$= -1256.6$	$= 23.02$	$= 56.97$	-28926.9	-71588.5
3)	semi-circle ①	$\frac{\pi \times 20^2}{2}$	$40 + \frac{4 \times 20}{3\pi}$	$40 + 20$		
		$= -628.3$	$= 48.5$	$= 60$	-30472.5	-37698
4)	semi-circle ②	$\frac{\pi \times 20^2}{2}$		$40 - \frac{4 \times 20}{3\pi}$		
		$= -628.3$	20	$= 31.51$	-12566	-19797.7

$\Sigma A_i = 2513.3$

$\Sigma A_i x_i = 129094.6$

$\Sigma A_i y_i = 71975.8$

$\bar{x} = 51.36$

$\bar{y} = 28.63$

f inertia

$I_{xx} = (I_{xx})_1 - (I_{xx})_2 - (I_{xx})_3 - (I_{xx})_4$

$= \left[(I_{xx})_{cg} + A_1(y_1-\bar{y})^2\right] - \left[(I_{xx})_{cg} + A_2(y_2-\bar{y})^2\right] - \left[(I_{xx})_{cg} + A_3(y_3-\bar{y})^2\right]$

$- \left[(I_{xx})_{cg} + A_4(y_4-\bar{y})^2\right]$ ~~$[(I_{xx})_{cg}+$~~

$= \left[\frac{\pi 80^4}{64} + 5026.5(40-28.63)^2\right] - \left[0.055\times 40^4 + 1256.6(56.97 - 28.63)^2\right]$

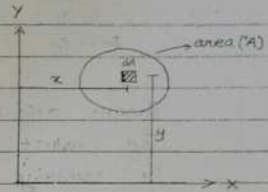
$- \left[\frac{\pi 20^4}{8} + 628.3(60-28.63)^2\right] - \left[0.11\times 20^4 + 628.3(31.51 - 28.63)^2\right]$

$=$

of inertia

...t of inertia with respect to co-ordinate axis is the ...uct of any elementary area with both of their ...ordinates (x, y) i.e. at a distance 'x' from y-axis and ...istance 'y' from x-axis.

Consider an elemental area 'dA' which comprises a plane area of 'A'. Let this area be at a distance 'x' from y-axis and 'y' from x-axis.

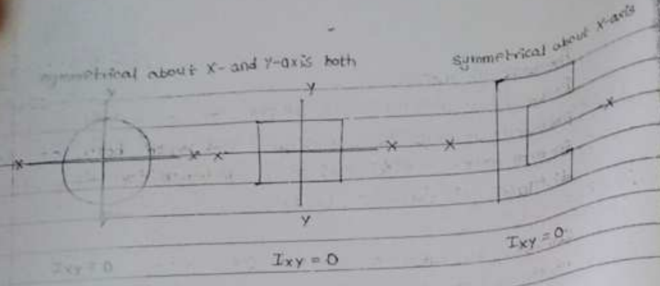


The first moment of area about X-axis is ydA and that about Y-axis is xdA. Further moment of 'y' dA about Y-axis is $xydA$. Then, the term $xydA$ is k/a the product of inertia of elemental area 'dA'. The integral $\int xydA$ is called product of inertia of entire area 'A'. Then, it is denoted by

$$I_{xy} = \int xy\, dA$$

Salient features

1) I_{xx} and I_{yy} are always positive but I_{xy} may be positive or negative or zero.
2) Product of inertia is positive, if area lies on 1st or 3rd quadrant.
3) Product of inertia is negative, if area lies on 2nd or 4th quadrant.
4) Product of inertia is zero if one or both of x and y axis are symmetrical on figure.

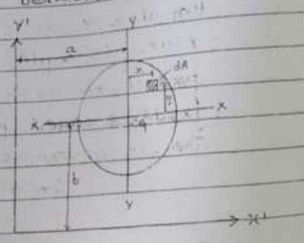


$I_{xy} = 0$ $I_{xy} = 0$ $I_{xy} = 0$

Parallel axis theorem for product of inertia

The product of inertia of plane area about any two mutually perpendicular centroidal axis is given by the sum of product of inertia about centroidal axis and area times distance between these parallel lines.

Consider an elementary area at a distance x and y from centroidal y- and x-axis respectively.



Product of inertia for centroidal xx and yy axis is given by

$$I_{xy} = \int xy\,dA$$

Consider any two axes X'X' and Y'Y' parallel to centroidal XX and YY axes respectively and at a distance 'b' from centroidal XX-axis and distance of 'a' from centroidal YY-axis.

Product of inertia about X'X' and Y'Y' is

$$I_{x'y'} = \Sigma(x+a)(y+b)\,dA$$

$$= \Sigma xy\,dA + \Sigma bx\,dA + \Sigma ay\,dA + \Sigma ab\,dA$$

The terms $\int x\,dA$ and $\int y\,dA$ are 1st moment of area about centroidal axis whose values are zero. Then,

$$I_{x'y'} = \Sigma xy\,dA + \Sigma ab\,dA$$

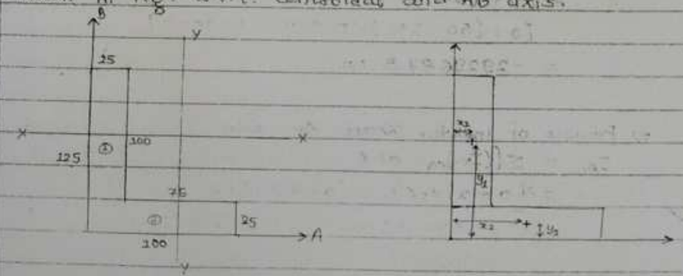
$$= \int xy\,dA + \int ab\,dA$$

$$\therefore \boxed{I_{x'y'} = I_{xy} + abA}$$

The above identity is called 'transfer formula' for product of inertia.

Q. Determine the product of inertia of the angle section shown in fig. w.r.t. centroidal and AB axis.

1)



Shape	Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1) rect ①	$25 \times 100 = 2500$	$25/2 = 12.5$	$\frac{100}{2} + 25 = 75$	31250	187500
2) rect ②	$100 \times 25 = 2500$	$100/2 = 50$	$25/2 = 12.5$	125000	31250

$\Sigma A_i = 5000$

$\Sigma A_i x_i = 156250$

$\Sigma A_i y_i = 218750$

$\bar{x} = 31.25$ mm

$\bar{y} = 43.75$ mm

⇒ Product of inertia about centroidal xy-axis

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2$$

$$= \left[(I_{xy})_{cg} + (x_1-\bar{x})(y_1-\bar{y})A_1\right] + \left[(I_{xy})_{cg} + (x_2-\bar{x})(y_2-\bar{y})A_2\right]$$

$$= \left[0 + (12.5-31.25)(75-43.75)\,2500\right] + \left[0 + (50-31.25)(12.5-43.75)\,2500\right]$$

$$= -2929687.5 \text{ mm}^4$$

⇒ Product of inertia about AB-axis

$$I_{AB} = \Sigma\left[(I_{xy})_{cg} + abA\right]$$

$$\left[0 + x_1 y_1 A\right]_1 + \left[0 + x_2 y_2 A\right]_2$$

$$= (12.5 \times 75 \times 2500) + (50 \times 12.5 \times 2500)$$

$$= 3906250 \text{ mm}^4$$

OR

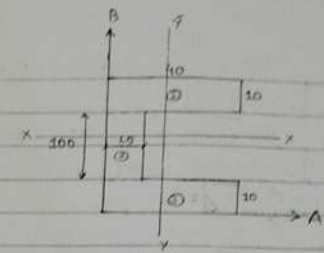
$$I_{AB} = I_{xy} + (abA)_{total}$$

$$= I_{xy} + \bar{x}\bar{y}A_{total}$$

$$= -2929687.5 + 31.25 \times 43.75 \times 5000$$

$$= 3906250 \text{ mm}^4$$

2)



Shape	Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1) rect ①	$40 \times 10 = 400$	$40/2 = 20$	$110 + 10/2 = 115$	8000	46000
2) rect ②	$10 \times 100 = 1000$	$10/2 = 5$	$10 + 50 = 60$	5000	60000
3) rect ③	$40 \times 10 = 400$	$40/2 = 20$	$10/2 = 5$	8000	2000

$\Sigma A_i = 1800$

$\Sigma A_i x_i = 21000$

$\Sigma A_i y_i = 108000$

$\bar{x} = 11.67$ mm

$\bar{y} = 60$ mm

⇒ Product of inertia about centroidal XY-axis

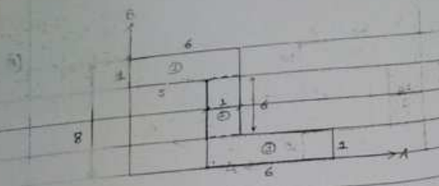
$$I_{xy} = \Sigma\left[(I_{xy})_{cg} + abA\right]$$

$$= \left[0 + (x_1-\bar{x})(y_1-\bar{y})A\right]_1 + \left[(y_2-\bar{y})(x_2-\bar{x})A\right]_2 + \left[(x_3-\bar{x})(y_3-\bar{y})A\right]_3$$

$$= \left[(20-11.67)(115-60)\,400\right] + \left[(5-11.67)\overset{0}{(60-60)}\,1000\right] + \left[(20-11.67)(5-60)\,400\right]$$

$$= 0$$

3)



	Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1) rect ①	$6\times1 = 6$	$6/2 = 3$	$7+1/2 = 7.5$	18	45
2) rect ②	$1\times6 = 6$	$5+1/2 = 5.5$	$1+6/2 = 4$	33	24
3) rect ③	$6\times1 = 6$	$5+6/2 = 8$	$1/2 = 0.5$	48	3

$\Sigma A_i = 18$

$\Sigma A_i x_i = 99$

$\Sigma A_i y_i = 72$

$\bar{x} = 5.5$ mm

$\bar{y} = 4$ mm

⇒ Product of inertia about centroidal xy-axis

$I_{xy} = \Sigma[(I_{xy})_{cg} + abA]$

$= [0 + (x_1-\bar{x})(y_1-\bar{y})A]_1 + [(x_2-\bar{x})(y_2-\bar{y})A]_2 + [(x_3-\bar{x})(y_3-\bar{y})A]_3$

$= [(3-5.5)(7.5-4)6] + [(5.5-\cancel{5.5})(4-4)6] + [(8-5.5)(0.5-4)6]$

$= -52.5 + (-52.5)$

$= -105 \text{ mm}^4$

⇒ Product of inertia about AB-axis

$I_{AB} = \Sigma[(I_{xy})_{cg} + abA]$

$= [0 + x_1 y_1 A]_1 + [x_2 y_2 A]_2 + [x_3 y_3 A]_3$

$= (3\times7.5\times6) + (5.5\times4\times6) + (8\times0.5\times6)$

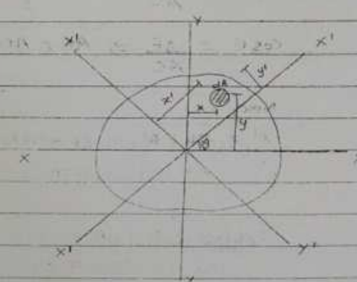
$= 291 \text{ mm}^4$

Principal moment of inertia and location of principal axis

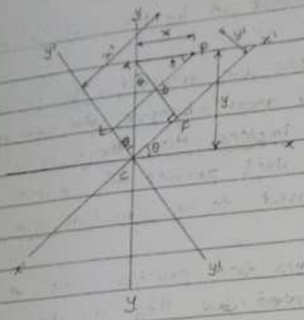
When the axes are rotated, the product of inertia for the plane area changes the sign and becomes negative. This implies that there is a certain direction of axes for which product of inertia is zero, such axes are referred to as principal axes of area.

There is always two principal axes at a given point in a given area and they will be mutually perpendicular to each other. The moment of inertia about principal axes is called principal moment of inertia. The moment of inertia about one of principal axes will be maximum and that about other will be minimum.

The maximum moment of inertia is called major principal moment of inertia and the minimum moment of inertia is called minor principal moment of inertia.



- xx and yy are the axes through centroid of plane area A
- x'x' and y'y' are the axes inclined at θ to the xx and yy.
- dA is the elemental area considered whose co-ordinate is (x, y) from xx and yy axes and is (x', y') from x'x' and y'y'



From $\triangle ABP$,

$\cos\theta = \frac{BP}{AP} \Rightarrow BP = AP\cos\theta = x\cos\theta$

$\sin\theta = \frac{AB}{AP} \Rightarrow AB = AP\sin\theta = x\sin\theta$

From $\triangle ACF$,

$\sin\theta = \frac{CF}{AC} \Rightarrow CF = AC\sin\theta = y\sin\theta$

$\cos\theta = \frac{AF}{AC} \Rightarrow AF = AC\cos\theta = y\cos\theta$

Now,

$x' = EB + BP = CF + BP = y\sin\theta + x\cos\theta$

$y' = AF - AB = y\cos\theta - x\sin\theta$

$$\begin{aligned}
I_{x'y'} &= \int x'y'\,dA \\
&= \int (y\sin\theta + x\cos\theta)(y\cos\theta - x\sin\theta)\,dA \\
&= \int (y^2\sin\theta\cos\theta - xy\sin^2\theta + xy\cos^2\theta - x^2\sin\theta\cos\theta)\,dA \\
&= \int y^2\sin\theta\cos\theta\,dA - \int xy\sin^2\theta\,dA + \int xy\cos^2\theta\,dA - \int x^2\sin\theta\cos\theta\,dA \\
&= I_{xx}\sin\theta\cos\theta - I_{xy}\sin^2\theta + I_{xy}\cos^2\theta - I_{yy}\sin\theta\cos\theta \quad \text{— i)}
\end{aligned}$$

$$\begin{aligned}
I_{x'x'} &= \int y'^2\,dA = \int (y\cos\theta - x\sin\theta)^2\,dA \\
&= \int (y^2\cos^2\theta - 2xy\sin\theta\cos\theta + x^2\sin^2\theta)\,dA \\
&= \int y^2\cos^2\theta\,dA - 2\int xy\sin\theta\cos\theta\,dA + \int x^2\sin^2\theta\,dA \\
&= I_{xx}\cos^2\theta - 2I_{xy}\sin\theta\cos\theta + I_{yy}\sin^2\theta \quad \text{— ii)}
\end{aligned}$$

$$\begin{aligned}
I_{y'y'} &= \int x'^2\,dA = \int (y\sin\theta + x\cos\theta)^2\,dA \\
&= \int (y^2\sin^2\theta + 2xy\sin\theta\cos\theta + x^2\cos^2\theta)\,dA \\
&= \int y^2\sin^2\theta\,dA + 2\int xy\sin\theta\cos\theta\,dA + \int x^2\cos^2\theta\,dA \\
&= I_{xx}\sin^2\theta + 2I_{xy}\sin\theta\cos\theta + I_{yy}\cos^2\theta \quad \text{— iii)}
\end{aligned}$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$ $\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\quad 2\sin\theta\cos\theta = \sin 2\theta$

Substituting these values in i), ii) & iii),

$$\begin{aligned}
I_{x'y'} &= I_{xx}\left(\frac{\sin 2\theta}{2}\right) - I_{xy}\left(\frac{1-\cos 2\theta}{2}\right) + I_{xy}\left(\frac{1+\cos 2\theta}{2}\right) - I_{yy}\left(\frac{\sin 2\theta}{2}\right) \\
&= \left(\frac{I_{xx}-I_{yy}}{2}\right)\sin 2\theta + \frac{I_{xy}}{2}(\cancel{1} + \cos 2\theta - \cancel{1} + \cos 2\theta)
\end{aligned}$$

$$\Rightarrow \boxed{I_{x'y'} = \left(\frac{I_{xx}-I_{yy}}{2}\right)\sin 2\theta + I_{xy}\cos 2\theta}$$

$$\cancel{I_{x'x'} = I_{xx}\left(\frac{1+\cos 2\theta}{2}\right) - I_{xy}\sin 2\theta + I_{yy}\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$\cancel{I_{y'y'} = I_{xx}\left(\frac{1-\cos 2\theta}{2}\right) + I_{xy}\sin 2\theta + I_{yy}\left(\frac{1+\cos 2\theta}{2}\right)}$$

$$I_{x'x'} = I_{xx}\left(\frac{1+\cos 2\theta}{2}\right) - I_{xy}\sin 2\theta + I_{yy}\left(\frac{1-\cos 2\theta}{2}\right)$$

$$\Rightarrow I_{x'x'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) + \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'y'} = I_{xx}\left(\frac{1-\cos 2\theta}{2}\right) + I_{xy}\sin 2\theta + I_{yy}\left(\frac{1+\cos 2\theta}{2}\right)$$

$$\Rightarrow I_{y'y'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) \mp \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos 2\theta + I_{xy}\sin 2\theta$$

It is observed that

$$\boxed{I_{x'x'} + I_{y'y'} = I_{xx} + I_{yy}}$$

i.e. sum of moment of inertia about any two perpendicular axis remains same.

To determine principal axis:

Product of inertia is zero about $x'x'$ & $y'y'$ axis.

$$I_{x'y'} = 0$$

$$\left(\frac{I_{xx}-I_{yy}}{2}\right)\sin 2\theta + I_{xy}\cos 2\theta = 0$$

$$\therefore \tan 2\theta = \frac{-I_{xy}}{\left(\frac{I_{xx}-I_{yy}}{2}\right)} = \left(\frac{-2I_{xy}}{I_{xx}-I_{yy}}\right)$$

Location of principal axis

$$\tan 2\theta = \left(\frac{-2I_{xy}}{I_{xx}-I_{yy}}\right)$$

$$I_{x'x'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) + \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos 2\theta - I_{xy}\sin 2\theta$$

$$= \left(\frac{I_{xx}+I_{yy}}{2}\right) + \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos 2\theta + \left(\frac{I_{xx}-I_{yy}}{2}\right)\sin 2\theta \tan 2\theta$$

$$= \left(\frac{I_{xx}+I_{yy}}{2}\right) + \left(\frac{I_{xx}-I_{yy}}{2}\right)\left[\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}\right]$$

$$= \left(\frac{I_{xx}+I_{yy}}{2}\right) + \left(\frac{I_{xx}-I_{yy}}{2}\right)\left(\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}\right)$$

$$= \left(\frac{I_{xx}+I_{yy}}{2}\right) + \left(\frac{I_{xx}-I_{yy}}{2}\right)\sec 2\theta \qquad \text{iv)}$$

We have,

$$\sec^2 2\theta = 1 + \tan^2 2\theta$$

$$\sec 2\theta = \sqrt{1+\tan^2 2\theta}$$

$$= \sqrt{1 + \frac{4I_{xy}^2}{(I_{xx}-I_{yy})^2}}$$

$$= \sqrt{\frac{(I_{xx}-I_{yy})^2 + 4I_{xy}^2}{(I_{xx}-I_{yy})^2}}$$

$$= \frac{\sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}}{\left(\frac{I_{xx}-I_{yy}}{2}\right)}$$

Subs. this value in iv),

$$I_{x'x'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) + \cancel{\left(\frac{I_{xx}-I_{yy}}{2}\right)} \cdot \frac{\sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}}{\cancel{\left(\frac{I_{xx}-I_{yy}}{2}\right)}}$$

$$\boxed{I_{max} = \left(\frac{I_{xx}+I_{yy}}{2}\right) + \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}}$$

$$I_{y'y'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) - \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos 2\theta + I_{xy}\sin 2\theta$$

$$= \left(\frac{I_{xx}+I_{yy}}{2}\right) - \left(\frac{I_{xx}-I_{yy}}{2}\right)\cos 2\theta - \left(\frac{I_{xx}-I_{yy}}{2}\right)\sin 2\theta \tan 2\theta$$

$$= \left(\frac{I_{xx}+I_{yy}}{2}\right) - \left(\frac{I_{xx}-I_{yy}}{2}\right)\left[\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}\right]$$

$$= \left(\frac{I_{xx}+I_{yy}}{2}\right) - \left(\frac{I_{xx}-I_{yy}}{2}\right)\sec 2\theta \qquad \text{v)}$$

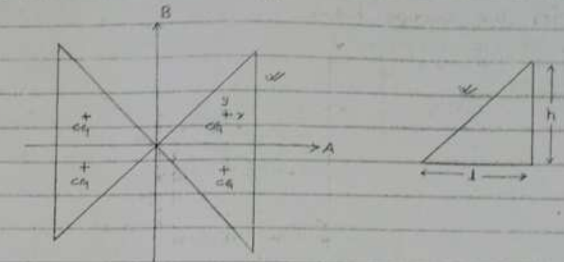
Subs. value of $\sec 2\theta$ in v),

$$I_{y'y'} = \left(\frac{I_{xx}+I_{yy}}{2}\right) - \cancel{\left(\frac{I_{xx}-I_{yy}}{2}\right)} \cdot \frac{\sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}}{\cancel{\left(\frac{I_{xx}-I_{yy}}{2}\right)}}$$

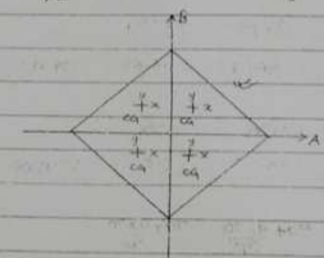
$$\boxed{I_{min} = \left(\frac{I_{xx}+I_{yy}}{2}\right) - \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}}$$

$$\therefore \quad I_{\substack{max \\ min}} = \left(\frac{I_{xx}+I_{yy}}{2}\right) \pm \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

Product of inertia for triangle

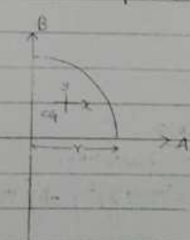


$I_{xy} = \frac{l^2h^2}{72}$ $\qquad$ $I_{AB} = \frac{l^2h^2}{8}$



$I_{xy} = \frac{-l^2h^2}{72}$ $\qquad$ $I_{AB} = \frac{l^2h^2}{24}$

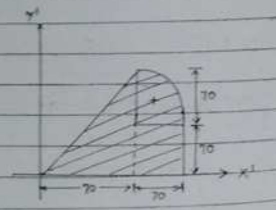
Product of inertia for quarter circle



$I_{xy} = -0.0165r^4$

$I_{AB} = \frac{r^4}{8}$

6. Determine the principal axes and principal moment of inertia for the section below.



shape	Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1) triangle	$\frac{1}{2}\times 70\times 140$ $= 4900$	$\frac{2}{3}\times 70$ $= 46.67$	$\frac{1}{3}\times 140$ $= 46.67$	228683	228683
2) square	70×70 $= 4900$	$70+35$ $= 105$	$70/2$ $= 35$	514500	171500
3) quarter circle	$\frac{\pi\times 70^2}{4}$ $= 3848.45$	$70+\frac{4\times 70}{3\pi}$ $= 99.70$	$70+\frac{4\times 70}{3\pi}$ $= 99.70$	383690.46	383690.46

$\Sigma A_i = 13648.45$

$\Sigma A_i x_i = 1126873.46$

$\Sigma A_i y_i = 783873.46$

$\bar{x} = 82.56$

$\bar{y} = 57.43$

- $I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3$

$$= \left[(I_{xx})_{cg} + A_1(y_1-\bar{y})^2\right] + \left[(I_{xx})_{cg} + A_2(y_2-\bar{y})^2\right] + \left[(I_{xx})_{cg} + A_3(y_3-\bar{y})^2\right]$$

$$= \left[\frac{70\times 140^3}{36} + 4900(46.67-57.43)^2\right] + \left[\frac{70\times 70^3}{12} + 4900(35-57.43)^2\right]$$

$$+ \left[0.055\times 70^4 + 3848.45(99.70-57.43)^2\right]$$

$$= 18565682.34 \text{ mm}^4$$

- $I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3$

$$= \left[\frac{140\times 70^3}{36} + 4900(46.67-82.56)^2\right] + \left[\frac{70\times 70^3}{12} + 4900(105-82.56)^2\right]$$

$$+ \left[0.055\times 70^4 + 3848.45(99.70-82.56)^2\right]$$

$$= 14564932.25 \text{ mm}^4$$

- $I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$

$$= \left[(I_{xy})_{cg} + (x_1-\bar{x})(y_1-\bar{y})A_1\right] + \left[(I_{xy})_{cg} + (x_2-\bar{x})(y_2-\bar{y})A_2\right] +$$

$$\left[(I_{xy})_{cg} + (x_3-\bar{x})(y_3-\bar{y})A_3\right]$$

$$= \left[\frac{b^2h^2}{72} + (46.67-82.56)(46.67-57.43)\times 4900\right] +$$

$$\left[0 + (105-82.56)(35-57.43)\times 4900\right] + \left[-0.0165\times 70^4 + (99.7-82.56)(99.7-57.43)\times 3848.45\right]$$

$$= 3151907.212 \text{ mm}^4$$

$\Rightarrow$ $$\tan 2\theta = \frac{-2I_{xy}}{I_{xx}-I_{yy}} = \frac{-2\times 3151907.212}{18565682.34 - 14564933.25}$$

$$2\theta = \tan^{-1}(-1.5756)$$

$$\theta_1 = -28.79^\circ \qquad \theta_2 = -28.79 + 90 = 61.21^\circ$$

$\Rightarrow$ $$I_{max} = \frac{I_{xx}+I_{yy}}{2} + \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$= 16565312.3 + 3733100.507$$

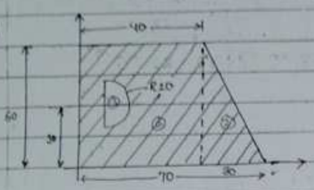
$$= 20298412.81 \text{ mm}^4$$

$$I_{min} = 16565312.3 - 3733100.507$$

$$= 12832211.79 \text{ mm}^4 \quad //$$

Q. Find principal ~~moment~~ moment of inertia and location ... axes.

a)



$\bar{X} = 28.88$ mm

$\bar{Y} = 27.14$ mm

$I_{xx} = 860300.84\ mm^4$

$I_{yy} = 820932.63\ mm^4$

$I_{xy} = (I_{xy})_1 + (I_{xy})_2 - (I_{xy})_3$

$= [(I_{xy})_{cg} + (x_1-\bar{x})(y_1-\bar{y})A_1] + [(I_{xy})_{cg} + (x_2-\bar{x})(y_2-\bar{y})A_2] -$

$[(I_{xy})_{cg} + (x_3-\bar{x})(y_3-\bar{y})A_3]$

$= [0 + (20-28.88)(30-27.14)2400] + \left[-\frac{30^2 \times 60^2}{72} + (50-28.88)\right.$

$\left.(20-27.14)900\right] + [0 + (\overset{14.24}{\cancel{115}}-28.88)(\overset{30}{\cancel{10.61}}-27.14)\overset{157.08}{\cancel{881.7}}]$

$= \cancel{1455842.94}\ mm^4$

-235094.27

⇒ Principal moment of inertia

$I_{max} = \frac{I_{xx}+I_{yy}}{2} + \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$

$= 940616.7 + 1156010.539$

$= \underline{2096627.239}\ mm^4$

1176591.76

$I_{min} = 940616.7 - 1156010.539$

$= \cancel{-215393.839}\ mm^4$

704768.79

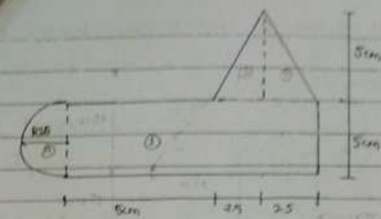
⇒ Principal axes

$\tan 2\theta = \frac{-2I_{xy}}{I_{xx}-I_{yy}}$

$2\theta = \tan^{-1}(-58.7196)$

$\theta_1 = \cancel{-44.51°}$ $42.51°$

$\theta_2 = \cancel{45.49°}$ // $132.61°$



$\bar{x} = 4.61$ cm

$\bar{y} = 3.22$ cm

$I_{xx} = 317.51\ cm^4$

$I_{yy} = 861.30\ cm^4$

$I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3 + (I_{xy})_4$

$= [(I_{xy})_{cg} + (x_1-\bar{x})(y_1-\bar{y})A_1] + [(I_{xy})_{cg} + (x_2-\bar{x})(y_2-\bar{y})A_2] +$

$[(I_{xy})_{cg} + (x_3-\bar{x})(y_3-\bar{y})A_3] + [(I_{xy})_{cg} + (x_4-\bar{x})(y_4-\bar{y})A_4]$

$= [0 + (5-4.61)(2.5-3.22)50] + \left[\frac{b^2h^2}{72} + (6.67-4.61)(6.67-3.22)6.25\right]$

$+ \left[-\frac{b^2h^2}{72} + (8.33-4.61)(6.67-3.22)6.25\right] + [0 + (-1.06-4.61)$

$(2.5-3.22)9.81]$

$= 150.63\ cm^4$

⇒ Principal moment of inertia

$I_{max} = \frac{I_{xx}+I_{yy}}{2} + \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$

$= 589.405 + 310.831$

$= 900.236\ cm^4$

$I_{min} = 589.405 - 310.831$

$= 278.574\ cm^4$

⇒ Principal axes

$\tan 2\theta = \frac{-2I_{xy}}{I_{xx}-I_{yy}}$

$2\theta = \tan^{-1}(0.55400)$

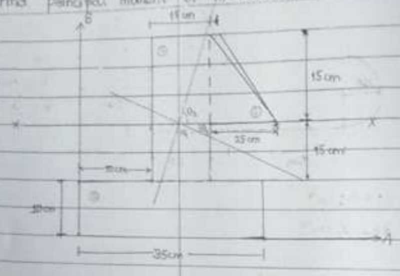
$\theta_1 = 14.49°$

$\theta_2 = 14.49 + 90$

$= 104.49°$

Q. Find principal moment of inertia and location of principal axes.

a)



shape	Area (A_i)	x_i	y_i	A_ix_i	A_iy_i
1) rect ①	35×10 $=350$	$35/2$ $=17.5$	$10/2$ $=5$	6125	1750
2) rect ②	15×30 $=450$	$10+15/2$ $=17.5$	$10+30/2$ $=25$	~~2275~~ 7875	11250 ~~6750~~
3) triangle ③	$\frac{1}{2}\times15\times15$ $=112.5$	$25+\frac{1}{3}\times15$ $=30$	$25+\frac{1}{3}\times15$ $=30$	3375	3375

$\Sigma A_i = 912.5$

$\Sigma A_ix_i = 17375$

$\Sigma A_iy_i = 16375$

$\bar{X} = 19.04$ cm

$\bar{Y} = 17.94$ cm

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3$$

$$= \left[(I_{xx})_{cg} + A_1(y_1-\bar{y})^2\right] + \left[(I_{xx})_{cg} + A_2(y_2-\bar{y})^2\right] + \left[(I_{xx})_{cg} + A_3(y_3-\bar{y})^2\right]$$

$$= \left[\frac{35\times10^3}{12} + 350(5-17.94)^2\right] + \left[\frac{15\times30^3}{12} + 450(25-17.94)^2\right] + \left[\frac{15\times15^3}{36} + 112.5(30-17.94)^2\right]$$

$$= 135470.20 \text{ cm}^4$$

$$I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3$$

$$= \left[\frac{10\times35^3}{12} + 350(17.5-19.04)^2\right] + \left[\frac{30\times15^3}{12} + 450(17.5-19.04)^2\right] + \left[\frac{15\times15^3}{36} + 112.5(30-19.04)^2\right]$$

$$= 60983.87 \text{ cm}^4$$

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$

$$= \left[(I_{xy})_{cg} + (x_1-\bar{x})(y_1-\bar{y})A_1\right] + \left[(I_{xy})_{cg} + (x_2-\bar{x})(y_2-\bar{y})A_2\right] + \left[(I_{xy})_{cg} + (x_3-\bar{x})(y_3-\bar{y})A_3\right]$$

$$= \left[0 + (17.5-19.04)(5-17.94)350\right] + \left[0 + (17.5-19.04)(25-17.94)450\right] + \left[-\frac{b^2h^2}{72} + (30-19.04)(30-17.94)112.5\right]$$

$$= \cancel{3037.175 \text{ cm}^4} \quad 16248.935 \text{ cm}^4$$

⇒ Principal moment of inertia

$$I_{max} = \frac{I_{xx}+I_{yy}}{2} + \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2} \qquad I_{min} = 60860.722 \text{ cm}^4$$

$$= 98227.035 + 37366.313$$

$$= 135593.348 \text{ cm}^4$$

⇒ Principal axes

$$\tan 2\theta = \frac{-2I_{xy}}{I_{xx} - I_{yy}}$$

$$2\theta = \tan^{-1}(-0.4362)$$

$$\theta_1 = -11.78^\circ \qquad \theta_2 = -11.78 + 90 = 78.22^\circ$$

⇒ Principal moment of inertia

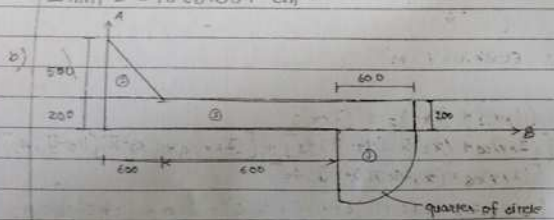
$$I_{max} = \frac{I_{xx}+I_{yy}}{2} + \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$= 98227.035 + 40633.498$$

$$= 138860.53\ cm^4$$

$$I_{min} = 57593.537\ cm^4$$

b)



	shape	Area (A_i)	x_i	y_i	A_ix_i	A_iy_i
1)	rect ①	1800 × 200	1800/2	−200/2		
		= 360000	= 900	= 100	324000000 ~~360000~~	36000000
2)	triangle ②	$\frac{1}{2}$ × 600 × 500	$\frac{1}{3}$ × 600	200 + $\frac{1}{3}$ × 500		
		= 150000	= 200	= 366.67	30000000	55000500
3)	quarter circle ③	$\frac{\pi \times 600^2}{4}$	1200 + $\frac{4 \times 600}{3\pi}$	$\frac{-4 \times 600}{3\pi}$		
		= 282743.33	= 1454.69	= −254.64	411289757.6	−71997761.55
		ΣA_i = 792743.33			ΣA_ix_i = 765289757.6	ΣA_iy_i = 19002738.45

$$\bar{X} = 965.36$$

$$\bar{Y} = 23.97$$

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3$$

$$= \left[(I_{xx})_{cg} + A_1(y_1-\bar{y})^2\right] + \left[(I_{xx})_{cg} + A_2(y_2-\bar{y})^2\right] + \left[(I_{xx})_{cg} + A_3(y_3-\bar{y})^2\right]$$

$$= \left[\frac{1800 \times 200^3}{12} + 360000(100-23.97)^2\right] + \left[\frac{600 \times 500^3}{36} + 150000\right.$$

$$\left.(366.67-23.97)^2\right] + \left[0.055 \times 600^4 + 282743.33(-254.64-23.97)^2\right]$$

$$= 5.097 \times 10^{10}\ mm^4 = 50.97 \times 10^5\ cm^4$$

$$I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3$$

$$= \left[\frac{200 \times 1800^3}{12} + 360000(900-965.36)^2\right] + \left[\frac{500 \times 600^3}{36} + 150000(200-965.36)^2\right]$$

$$+ \left[0.055 \times 600^4 + 282743.33(1454.69-965.36)^2\right]$$

$$= 2.644 \times 10^{11}\ mm^4$$

$$= 26.44 \times 10^6\ cm^4$$

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$

$$= \left[(I_{xy})_{cg} + (x_1-\bar{x})(y_1-\bar{y})A_1\right] + \left[(I_{xy})_{cg} + (x_2-\bar{x})(y_2-\bar{y})A_2\right] +$$

$$\left[(I_{xy})_{cg} + (x_3-\bar{x})(y_3-\bar{y})A_3\right]$$

$$= \left[0 + (900-965.36)(100-23.97)360000\right] + \left[-\frac{b^2h^2}{72} + (200-965.36)\right.$$

$$\left.(366.67-965.23.97)150000\right] + \left[0.0165 \times 600^4 + (1454.64-965.36)\right.$$

$$\left.(-254.64-23.97)282743.33\right]$$

$$=$$

Mohr circle for principal moment of inertia

Steps:

1) Take origin O and from it, measure $OL = I_{YY}$ and $OM = I_{XX}$ along the horizontal I-axis.

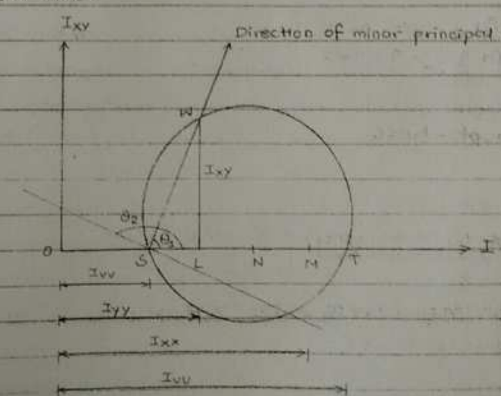
2) At L, draw upward perpendicular LW equal to I_{XY} if I_{XY} is positive (and downward perpendicular) if I_{XY} is negative.

3) Bisect LM at N.

4) With N as centre and NW as radius, draw a circle cutting OL and OM extended at S and T respectively.

5) This OS gives I_{VV} (minor principal moment of inertia, I_{min}), and OT gives I_{UU} (major principal moment of inertia, I_{max}).

6) The direction of SW gives the direction of minor principal axis and perpendicular to it gives the direction of major principal axis.



Chapter 3 : Stress & Strain Analysis

Stress

When a body is acted upon by a load or external force, it undergoes deformation (change in shape or dimension) which increases gradually. During deformation, the material of a body resist the tendency of the load to deform body, and when the load influence is taken over by the internal resistance of the material of the body, it becomes stable. This internal resistance which body offers to meet with the load is called stress.

$$\text{stress}(\sigma) = \frac{\text{Resistance offered by the body (P)}}{\text{load resisting area (A)}}$$

$\sigma = P/A = N/m^2$

$1\ N/m^2 = 1$ Pascal

1 KPa $= 10^3\ N/m^2$

1 MPa $= 10^6\ N/m^2$

1 GPa $= 10^9\ N/m^2$

Types of stress

a) Simple or direct stress

1) Normal stress — Tensile stress / Compressive stress

2) Shearing stress (acts tangentially)

3) Bearing stress

b) Indirect stress

1) Bending stress

2) Torsional stress

Strain

Any element in a material subjected to stress is called strained. Strain (ε) is deformation produced by stress.

Types of strain

a) Normal strain — Tensile strain / Compressive strain

b) Shear strain

c) Volumetric strain

Normal stress

In case of axial ~~in~~ loading, the resistance offered by material per unit area along the direction of force is called normal stress.

- Tensile stress

If the elongation of material along the direction of force takes place, then the resistance offered by material per unit area is k/a tensile stress.

- Compressive stress

If the contraction of material along the direction of force takes place, then the resistance offered by material per unit area is k/a compressive stress.

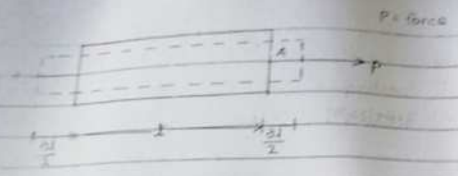


fig. Tensile stress (elongation) $\sigma = P/A$

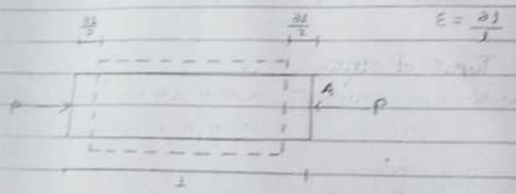


fig. Compressive stress (contraction)

Normal strain $\varepsilon = \dfrac{\text{change in length } (\partial l)}{\text{original length } (l)}$

- Tensile strain

A piece of material, with uniform cross-section, subjected to uniform axial tensile stress, will increase in length from l to $(l+\partial l)$ and the increment of length (∂l) is the actual deformation of the material. The fractional deformation or tensile strain is

$$\boxed{\varepsilon_t = \frac{\partial l}{l}}$$

- Compressive strain

Under compressive forces, a similar piece of material will be reduced in length from l to $(l-\partial l)$. The fractional deformation or compressive strain is

$$\boxed{\varepsilon_c = \frac{\partial l}{l}}$$

Hooke's law

It states 'when a material is loaded, within elastic limit, the stress is proportional to strain.'

Mathematically,

stress $\propto$ strain

$\sigma \propto \varepsilon$

$\sigma = E\varepsilon$

$$\therefore \boxed{E = \frac{\sigma \text{ (stress)}}{\varepsilon \text{ (strain)}}} \qquad E = \frac{F/A}{\partial l/l} = \frac{Fl}{A\,\partial l}$$

This means ratio of stress to corresponding strain bears a constant ratio called Young's modulus of elasticity (E).

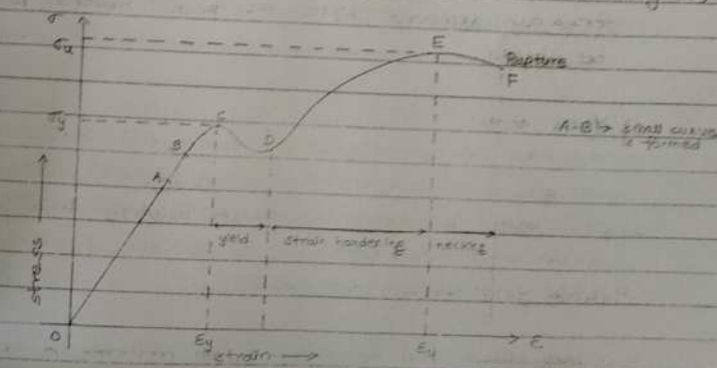


fig. Stress-strain relationship

A - Proportional limit
B - Elastic limit
C - Upper yield point (such point in which deformation is seen although load is not applied)
D - Lower yield point
E - Ultimate load point (point above which any load can't be bear)
F - Fracture (Rupture)

- In order to obtain the stress-strain curve, the tensile test of the specimen should be done.

a) Proportional limit
It is the maximum value of stress upto which stress-strain relationship is a straight line. Experimentally, it can be observed.

b) Elastic limit
It is the maximum value of stress upto which material returns to its original shape if externally applied forces are removed. Upto this point, Hooke's law can be applied.

c) Yield stress
The stress at which yield is initiated is called yield strength of material. After increasing the load steadily, it is observed the curve drops slightly to a lower value which is maintained for a certain period while test piece keeps elongating.

Yield starts due to slippage of particles of material.

SOM

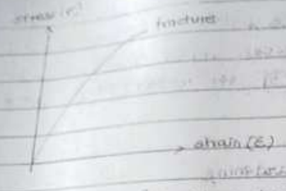
After lower yield, strain increases with increase in load. Between upper and lower yield point, strain hardening of ~~str~~ material takes place, it takes increasing load.

d) Strain hardening (materials recompose & strain hardening occurs)
It is the phenomenon which results in an increase in hardness and strength of material subjected to plastic deformation. It happens due to repositioning of molecules of metal. This increase the stress required to cause slip because of previous plastic deformation or the increase of strength of material due to mechanical working (material recomposition) is k/a strain hardening.

e) Ultimate strength

It is the maximum stress which can be applied on a material before fracture. At this point, neck formation starts and excessive deformation takes place which leads to fracture of material.

For brittle materials such as cast iron, stone, high carbon steel, glass, etc., yield point and elastic limit can't be determined experimentally. For these ~~meta~~ materials, rupture occurs without any noticiable prior change in rate of elongation.



stress-strain diagram of brittle material

Breaking point

The stress at which finally the specimen fails is called breaking point.

Poisson's Ratio (ν - nue)

- When an axial load is applied on a member, axial length increases and transverse dimension decreases.

Linear strain (axial strain) (ε) = $\frac{\partial l}{l}$
longitudinal

- Change in lateral dimension with respect to corresponding original dimension is called lateral strain.

Lateral strain (ε_{lat}) = $\frac{\partial b}{b}$ or $\frac{\partial t}{t}$

- For elastic material, within elastic limit, lateral strain is directly proportional to longitudinal (axial or normal) strain.

$\varepsilon_{lat} \propto \varepsilon$

$\varepsilon_{lat} = \nu \varepsilon$

SOM

ε_{lat} $\nu = \frac{\varepsilon_{lat}}{\varepsilon} = \frac{\text{lateral strain}}{\text{linear strain (longitudinal strain)}} = \frac{1}{m}$

$$\Rightarrow \boxed{\nu = \frac{\varepsilon_{lat}}{\varepsilon} = \frac{1}{m}}$$

ν or $\frac{1}{m}$ is called Poisson's ratio. Its values varies from $\frac{1}{3}$ to $\frac{1}{4}$ and $\ngtr \frac{1}{2}$.

8. A circular rod of diameter 20 mm and 500 mm long is subjected to a tensile force of 45 kN. The modulus of elasticity of steel may be taken as 200 kN/mm². Find stress, strain and elongation of bar due to applied load.

Solⁿ.

diameter of circular rod (d) = 20 mm = 0.02 m

length of " " (l) = 500 mm = 0.5 m

modulus of elasticity of steel (E) = 200 kN/mm² = 2×10^{11} N/m²

Tensile force (F) = 45 kN = 45×10^3 N

We have,

stress (σ) $= \frac{F}{A} = \frac{F}{\frac{\pi d^2}{4}} = \frac{4 \times 45 \times 10^3}{\pi \times (0.02)^2} = 143239448.8$ N/m² $= 0.1432$ kN/mm²

~~strain~~ E = $\frac{\text{stress } (\sigma)}{\text{strain } (\varepsilon)}$

or, $2 \times 10^{11} = \frac{143239448.8}{\text{strain}}$

∴ Strain (ε) = 0.000716

Elongation of bar
strain $= \frac{\partial l}{l}$

$\partial l = \varepsilon \times l = 0.000726 \times 0.5$

$\therefore$ elongation of bar $(\partial l) = 0.00036\,m$

Q. A hollow cylinder 2m long has outside diameter of 50mm and inside diameter of 30mm. If the cylinder is carrying a load of 25kN, find the stress in cylinder. Also find deformation of cylinder, value of modulus of elasticity of material of cylinder is 100 GPa.

Soln

length of cylinder $(l) = 2m$
outside diameter $(D) = 50\,mm = 0.05\,m$
inside diameter $(d) = 30\,mm = 0.03\,m$
load $(F) = 25\,kN = 25 \times 10^3\,N$
stress $(\sigma) = ?$
deformation of cylinder $= ?$
modulus of elasticity of material $(E) = 100\,GPa = 100 \times 10^9\,Pa$

We have,

$\Rightarrow$ stress $(\sigma) = \frac{F}{A} = \frac{F}{\frac{\pi(D^2-d^2)}{4}} = \frac{4 \times 25 \times 10^3}{\pi(0.05^2 - 0.03^2)}$

$= 19894367.89\,N/m^2$

$= 0.0188\,kN/mm^2$

$\Rightarrow$ deformation of cylinder is

$E = \frac{\sigma}{\varepsilon} \Rightarrow \varepsilon = \frac{\sigma}{E} = \frac{19894367.89}{100 \times 10^9} = 0.000198$

Now,

$\varepsilon\,(\text{strain}) = \frac{\partial l}{l}$

$0.000198 = \frac{\partial l}{2}$

$\therefore$ deformation of cylinder $(\partial l) = 0.00039\,m$

Q. Two wires one of steel and other of copper are of same length and are subjected to the same tension. If the diameter of cu wire is 2mm, find the diameter of steel wire, if they are elongated by same amount. Take E for steel as 200 GPa and that for Cu as 100 GPa.

Soln.

Let l be the length of steel & cu wire and F be the tension produced in them.

diameter of cu wire $(d_c) = 2\,mm = 0.002\,m$

" " steel wire $(d_s) = ?$

amount of elongation is same.

$E_s = 200\,GPa = 200 \times 10^9\,Pa$

$E_c = 100\,GPa = 100 \times 10^9\,Pa$

stress $(\sigma) = \frac{\text{force}\,(F)}{\text{area}\,(A)}$

$\cancel{E} = \frac{F}{A}$

$E = \frac{\sigma}{\varepsilon} = \frac{F/A}{\partial l/l}$

$\partial l = \frac{Fl}{AE}$

$(\partial l)_s = (\partial l)_c$

$\frac{F_s l_s}{A_s E_s} = \frac{F_c l_c}{A_c E_c}$

$A_c E_c = A_s E_s$

$\frac{\pi d_c^2}{4} \times 100 \times 10^9 = \frac{\pi d_s^2}{4} \times 200 \times 10^9$

$\frac{d_c^2}{2} = d_s^2$

$d_s^2 = \frac{0.002^2}{2} \Rightarrow d_s = 0.0014\,m$

∴ The diameter of steel wire is 0.0014 m //

Q. Determine the change in length, breadth & thickness of [illegible] which is 4m long, 30 mm wide and 20mm thick and is subjected to an axial pull of 30 kN in the direction of its length. Take $E = 2 \times 10^5\ N/mm^2$ and Poisson's ratio $(\nu) = 0.3$

Sol^n:

length of steel bar $(l) = 4m$

breadth " " " $(b) = 30\,mm = 0.03\,m$

thickness " " " $(t) = 20mm = 0.02\,m$

axial force $(F) = 30\,kN = 30 \times 10^3 N$ in dir^n of length

$E = 2 \times 10^5\ N/mm^2 = \cancel{2 \times 10^{11}\ N/m^2}\ \cancel{0.2\ N/m^2}\ 2 \times 10^{11}\ N/m^2$

Poisson's ratio $(\nu) = 0.3$

We have,

$$\text{Stress } (\sigma) = \frac{F}{A} = \frac{F}{b \times t} = \frac{30 \times 10^3}{0.03 \times 0.02} = 5 \times 10^7\ N/m^2$$

$$\text{strain } (\varepsilon) = \frac{\sigma}{E} = \frac{5 \times 10^7}{0.2 \times 10^{11}} = \cancel{0.00025}\ 2.5 \times 10^8$$

$$\varepsilon = \frac{\partial l}{l} \Rightarrow \partial l = \varepsilon \times l = 0.00025 \times 4 = 0.001\ m\ (1\ mm)$$

Then,

$$\text{Poisson's ratio } (\nu) = \frac{\varepsilon_{lat}}{\varepsilon} = \frac{\partial b / b}{0.00025}$$

or, $0.3 \times 0.00025 = \dfrac{\partial b}{0.03}$

∴ $\partial b = 0.$

$$\varepsilon = \frac{\partial l}{l} \Rightarrow \partial l = \varepsilon \times l = 2.5 \times 10^8 \times 4 = 1 \times 10^9\ m$$

Then,

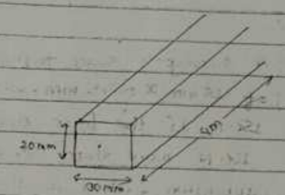
$$\text{Poisson's ratio } (\nu) = \frac{\varepsilon_{lat}}{\varepsilon} = \frac{\partial b / b}{2.5 \times 10^8}$$

$$0.3 \times 2.5 \times 10^8 = \frac{\partial b}{0.03}$$

∴ $\partial b = 2250000\ m$

$$\frac{\partial b}{b} = \frac{\partial t}{t}$$

$$\frac{2250000}{0.0}$$



⇒ change in length

$$E = \frac{\sigma}{\varepsilon} = \frac{F/A}{\partial l / l} = \frac{F \times l}{A\,\partial l}$$

or, $\partial l = \dfrac{F.l}{AE} = \dfrac{30 \times 10^3 \times 4}{(0.03 \times 0.02) \times 2 \times 10^{11}}$

∴ $\partial l = 0.001\ m\ \ (1\ mm)$

$$\nu = \frac{\varepsilon_{lat}}{\varepsilon}$$

linear strain $(\varepsilon) = \dfrac{\partial l}{l} = \dfrac{0.001}{4} = 0.00025$

$$0.3 = \frac{\varepsilon_{lat}}{0.00025}$$

∴ $\varepsilon_{lat} = 0.000075$

⇒ change in breadth

$\partial b = \varepsilon_{lat} \times b = 0.000075 \times 0.03 = 0.00000225$ m

⇒ change in thickness

$\partial t = \varepsilon_{lat} \times t = 0.000075 \times 0.02 = 0.0000015$ m //

Q. A surveyor's steel tape of 30 m long has a cross-section of 15 mm × 0.75 mm. With this, line AB is measured as 150 m. If the force applied during measurement is 100 N more than force applied at the time of calibration, what is the actual length of the line? Take E for steel as 200 kN/mm².

Soln.

length of steel tape (l) = 30 m

cross-section area " " " $= (15 \times 0.75)\ mm^2$

$= 1.125 \times 10^{-5}\ m^2$

force applied during measurement (F) = 100 N more

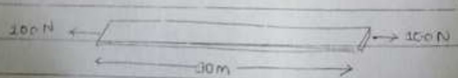
E for steel $= 200\ kN/mm^2$

$= 2 \times 10^{11}\ N/m^2$

actual length of line = ?

∂l $\quad \partial l + 30$

... = 30.00... m



Q. A steel wire 2 m long and 3 mm in diameter is extended by 0.75 mm when a wt. W is suspended from a wire. If the same wt. is suspended from a brass wire, 2.5 m long & 2 mm in diameter, it is elongated by 4.64 mm. Determine the modulus of elasticity of the bar if that of steel is $2 \times 10^5\ N/mm^2$.

Soln.

For steel wire,	For brass wire,
$l = 2$ m	$l = 2.5$ m
$d = 3$ mm = 0.003 m	$d = 2$ mm = 0.002 m
$\Delta l = 0.75$ mm $= 0.75 \times 10^{-3}$ m	$\Delta l = 4.64$ mm $= 4.64 \times 10^{-3}$ m
$E_{steel} = 2 \times 10^5\ N/mm^2$	$E_{brass} = ?$
$= 2 \times 10^{11}\ N/m^2$	

$$E_s = \left(\frac{F\,l}{A\,\Delta l}\right)_s \qquad E_b = \left(\frac{F\,l}{A\,\Delta l}\right)_b$$

$$F_s = \frac{E_s \times A_s \times (\Delta l)_s}{l_s} \quad \text{—— i)}$$

$$F_b = \frac{E_b \times A_b \times (\Delta l)_b}{l_b} \quad \text{—— ii)}$$

Since wt. suspended from both wires is same,

$$F_s = F_b$$

$$\frac{E_s \times A_s\,(\Delta l)_s}{l_s} = \frac{E_b \times A_b\,(\Delta l)_b}{l_b}$$

$$\frac{E_s}{E_b} = \left(\frac{\Delta l}{l}\right)_b \times \left(\frac{l}{\Delta l}\right)_s \times \frac{A_b}{A_s}$$

$$= \left(\frac{4.64 \times 10^{-3}}{2.5}\right) \times \left(\frac{2}{0.75 \times 10^{-3}}\right) \times \frac{\frac{\pi (0.002)^2}{4}}{\frac{\pi (0.003)^2}{4}}$$

$$\frac{2 \times 10^{11}}{E_b} = 2.1997$$

$$\frac{2 \times 10^{11}}{2.1997} = E_b$$

$$E_b = 0.909 \times 10^{11}\ N/m^2$$

$$= 0.909 \times 10^{5}\ N/mm^2$$

Hence, the modulus of elasticity of brass bar is $0.909 \times 10^5\ N/mm^2$ //

Q. The following observations were made during the tensile test on a mild steel specimen 40 mm in diameter and 200 mm long.

Elongation with 40 kN load (within limit of proportionality)

$\delta l = 0.0304$ mm

Yield load = 161 kN

Maximum load = 242 kN

Length of specimen at fracture = 249 mm.

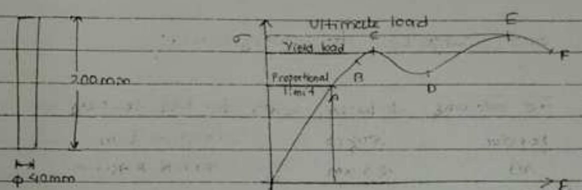
Determine : a) Young's modulus of elasticity

b) Yield point stress

c) Ultimate stress

d) Percentage elongation

Soln.



a) Young's modulus of elasticity

$$E = \frac{\sigma}{\varepsilon} = \frac{F/A}{\Delta l / l} = \frac{40 \times 10^3}{\frac{\pi (40 \times 10^{-3})^2}{4}} \times \frac{200 \times 10^{-3}}{0.0304 \times 10^{-3}}$$

$$= 2.094 \times 10^{11}\ N/m^2$$

$$= 2.094 \times 10^{5}\ N/mm^2$$

b) Yield point stress

$$\sigma_y = \frac{\text{yield load}}{\text{Area}} = \frac{161 \times 10^3}{\frac{\pi (40 \times 10^{-3})^2}{4}} = 1.28 \times 10^{8}\ N/m^2$$

$$= 1.28 \times 10^{2}\ N/mm^2$$

c) Ultimate stress

$$\sigma_u = \frac{\text{ultimate load}}{\text{Area}} = \frac{242 \times 10^3}{\frac{\pi (40 \times 10^{-3})^2}{4}} = 1.92 \times 10^8 \ N/m^2 = 1.92 \times 10^2 \ N/mm^2$$

d) Percentage elongation

$$= \frac{\text{length at fracture} - \text{original length}}{\text{original length}} \times 100\%$$

$$= \frac{249 \times 10^{-3} - 200 \times 10^{-3}}{200 \times 10^{-3}} \times 100\%$$

$$= 0.245 \times 100\%$$

$$= 24.5\% \quad //$$

Bars subjected to varying loads

Q. The following details refer to bar section

portion	length	Cross section
AB	600 mm	40 mm * 40 mm
BC	800 mm	30 mm * 30 mm
CD	1000 mm	20 mm * 20 mm

If the loads $P_1 = 80$ kN, $P_2 = 60$ kN and $P_3 = 40$ kN, find the extension of bar. $E = 2 \times 10^5 \ N/mm^2$.

Soln.

At equilibrium of bar,

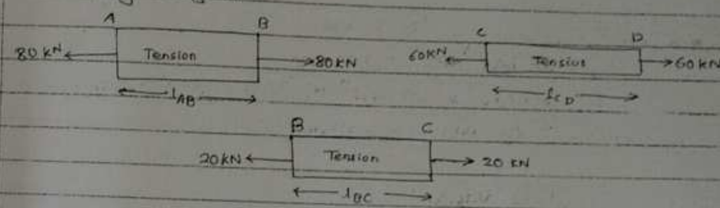
$\Sigma F_x = 0 \quad (\xrightarrow{+ve})$

$-P_1 + P_2 - P_3 + P_4 = 0$

$-80 + 60 - 40 + P_4 = 0$

$P_4 = 60$ kN

Free body diagram



$$\text{Total elongation } (\partial l) = (\partial l)_{AB} + (\partial l)_{BC} + (\partial l)_{CD}$$

$$= \frac{F_{AB} \times l_{AB}}{A_{AB} \cdot E} + \frac{F_{BC} \times l_{BC}}{A_{BC} \cdot E} + \frac{F_{CD} \times l_{CD}}{A_{CD} \cdot E}$$

$$= \frac{80 \times 10^3 \times 600}{(40 \times 40) \cdot 2 \times 10^5} + \frac{20 \times 10^3 \times 800}{(30 \times 30) \cdot 2 \times 10^5} + \frac{60 \times 10^3 \times 1000}{(20 \times 20) \cdot 2 \times 10^5}$$

$$= 0.988 \ mm \quad //$$

Q. A bar of 800mm length is attached rigidly at A and B as shown in fig. Forces of 30 kN and 60 kN act as shown in fig. If E = 200 GPa, determine the reactions at the two ends. If the bar diameter is 25 mm, find the stress and change in length of each portion.

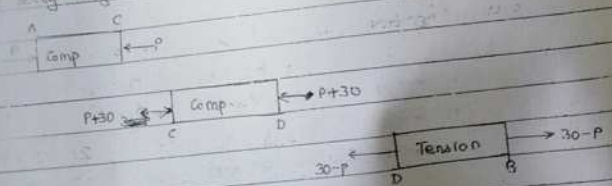
Soln.

Length of bar AB = 800 mm

$E = 200\,GPa = 200\times10^{9}\ N/m^2 = 2\times10^{5}\ N/mm^2$

diameter of bar = 25 mm

Free body diagram



Since Total elongation

$\Delta l = -(\Delta l)_{AC} - (\Delta l)_{CD} + (\Delta l)_{DB}$

$0 = -\dfrac{F_{AC}\, l_{AC}}{A_{AC}\cdot E} - \dfrac{F_{CD}\, l_{CD}}{A_{CD}\cdot E} + \dfrac{F_{DB}\, l_{DB}}{A_{DB}\, E}$ $\quad \boxed{\because A_{AC} = A_{CD} = A_{DB}}$

$-F_{AC} l_{AC} - F_{CD} l_{CD} + F_{DB} l_{DB} = 0$

$-P\times275 - (P+30)\times150 + (30-P)\times375 = 0$

$-275P - 150P - 4500 + 11250 - 375P = 0$

$6750 = 800P$

$\therefore P = 8.44\ kN$

Reaction at end A = P = 8.44 kN

" " " B = 30 - P = 30 - 8.44 = 21.56 kN //

i) For portion AC

$\underset{(\sigma)}{\text{stress}} = \dfrac{F_{AC}}{A} = \dfrac{8.44\times10^{3}}{\frac{\pi(25)^2}{4}} = 17.19\ N/mm^2$ (C)

$\Delta l = \dfrac{F_{AC}\, l_{AC}}{A\cdot E} = \dfrac{8.44\times10^{3}\times275}{\frac{\pi(25)^2}{4}\times 2\times10^{5}} = \cancel{4.12\times10^{-4}\ mm} = 0.024\ mm$ (contraction ~~compression~~)

ii) For portion CD

$\underset{(\sigma)}{\text{stress}} = \dfrac{F_{CD}}{A} = \dfrac{(\cancel{P+30})\,38.44\times10^{3}}{\frac{\pi(25)^2}{4}} = 78.30\ N/mm^2$ (C)

$\Delta l = \dfrac{F_{CD}\, l_{CD}}{A\cdot E} = \dfrac{38.44\times10^{3}\times150}{\frac{\pi(25)^2}{4}\times 2\times10^{5}} = 0.058\ mm$ (contraction ~~compression~~)

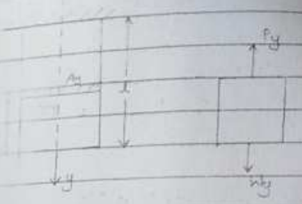
iii) For portion DB

$\underset{(\sigma)}{\text{stress}} = \dfrac{F_{DB}}{A} = \dfrac{21.56\times10^{3}}{\frac{\pi(25)^2}{4}} = 43.92\ N/mm^2$ (T)

$\Delta l = \dfrac{F_{DB}\, l_{DB}}{A\cdot E} = \dfrac{21.56\times10^{3}\times375}{\frac{\pi(25)^2}{4}\times 2\times10^{5}} = 0.082\ mm$ (elongation)

- $F_{AC} = P$
- $F_{CD} = P+30$
- $F_{DB} = 30-P$

Elongation of bar due to self weight



a) Bar of uniform section

Consider a uniform bar of length l, cross-sectional area A, unit weight of material be γ.

From fig, $P_y = W_y$

$\gamma = \frac{W}{V}$

$\Rightarrow W = \gamma V$

$\Rightarrow P_y = W_y = \gamma V_y$

Elongation due to self weight

$$\delta l = \int_0^l \frac{P_y\, dy}{A_y E} = \int_0^l \frac{\gamma V_y\, dy}{A_y E}$$

$$= \int_0^l \frac{\gamma \cancel{A_y}\, y\, dy}{\cancel{A_y}\, E}$$

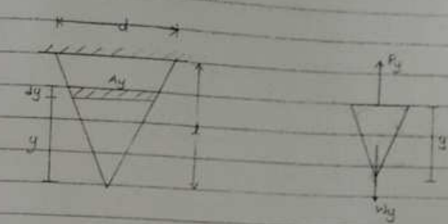
$$= \frac{\gamma}{E}\int_0^l y\, dy$$

$$= \frac{\gamma}{E}\left[\frac{y^2}{2}\right]_0^l$$

$$= \frac{\gamma}{E}\frac{l^2}{2} = \frac{W}{VE}\frac{l^2}{2} = \frac{W}{A l E}\cdot\frac{l^2}{2}$$

$$\boxed{\delta l = \frac{Wl}{2AE}}$$

b) Bar of tappered section



Consider a tappered bar of base diameter 'd', length 'l'. Consider an elemental area A_y at a distance y.

From free body diagram,

$P_y = W_y$

Here,

$$W_y = \gamma V_y = \frac{\gamma A_y \cdot y}{3} \qquad \left[\because V_y = \frac{1}{3}A_y\cdot y \quad \text{Volume of cone} = \frac{1}{3}(\pi r^2)h = \frac{1}{3}Ah\right]$$

Elongation due to self weight

$$\delta l = \int_0^l \frac{P_y\, dy}{A_y E} = \int_0^l \frac{\gamma \cancel{A_y}\cdot y\, dy}{3E\cancel{A_y}} = \frac{\gamma}{3E}\int_0^l y\, dy$$

$$= \frac{\gamma}{3E}\left[\frac{y^2}{2}\right]_0^l$$

$$= \frac{\gamma}{3E}\frac{l^2}{2}$$

$$= \frac{\gamma l^2}{6E}$$

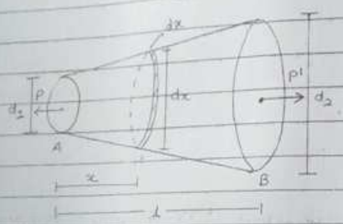
But $\gamma = W/V = \dfrac{W}{\frac{1}{3}\frac{\pi d^2}{4}l} = \dfrac{12W}{\pi d^2 l}$

$$\therefore \boxed{\delta l = \frac{Wl}{2AE}}$$

$$\delta l = \frac{12Wl^2}{\pi d^2 l\, 6E} = \frac{2Wl}{4\frac{\pi d^2}{4}E} = \frac{Wl}{2AE}$$

V. Imp

Deformation of bar with tappering section



Consider a tappered bar of diameter 'd_1' at end A and diameter 'd_2' at end B of total length 'L'. Let 'P' be the force applied at the ends. Consider an elemental strip of thickness 'dx' at a distance 'x' from end A.

Linear interpolation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

at $x_1 = 0$, $y_1 = d_1$

at $x_2 = l$, $y_2 = d_2$

at $x = x$, $y = d_x$

$$d_x - d_1 = \frac{d_2 - d_1}{l - 0}(x - 0)$$

$$d_x = d_1 + \frac{d_2 - d_1}{l}(x - 0)$$

Let $\left(\frac{d_2 - d_1}{l}\right) = k$, $\quad d_x = d_1 + kx$

Elongation on strip due to load

$$\partial l = \frac{P\,dx}{A_x E} \quad \text{—— i)}$$

Total elongation is

$$\partial l = \int_0^l \frac{P\,dx}{A_x E}$$

$$= \int_0^l \frac{P\,dx}{\frac{\pi d_x^2}{4} \cdot E}$$

$$= \int_0^l \frac{4P\,dx}{\pi(d_1 + kx)^2\,E}$$

$$= \frac{4P}{\pi E}\int_0^l \frac{1}{(d_1 + kx)^2}\,dx$$

$$= \cancel{\frac{4P}{\pi E}}\left[\cancel{\frac{1}{(d_1 + kx)}}\right]$$

$$= \frac{4P}{\pi E}\int_0^l (d_1 + kx)^{-2}\,dx$$

$$= \frac{4P}{\pi E}\left[\frac{(d_1 + kx)^{-1}}{-1 \cdot k}\right]_0^l$$

$$= \frac{-4P}{\pi E k}\left[\frac{1}{d_1 + kx}\right]_0^l$$

$$= \frac{-4P}{\pi E k}\left[\frac{1}{d_1 + kl} - \frac{1}{d_1}\right]$$

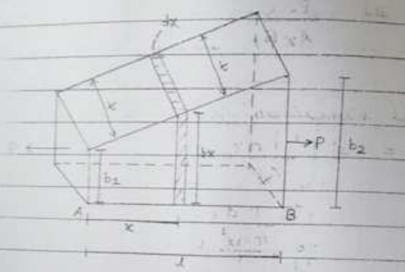
$$= \frac{-4P}{\pi E k}\left[\frac{1}{d_1 + \frac{(d_2 - d_1)}{l} \cdot l} - \frac{1}{d_1}\right]$$

$$= \frac{-4P}{\pi E k}\left[\frac{d_1 - d_2}{d_1 d_2}\right]$$

$$= \frac{-4Pl}{\pi E\,\cancel{(d_2 - d_1)}} \times \frac{-\cancel{(d_2 - d_1)}}{d_1 d_2}$$

$$\therefore \boxed{\partial l = \frac{4PL}{\pi E d_1 d_2}}$$

Bar of trapezoidal section



Consider a trapezoidal bar of thickness 't' having base b_1 at end A and base 'b_2' at end B.

Linear interpolation

at $x_1 = 0$, $y_1 = b_1$

at $x_2 = l$, $y_2 = b_2$

at $x = x$, $y = b_x$

$$b_x - b_1 = \frac{b_2 - b_1}{l - 0}(x - 0)$$

$$b_x = b_1 + \frac{(b_2 - b_1)}{l}.x$$

Let $\frac{b_2 - b_1}{l} = k$, $b_x = b_1 + kx$

Elongation on strip due to load P

$$\delta l = \frac{P\,bx}{A_x\,E} \quad \text{—— i)}$$

Total elongation is

$$\partial l = \int_0^l \frac{P.dx}{A_x.E}$$

$$= \int_0^l \frac{P.dx}{b_x.t\,E}$$

$$= \frac{P}{tE}\int_0^l \frac{1}{(b_1 + kx)}\,dx$$

$$= \frac{P}{tE}\int_0^l (b_1 + kx)^{-1}\,dx$$

$$= \frac{P}{tE}\left[\frac{\log(b_1 + kx)}{k}\right]_0^l$$

$$= \frac{P}{tkE}\left[\log(b_1 + kl) - \log b_1\right]$$

$$= \frac{P}{tkE}\log\left(\frac{b_1 + kl}{b_1}\right)$$

$$= \frac{Pl}{t(b_2 - b_1)E}\log\left(\frac{b_1 + \frac{b_2 - b_1}{l}.l}{b_1}\right) \quad \left[\because k = \frac{b_2 - b_1}{l}\right]$$

$$\therefore \boxed{\partial l = \frac{Pl}{(b_2 - b_1)tE}\log_e\left(\frac{b_2}{b_1}\right)}$$

5. A circular bar 1.75 m long uniformly tappers from 125 mm diameter at one end to 100 mm diameter at other end. If axial load applied is 100 kN, what would be the elongation on bar? Take E = 100 GPa.

Soln.

length of circular bar (L) = 1.75 m

d_1 = 125 mm = 0.125 m

d_2 = 100 mm = 0.1 m

axial load applied (P) = 100 kN = 1×10^5 N

E = 100 GPa = 100×10^9 N/m²

We have,

$$\delta l = \frac{4PL}{\pi E d_1 d_2}$$

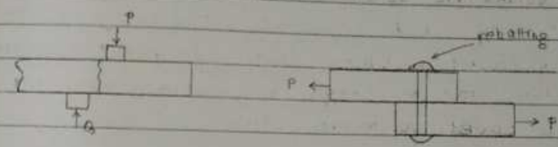
$$= \frac{4 \times 1 \times 10^5 \times 1.75}{\pi \times 100 \times 10^9 \times 0.125 \times 0.1}$$

$$= 1.78 \times 10^{-4} \text{ m}$$

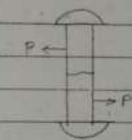
∴ The elongation on bar would be 1.78×10^{-4} m.

Shear stress (τ)

The stresses induced in a body when subjected to two equal and opposite forces which are acting tangentially across the resisting section, as a result of which body tends to shear (कतरना) off across the section is known as shear stress.



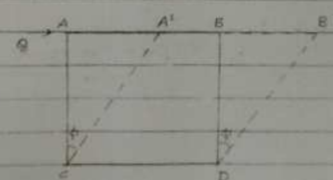
$$\underset{(\tau)}{\text{Shear stress}} = \frac{\text{shear resistance}}{\text{shearing area}}$$



Shearing strain (γ)

Due to application of shear forces, shearing stresses will develop on the body, this will result shear strain and it is measured by the angle through which the body distorts. It is change in angle measured in radian.

Within the elastic limit, shear stress is directly proportional to the strain.



$\tau \propto \gamma$

$\tau = G\gamma$

$$\boxed{G = \frac{\tau}{\gamma}}$$

$$\tan\phi = \frac{BB'}{BD}$$

$$\varepsilon_s = \tan\phi = \gamma$$

G is proportionality constant called shear modulus of elasticity or modulus of rigidity. It is also denoted by C.

Volumetric strain (ϵ_v)

A body, which is subjected to stresses in all directions is called volumetric stress or hydrostatic pressure. The ratio of change in volume per unit original volume is called volumetric strain.

$$\epsilon_v = \frac{\text{change in volume } (\Delta V)}{\text{original volume } (V)}$$

Within the elastic limit, volumetric stress (σ_v) is directly proportional to volumetric strain (ϵ_v)

$P = \rho g h$

i.e. $\sigma_v \propto \epsilon_v$

$\sigma_v = K\epsilon_v$

$$\boxed{K = \frac{\sigma_v}{\epsilon_v}}$$

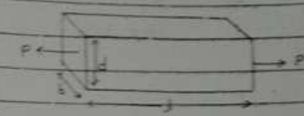
K is proportionality constant called bulk modulus.

Volumetric strain on a body subjected to axial load P

Let l = length of bar

b = breadth of bar

d = depth of bar



P = axial pull applied in the direction of length

Final length = $(l + \partial l)$

Final breadth = $(b - \partial b)$

Final depth = $(d - \partial d)$

where, ∂l, ∂b and ∂d are change in length, breadth & depth respectively.

Original volume = lbd

Final volume $= (l+\partial l)(b-\partial b)(d-\partial d)$

$= (lb - l\partial b + b\partial l - \partial l\partial b)(d - \partial d)$

$= lbd - ld\partial b + bd\partial l - d\partial l\partial b - lb\partial d + l\partial b\partial d - b\partial l\partial d + \partial l\partial b\partial d$

Since ∂l, ∂b & ∂d are very small, their product can be neglected. Then,

Final volume $= lbd - ld\partial b + bd\partial l - lb\partial d$

$\therefore$ change in volume = final vol − initial vol

$= \cancel{lbd} - ld\partial b + bd\partial l - lb\partial d - \cancel{lbd}$

$= bd\partial l - ld\partial b - lb\partial d$

Volumetric strain (ε_v) = $\dfrac{\text{change in volume}}{\text{original vol}}$

$= \dfrac{bd\partial l - ld\partial b - lb\partial d}{lbd}$

$$\varepsilon_v = \frac{\partial l}{l} - \frac{\partial b}{b} - \frac{\partial d}{d} \quad \text{———— (i)}$$

Since $\varepsilon = \dfrac{\partial l}{l}$ & $\varepsilon_{lat} = \dfrac{\partial b}{b}$ or $\dfrac{\partial d}{d}$, above eq^n becomes

$\varepsilon_v = \dfrac{\partial l}{l} - 2\varepsilon_{lat}$

Poisson's ratio

$= \dfrac{\partial l}{l} - 2 \times \dfrac{\varepsilon}{m}$ $\quad \left[\because \nu = \dfrac{1}{m} = \dfrac{\varepsilon_{lat}}{\varepsilon} \right]$

$= \dfrac{\partial l}{l} - \dfrac{2}{m}\dfrac{\partial l}{l}$

$$\boxed{\varepsilon_v = \frac{\partial l}{l}\left(1 - \frac{2}{m}\right)}$$

$\Rightarrow \varepsilon_v = \dfrac{Pl}{AEl}\left(1 - \dfrac{2}{m}\right)$

$\Rightarrow \varepsilon_v = \dfrac{\sigma}{E}\left(1 - \dfrac{2}{m}\right)$

$\Rightarrow \varepsilon_v = \dfrac{\sigma}{E}(1 - 2\nu)$

Q. A steel bar 50 mm x 50 mm in cross-section is 1.2 m long. It is subjected to an axial pull of 200 kN. What is the change in length, width & volume of bar if the value of poison's ratio is 0.3? Take E as 200 GPa.

Sol^n.

cross-sectional area $(A) = (50 \times 50)\,mm^2$

$= 0.0025\ m^2$

length of bar $(l) = 1.2\ m$

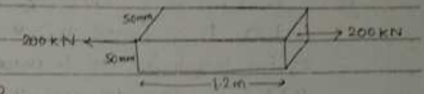
axial force $(F) = 200\ kN = 200 \times 10^3\ N$

$\Delta l, \Delta d, \Delta V = ?$

Poisson's ratio $(\nu) = 0.3$

$E = 200 \times 10^9\ Pa$

$\Rightarrow \partial l = \dfrac{Fl}{AE}$



$\partial l = \dfrac{200 \times 10^3 \times 1.2}{0.0025 \times 200 \times 10^9}$

$\partial l = 0.00048\ m$ (change in length)

$\Rightarrow \nu = \varepsilon_{lat} = \dfrac{\Delta d}{d}$

$\varepsilon \nu = \dfrac{\Delta d}{d}$ $\quad \left[\because \nu = \dfrac{\varepsilon_{lat}}{\varepsilon} \right]$

$\dfrac{\Delta l}{l} \times 0.3 = \dfrac{\Delta d}{d}$

$\dfrac{0.00048}{1.2} \times 0.3 = \dfrac{\Delta d}{0.05}$

$\Delta d = \cancel{0.0000144}\ m$ (change in width)

6×10^{-6}

$\Rightarrow$ Volumetric strain $(\varepsilon_v) = \dfrac{\text{change in volume } (\Delta V)}{\text{original volume } (V)}$

$$\frac{\partial l}{l}(1-2\nu) = \frac{\Delta V}{lbd}$$

$$\frac{0.00048}{1.2}(1-2\times0.3) = \frac{\Delta V}{0.05\times0.05\times1.2}$$

$$0.0016 = \frac{\Delta V}{0.003}$$

$\therefore$ $\Delta V = \cancel{0.0000115\,2\,m^3}$
$= \cancel{1.15\times10^{-5}}\,m^3$ (change in volume)
$= 4.8\times10^{-7}\,m^3$ //

Q. A copper bar 250 mm long and 50 mm * 50 mm in cross-section is subjected to an axial pull in the direction of its length. If the increase in volume of the bar is 37.5 mm^3, find the magnitude of pull. Take m = 4 and E = 100 GPa.

Soln.
length of Cu bar (l) = 250 mm
= 0.25 m
cross-sectional area $(A) = (50\times50)\,mm^2$
$= 0.0025\,m^2$
increase in volume $(\Delta V) = 37.5\,mm^3$
$= 3.75\times10^{-8}\,m^3$
magnitude of axial pull (F) = ?
m = 4 E = 100×10^9 Pa

We have,

$$\varepsilon_V = \frac{F}{AE}\left(1-\frac{2}{m}\right)$$

$V = lbd = 0.25\times0.05\times0.05$
$= 0.000625\,m^3$

or, $$\frac{\Delta V}{V} = \frac{F}{0.0025\times100\times10^9}\left(1-\frac{2}{4}\right)$$

or, $$\frac{3.75\times10^{-8}}{0.000625} = \frac{F}{250000000}\times0.5$$

$\therefore$ F = 30000 N = 30 kN

Hence;
The magnitude of pull is 30 kN //

Thermal stress in simple bars

change in length $\partial l = l\alpha t$
where,
l = original length
α = coefficient of ~~(linear)~~ thermal expansion
t = change in temperature

if temp. increases, bar will elongate otherwise shorten.

$$\text{strain } (\varepsilon) = \frac{\partial l}{l} = \frac{l\alpha t}{l} = \alpha t$$

$$\text{stress } (\sigma) = E\varepsilon = \alpha t E$$

if supports yield by 'Δ' amount, then actual expansion will be $\Delta l = \partial l - \Delta = l\alpha t - \Delta$

$$\text{strain } (\varepsilon) = \frac{\Delta l}{l} = \left(\frac{l\alpha t - \Delta}{l}\right) = \left(\alpha t - \Delta/l\right)$$

$\therefore$ stress $(\sigma) = \delta E = \left(\alpha t - \frac{\Delta}{l}\right) E$

Q. The two parallel walls 6m apart are stayed together by a steel rod 25mm diameter passing through metal pieces and nuts at each end. The nuts are tightened, when the rod is at a temp. of 100°C. Determine stress in rod if temp. falls down to 60°C if

a) end do not yield

b) the end yield by 1mm

Take $\alpha = 12 \times 10^{-6}/°C$

$E = 200\,GPa$

Soln.

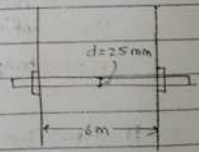
Separation of 2 parallel walls $(l) = 6\,m$

diameter of steel rod $(d) = 25\,mm = 0.025\,m$

Stress in rod $(\sigma) = ?$ when temp. falls to 60°C.

change in temp $(t) = 100°C - 60°C = 40°C$

$E = 200 \times 10^9\,Pa$



a) When end do not yield,

$$\text{stress}(\sigma) = \alpha t E$$

$$= 12 \times 10^{-6} \times 40 \times 200 \times 10^9$$

$$= 96000000\ N/m^2 \quad (96\ MPa)$$

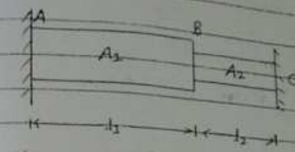
b) When end yield by $\Delta = 1mm = 1 \times 10^{-3}\,m$

$$\text{stress}(\sigma) = \left(\alpha t - \frac{\Delta}{l}\right) E$$

$$= \left(12 \times 10^{-6} \times 40 - \frac{1 \times 10^{-3}}{6}\right) \times 200 \times 10^9$$

$$= 62666666.67\ N/m^2 \quad (62.6\ MPa)$$

Thermal stress in a bar of various sections



Thermal load is shared equally by both portion of bars.

$$\sigma_1 A_1 = \sigma_2 A_2$$

$$\partial l = \partial l_1 + \partial l_2 = \frac{P_1 l_1}{A_1 E_1} + \frac{P_2 l_2}{A_2 E_2}$$

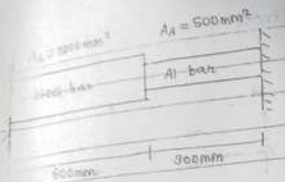
$$\therefore \boxed{\partial l = \frac{\sigma_1 l_1}{E_1} + \frac{\sigma_2 l_2}{E_2}}$$

Q. A composite bar made up of aluminium & steel is held betn 2 supports as shown. The bars are stress free at temp. of 38°C. What will be the stress in 2 bars, when the temp. is 21°C? If

a) the supports are unyielding

b) the supports come nearer to each other by 0.01 mm

It can be assumed that the change of temp. is uniform along the length of bar. Take E for steel as 200GPa, and E for aluminium as 75 GPa, coeff of expansion of steel as $11.7 \times 10^{-6}/°C$ and coeff. of expansion for Al as $23.4 \times 10^{-6}/°C$.



Soln:

$E_s = 200$ GPa

$E_A = 75$ GPa

For steel

$l_s = 0.6$ m

$\alpha_s = 11.7 \times 10^{-6}/°C$

$E_s = 200 \times 10^9$ Pa

$A_s = 0.001$ m^2

For Aluminium

$l_A = 0.3$ m

$\alpha_A = 23.4 \times 10^{-6}/°C$

$E_A = 75 \times 10^9$ Pa

$A_A = 0.0005$ m^2

change in temp $(t) = 38 - 21 = 17°C$

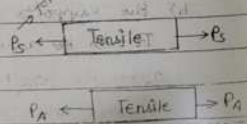
a) When supports are unyielding

Contraction on bars due to temp. fall

$\delta l_s = l_s \alpha_s t$

$= 0.6 \times 11.7 \times 10^{-6} \times 17$

$= 0.000119$ m



$\delta l_A = l_A \alpha_A t$

$= 0.3 \times 23.4 \times 10^{-6} \times 17$

$= 0.000119$ m

Total contraction of bars due to temp fall

$\delta l = \delta l_s + \delta l_A = 0.000239$ m

0.000239 m should elongate due to the tensile force acting on supports. And we know that thermal load is equally shared by both materials.

$$P_s = P_A$$

i.e. $\sigma_s A_s = \sigma_A A_A \Rightarrow 0.001 \sigma_s = 0.0005 \sigma_A$

$2\sigma_s = \sigma_A$ ___ i)

⇒ change in length due to tensile force

$$\delta l = \frac{\sigma_s l_s}{E_s} + \frac{\sigma_A l_A}{E_A}$$

$$0.000239 = \frac{\sigma_s \times 0.6}{200 \times 10^9} + \frac{2\sigma_s \times 0.3}{75 \times 10^9}$$

$0.000239 = 3 \times 10^{-12} \sigma_s + 8 \times 10^{-12} \sigma_s$

$0.000239 = 1.1 \times 10^{-11} \sigma_s$

$\therefore \sigma_s = 21727272.73$ N/m^2 $(21.72$ $N/mm^2)$

$\sigma_A = 2\sigma_s = 43454545.45$ N/m^2 $(43.45$ $N/mm^2)$

b) When supports come nearer to each other by 0.1 mm

Total contraction on bar $= 0.000239 - 0.1 \times 10^{-3}$

$= 0.000139$ m

Tensile force on support should elongate the bars by 0.000139 m

$$\delta l = \frac{\sigma_s l_s}{E_s} + \frac{\sigma_A l_A}{E_A}$$

$$0.000139 = \frac{\sigma_s \times 0.6}{200 \times 10^9} + \frac{2\sigma_s \times 0.3}{75 \times 10^9}$$

$0.000139 = 1.1 \times 10^{-11} \sigma_s$

$\therefore \sigma_s = 12636363.64$ N/m^2 $(12.63$ $N/mm^2)$

$\sigma_A = 2\sigma_s = 25272727.27$ N/m^2 $(25.27$ $N/mm^2)$

Q. At room temp, the gap betn bar A and bar B as shown in figure is 0.25 mm. What are the stresses induced in the bar if temp rise is 35°C? Area of bar A = 1000 mm², Area of bar B = 800 mm², $E_A = 2\times10^5$ N/mm², $E_B = 1\times10^5$ N/mm², $\alpha_A = 12\times10^{-6}/°C$, $\alpha_B = 23\times10^{-6}/°C$

Soln.

Bar A

$A_A = 1000\ mm^2$ $\quad l_A = 0.4\,m$

$= 0.001\ m^2$

$E_A = 2\times10^5\ N/mm^2$

$= 2\times10^{11}\ N/m^2$

$\alpha_A = 12\times10^{-6}/°C$

Bar B

$A_B = 800\ mm^2$ $\quad l_B = 0.3\,m$

$= 0.0008\ m^2$

$E_B = 1\times10^5\ N/mm^2$

$= 1\times10^{11}\ N/m^2$

$\alpha_B = 23\times10^{-6}/°C$

change in temp. (t) = 35°C (temp rise)

⇒ Elongation on bars due to temp. rise

$\partial l_A = l_A\alpha_A t$

$= 0.4\times12\times10^{-6}\times35$

$= 0.000168\ m$

$\partial l_B = l_B\alpha_B t$

$= 0.3\times23\times10^{-6}\times35$

$= 0.0002415\ m$

Total ~~contraction~~ elongation on bars $\partial l = \partial l_A + \partial l_B = 0.00041\ m$

$= 0.41\ mm$

Since there is gap of 0.25 mm betn bar A and bar B, the bars can elongate upto 0.25 mm and the rest $(0.41-0.25) = 0.16$ mm (?) should contract due to compressive force acting on supports.

We know that thermal load is equally shared by both bars.

$F_A = F_B$

$\sigma_A A_A = \sigma_B A_B$

$0.001\,\sigma_A = 0.0008\,\sigma_B$

$1.25\,\sigma_A = \sigma_B$ —— i)

change in length due to compressive force

$\partial l = (\partial l)_A + (\partial l)_B$

$0.16 = \dfrac{\sigma_A l_A}{E_A} + \dfrac{\sigma_B l_B}{E_B}$

$0.16 = \dfrac{\sigma_A\times0.4}{2\times10^{11}} + \dfrac{1.25\sigma_A\times0.3}{1\times10^{11}}$

$0.16 = 5.75\times10^{-12}\,\sigma_A$

$\therefore \sigma_A = 2.78\times10^{10}\ N/m^2$

$= 27826.08\ N/mm^2$

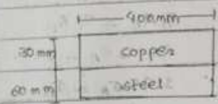
$\sigma_B = 1.25\,\sigma_A = 34782.60\ N/mm^2$ //

[illegible] flat measuring 60 mm × 30 mm is brazed to [illegible] 60 mm × 60 mm mild steel flat as shown in fig. [illegible] the combination is heated through 120°C, [illegible]

[illegible] produced in each bar

[illegible] force which tends to rupture the brazing

[illegible] stress



$\alpha_c = 18.5 \times 10^{-6} /°C$

$\alpha_s = 12 \times 10^{-6} /°C$

$E_c = 110\ GN/m^2$

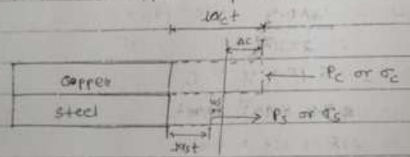
$E_s = 220\ GN/m^2$

$t = 120°C$

$A_c = (60 \times 30)mm^2 = 0.0018\ m^2$

$A_s = (60 \times 60)mm^2 = 0.0036\ m^2$

[illegible] coeff. of linear expansion of copper is greater than that of steel, copper expands more than steel



From fig,

$\Delta_c + \Delta_s = l\alpha_c t - l\alpha_s t$

$\frac{\sigma_c l_c}{E_c} + \frac{\sigma_s l_s}{E_s} = lt(\alpha_c - \alpha_s)$ —— i)

Since thermal load is equally shared by each material

$\sigma_c A_c = \sigma_s A_s \quad [\because F_c = F_s]$

$\sigma_c \times 0.0018 = \sigma_s \times 0.0036$

$0.5\sigma_c = \sigma_s$

$\sigma_c = 2\sigma_s$

Putting value of σ_c in i),

$\frac{2\sigma_s \times 0.4}{110 \times 10^9} + \frac{\sigma_s \times 0.4}{220 \times 10^9} = 0.4 \times 120 (18.5 \times 10^{-6} - 12 \times 10^{-6})$

$9.09 \times 10^{-12} \sigma_s = 0.000312$

$\sigma_s = 34323432.34\ N/m^2$

$= 34.32\ N/mm^2$

$\sigma_c = 2\sigma_s = 68.64\ N/mm^2 \quad (68646864.68\ N/m^2)$

2) $F_c = \sigma_c A_c = \cancel{68.64}\ 68646864.68 \times 0.0018 = 123564.35\ N$

shear force which tends to rupture the brazing is

$F_c = 123564.35\ N$

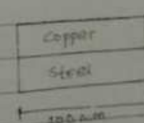
3) shear stress $= \frac{\text{shear force}}{\text{shear area}}$

Final length $= l + l\alpha_s t + \Delta s$ or $l + l\alpha_c t - \Delta c$

$= \frac{123564.35\ N}{(0.4 \times 0.6)\ m^2}$

$= 514851.48\ N/m^2$ //

5. Find the stress in each of the composite bar of length 100mm and hence calculate the final length of composite section if temp. is decreased by 100°C.



$A_c = 60\ mm^2 = 60 \times 10^{-6}\ m^2$

$\alpha_c = 15 \times 10^{-6} /°C$

$E_c = 180\ kN/mm^2$

$= 1.8 \times 10^{11}\ N/m^2$

$A_s = 40\ mm^2 = 40 \times 10^{-6}\ m^2$

$\alpha_s = 12 \times 10^{-6} /°C$

$l = 100\ mm = 0.1\ m$

$E_s = 220\ kN/mm^2$

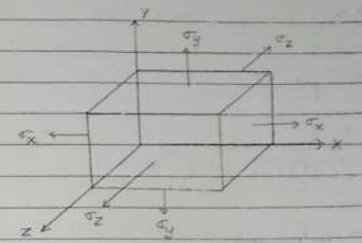
$= 2.2 \times 10^{11}\ N/m^2$

A steel bar is placed between 2 copper bars each having the same area and length as steel bar at 20°c. At this stage, they are tightly connected together at both ends. When the temp. is raised to 320°c, the length of the bar increases by 1.5mm. Determine the original length and final stresses in the bar.

Given: $E_s = 220\ GN/m^2$ $\qquad E_c = 110\ GN/m^2$

$\alpha_s = 0.000012/°c$ $\qquad \alpha_c = 0.0000175/°c$

Relationship between modulus of elasticity (E), Poisson's ratio and Bulk modulus (K)



Consider a block subjected to three mutually-perpendicular stresses σ_x, σ_y & σ_z along x, y and z respectively. Let $\varepsilon_x, \varepsilon_y$ & ε_z be the total strains along x, y and z axes.

Total strain along x-axis

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

Similarly,

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z)$$

Volumetric strain

$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$= \frac{\sigma_x + \sigma_y + \sigma_z}{E}(1 - 2\nu)$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

When the body is subjected to equal stresses from all directions, then $\sigma_x = \sigma_y = \sigma_z = \sigma$

Bulk modulus $= \frac{\sigma}{\varepsilon_v} = K$

$K = \frac{\sigma}{\left(\frac{\sigma+\sigma+\sigma}{E}\right)(1-2\nu)}$

$K = \frac{E}{3(1-2\nu)}$

$\boxed{E = 3K(1-2\nu)}$

Relationship betn modulus of rigidity (G), modulus of elasticity (E) and Poisson's ratio (ν)

$G = \frac{E}{2(1+\nu)}$

Relationship betn modulus of elasticity (E), bulk modulus (K) and Poisson's ratio (ν)

$E = 3K(1-2\nu)$

Relationship betn E, G and K

$E = \frac{9KG}{G+3K}$

Q. A bar of 20 mm dia is tested in tension. It is observed that when a load of 37.7 kN is applied, the extension measured over the gauge length of 200 mm is 0.12 mm and contraction in diameter is 0.0036 mm. Find the Poisson's ratio and elastic constants E, G and K.

Soln.

diameter of bar (d) = 20 mm = 0.02 m

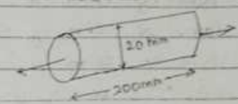
load (F) = 37.7 kN = 37.7×10^3 N

length (l) = 200 mm = 0.2 m

extension (∂l) = 0.12 mm = 0.00012 m

contraction (∂d) = 0.0036 mm = 0.0000036 m

⇒ Poisson's ratio



$(\nu) = \frac{\varepsilon_{lat}}{\varepsilon}$

$= \frac{\partial d/d}{\partial l/l}$

$= \frac{\partial d \times l}{\partial l \times d}$

$= \frac{0.0036 \times 200}{0.12 \times 20}$

$= 0.3$

⇒ Young's modulus of elasticity

$E = \frac{\sigma}{\varepsilon} = \frac{Fl}{A.\partial l} = \frac{37.7 \times 10^3 \times 0.2}{\frac{\pi (0.02)^2}{4} \times 0.00012}$

$= 2 \times 10^{11}$ N/m^2

$\Rightarrow E = 3K(1-2\nu)$

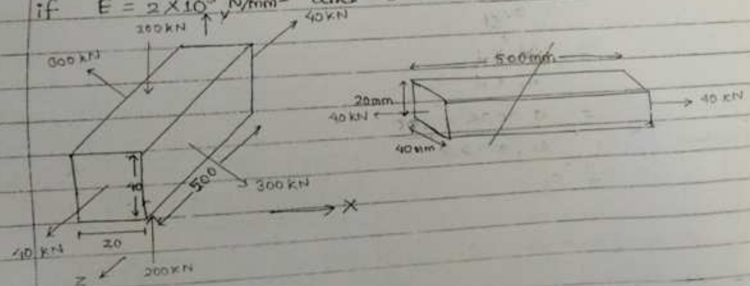
$2\times10^{11} = 3K(1-2\times0.3)$

$2\times10^{11} = 1.2\times K$

$\therefore K = 1.67\times10^{11}\ N/m^2$

$\Rightarrow G = \dfrac{E}{2(1+\nu)} = \dfrac{2\times10^{11}}{2(1+0.3)} = 7.69\times10^{10}\ N/m^2$ //

Q. A 500 mm long bar has rectangular cross-section of 20mm by 40mm. This bar is subjected to i) 40 kN tensile force on 20mm by 40mm face ii) 200 kN compressive force on 20mm by 500mm face. iii) 300 kN tensile force on 40 mm by 500 mm face. Find the change in volume if $E = 2\times10^5\ N/mm^2$ and $\nu = 0.3$.



Solution

Original volume $(V) = (20\times40\times500)\ mm^3 = 4\times10^5\ mm^3$

$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$\varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E}$

$\sigma_x = \dfrac{40\times10^3}{20\times40} = 50\ N/mm^2$ (tensile)

$\sigma_y = \dfrac{200\times10^3}{20\times500} = 20\ N/mm^2$ (compression)

$\sigma_z = \dfrac{300\times10^3}{40\times500} = 15\ N/mm^2$ (tension)

$\varepsilon_x = \dfrac{50}{2\times10^5} + \dfrac{0.3\times20}{2\times10^5} - \dfrac{0.3\times15}{2\times10^5}$ ($\sigma_y = -20\ N/mm^2$)

$= 0.0002575$

$\varepsilon_y = \dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_z}{E}$

$= \dfrac{-20}{2\times10^5} - \dfrac{0.3\times50}{2\times10^5} - \dfrac{0.3\times15}{2\times10^5}$

$= -0.0001975$ mm

$\varepsilon_z = \dfrac{\sigma_z}{E} - \nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E}$

$= \dfrac{15}{2\times10^5} - \dfrac{0.3\times50}{2\times10^5} + \dfrac{0.3\times20}{2\times10^5}$

$= 0.00003$ mm

$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = 0.00009$ mm

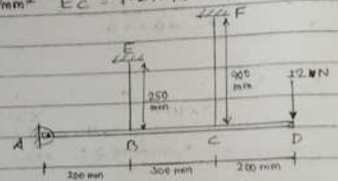
$\Rightarrow$ Volumetric strain $(\varepsilon_v) = \dfrac{\delta V}{V}$

$0.00009 = \dfrac{\delta V}{4\times10^5}$

$\therefore \delta V = 36\ mm^3$ is the required change in volume //

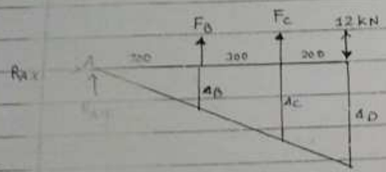
Q. The bar ABCD is very rigid. It has pin-jointed support at A and is supported by a steel wire BE and copper wire CF as shown. Find the stresses produced in steel and copper wires when a load of 12 kN acts at free end. Take $A_s = 400\ mm^2$ $A_c = 600 mm^2$

$E_s = 2\times10^5\ N/mm^2$ $E_c = 1.2\times10^5\ N/mm^2$



Solution

Deflected shape :



Taking moment about A (+ve)

$-F_B\times 200 - F_C\times 500 + 12\times 700 = 0$

$-2F_B - 5F_C + 84 = 0$

$2F_B + 5F_C = 84$ —— i)

~~From deflected shape,~~

$\cancel{\frac{\Delta D}{700} = \frac{\Delta C}{500}}$ $\cancel{\frac{\Delta D}{700} = \frac{\Delta B}{200}}$

$\Delta B = \frac{F_B\, l_{BE}}{A_S\times E_S} = \frac{F_B\times 250}{400\times 2\times10^5} = 3.125\times10^{-6}\, F_B$

$\Delta C = \frac{F_C\, l_{CF}}{A_C\times E_C} = \frac{F_C\times 400}{600\times 1.2\times10^5} = 5.555\times10^{-6}\, F_C$

From deflected shape,

$\frac{\Delta B}{200} = \frac{\Delta C}{500}$

$\frac{3.125\times10^{-6}\, F_B}{200} = \frac{5.555\times10^{-6}\, F_C}{500}$

$F_B = 0.7111\, F_C$

$F_B - 0.7111\, F_C = 0$ —— ii)

Solving i) & ii),

$F_B = 9.30\ N$

$F_C = 13.07\ N$

Stress produced in steel wire $(\sigma_B) = \frac{F_B}{A_S} = \frac{9.30}{400}$

$= 0.02325\ N/mm^2$

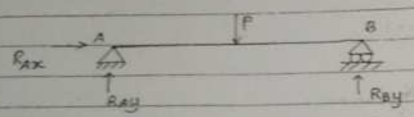
Stress produced in copper wire $(\sigma_C) = \frac{F_C}{A_C} = \frac{13.07}{600}$

$= 0.02178\ N/mm^2$ //

Chapter 4 : Analysis of Beam & Frame

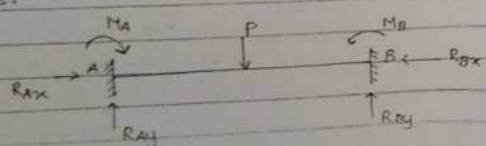
Determinate & Indeterminate structures

⇒ If the structure whose unknown reaction components can be solved by given conditions of static equilibrium, then such structure is known as determinate structure.



no. of unknowns = 3
Eqn. of static eqlm = 3 (i.e. $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$)
Hence, the above simply supported beam is determinate.

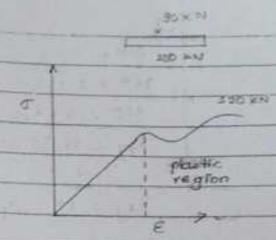
⇒ If the unknown reaction components on a structure cannot be solved by given equations of static equilibrium, such structure is known as indeterminate structure.



no. of unknowns = 6
Eqn. of static eqlm = 3
Degree of static indeterminacy = 6 − 3 = 3

Hence, the above fixed beam consists of 3 degree of static indeterminacy.

Linear & Non-linear structures

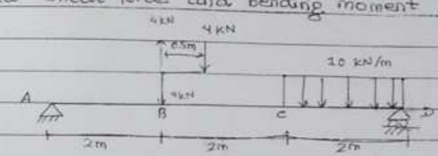


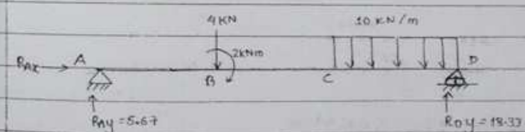
Linear structural analysis
- more cost
- more safety

Non-linear structural analysis
- used nowadays
- low cost
- less safety [from engineering point of view]

Q. Draw shear force and bending moment diagram.

1)





Using eqn of static eqlm,

$\Sigma F_x = 0$ ($\xrightarrow{+ve}$)

$R_{Ax} = 0$

$\Sigma F_y = 0$ (↑+ve)

$R_{Ay} + R_{Dy} - 4 - (10 \times 2) = 0$

$R_{Ay} + R_{Dy} = 24$ ____ 1)

$\Sigma M_A = 0$ (↻ +ve)

$4 \times 2 + 2 + 10 \times 2 \times 5 - R_{Dy} \times 6 = 0$

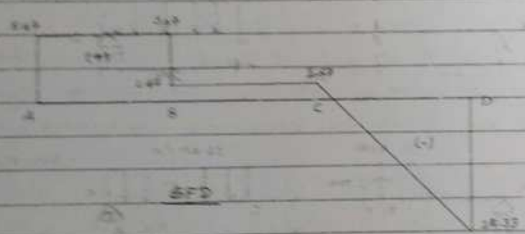
$110 = 6 \times R_{Dy}$

$R_{Dy} = 18.33$ kN

$R_{Ay} = 24 - R_{Dy} = 5.67$ kN

Shear force calculation

	SFD	BMD
	st. line (const)	oblique st. line
	oblique st. line	square parabola
	square parabola	cubic parabola



SFD

Bending moment calculation

portion AB

$BM_x = 5.67 \times x$

At $x=0$, $BM_A = 0$

At $x = 2m$, $BM_B = 11.34$ kNm

portion BC

$BM_x = 5.67(2+x) + 2 - 4 \times x$

At $x=0$, $BM_B = 13.34$ kNm

At $x = 2m$, $BM_C = 16.68$ kNm

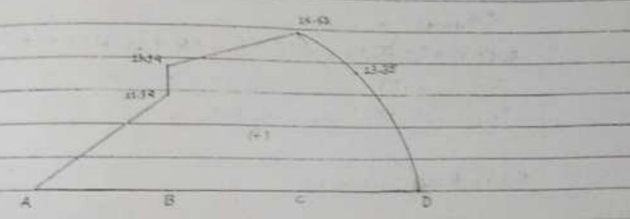
portion CD

$BM_x = 5.67(4+x) + 2 - 4(2+x) - 10 \times x \times \frac{x}{2}$

At $x=0$, $BM_C = 16.68$ kNm

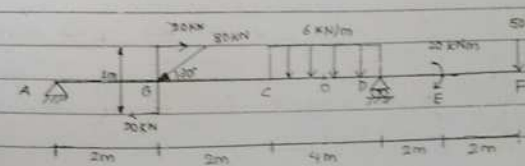
At $x=1$, BM = 13.35 kNm

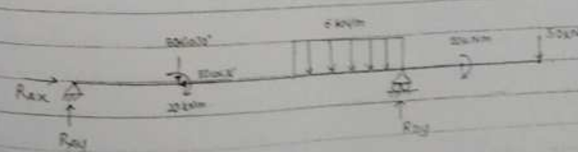
At $x = 2m$, $BM_D = 0.02$ kNm ≈ 0



BMD

2)





Using eqⁿ of eqlᵐ,

$\Sigma F_x = 0$ (+ve →)

$R_{Ax} - 80\cos 30^\circ = 0$

$R_{Ax} = 69.28$ KN

$\Sigma F_y = 0$ (↑+ve)

$R_{Ay} + R_{Dy} - 80\sin 30^\circ - 6\times 4 - 50 = 0$

$R_{Ay} + R_{Dy} = 114$ ______ i)

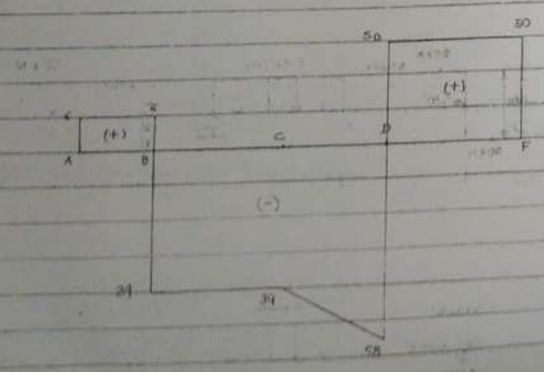
$\Sigma M_{about A} = 0$ (↻+ve)

$80\sin 30^\circ \times 2 + 20 + (6\times 4)\times 6 - R_{Dy}\times 8 + 20 + 50\times 12 = 0$

$864 = R_{Dy}\times 8$

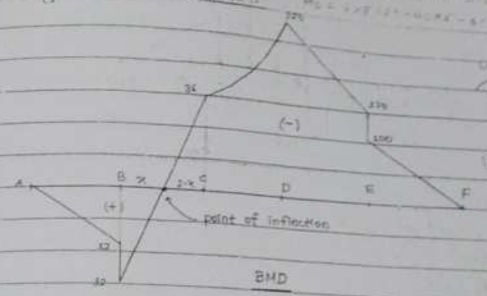
$R_{Dy} = 108$ KN $\qquad R_{Ay} = 6$ KN

Shear force calculation



SFD

Bending moment calculation



BMD

Point of contraflexure / inflection

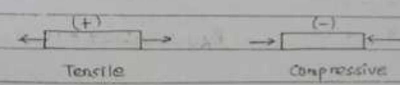
$\frac{\cancel{32}\,8}{x} = \frac{\cancel{36}\,9}{2-x}$

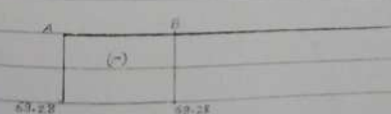
$16 - 8x = 9x$

~~$x = 16$ m~~ $\quad 16 = 17x$

$\therefore x = 0.94$ m

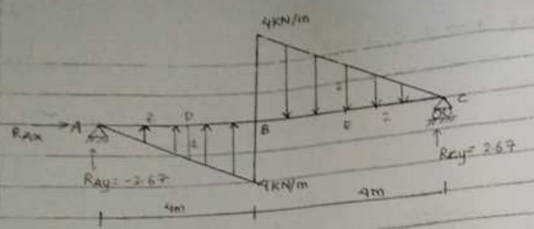
Axial force calculation





AFD

3)



Using eqn. of eqlm,

$\frac{4}{4} = \frac{x_0}{x}$

$\Sigma F_x = 0 \; (\xrightarrow{+ve})$

$x = \omega$

$R_{Ax} = 0$

$\Sigma F_y = 0 \; (\uparrow +ve)$

$R_{Ay} + R_{cy} + \frac{1}{2} \times 4 \times 4 - \frac{1}{2} \times 4 \times 4 = 0$

$R_{Ay} + R_{cy} = 0$ ——— i)

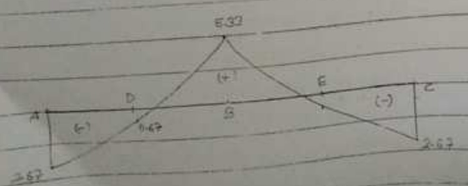
$\Sigma M_{about\ A} = 0$ (↺ve)

$-R_{cy} \times 8 - \frac{4 \times 4}{2} (8/3) + \frac{4}{2} \times 4 \, (4 + 4/3) = 0$

$-R_{cy} \times 8 = -21.33$

$R_{cy} = 2.67$ KN $\qquad R_{Ay} = -2.67$ KN

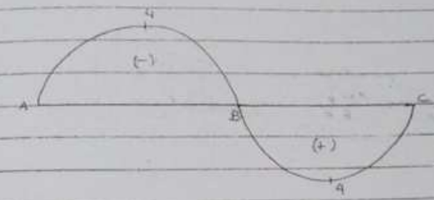
Shear force calculation



$Sf_D = -2.67 + \frac{1}{2} \times 2 \times 2 = -0.67$ $\qquad Sf_E = -2.67 + \frac{1}{2} \times 2 \times 2 = -0.67$

$Sf_B = -2.67 + \frac{1}{2} 4 \times 4 = 5.33$ $\qquad Sf_B = -2.67 + \frac{1}{2} \times 4 \times 4 = 5.33$

Bending moment calculation



portion AB

L-R $\quad BM_x = -2.67 \times x + \frac{1}{2} \times x \times x \, (x/3)$

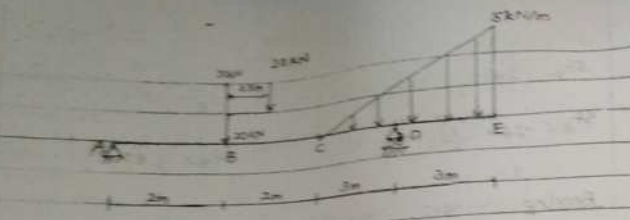
$BM_D = -4.004$ kNm

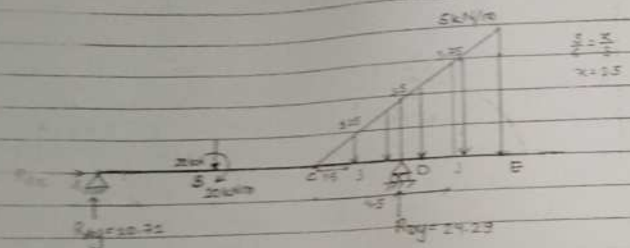
$BM_B = -0.018$ kNm ≈ 0

R-L $\quad$ portion BC

$BM_E = 2.67 \times 2 + -\frac{1}{2} \times 2 \times 2 \times 2/3 = 4.0$ kNm

$BM_B = 2.67 \times 4 - \frac{1}{2} \times 4 \times 4 \times 4/3 = 0.01$ kNm ≈ 0





using eqⁿ of eqlᵐ,

$\Sigma F_x = 0$ ($\xrightarrow{+ve}$)

$R_{Ax} = 0$

$\Sigma F_y = 0$ ($\uparrow +ve$)

$R_{Ay} + R_{Dy} - 20 - \frac{1}{2} \times 6 \times 5 = 0$

$R_{Ay} + R_{Dy} = 35$ ——— i)

$\Sigma M_{about\,A} = 0$

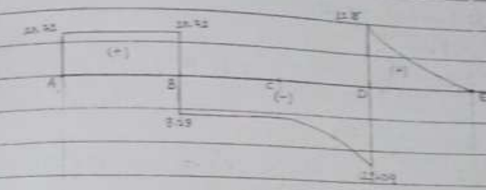
$20 \times 2 + 10 - R_{Dy} \times 7 + \frac{1}{2} \times \overset{3}{\cancel{6}} \times 5 \left(4 + \frac{2}{\cancel{3}} \times \overset{2}{\cancel{6}}\right) = 0$

$40 + 10 + 120 = R_{Dy} \times 7$

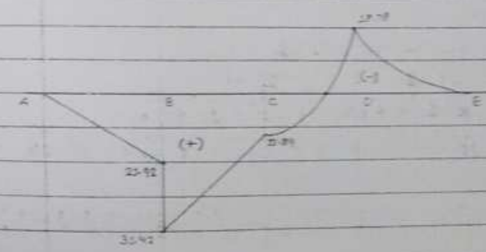
$170 = R_{Dy} \times 7$

$\therefore R_{Dy} = 24.28$ kN $\quad R_{Ay} = 10.71$ kN

Shear force calculation



SFD



BMD

Shear force calculation

$Sf_{bet^n D \& F} = 11.25 - \left(\frac{1}{2} \times 6 \times 5 - \frac{1}{2} \times 4.5 \times 3.75\right)$

$= 4.68$

$\frac{11.25}{2} = 5.62$

$Sf_{LA} = 0$

$Sf_{RA} = 10.71$ kN

$Sf_{LB} = 10.71$ kN

$Sf_{RB} = 10.71 - 20 = -9.29$ kN

$Sf_{C} = -9.29$ kN

$Sf_{LD} = -9.29 - \frac{1}{2} \times 3 \times 2.5 = -13.04$ kN

$Sf_{RD} = -13.04 + 24.29 = 11.25$ kN

$Sf_{E} = 11.25 - \left(\frac{1}{2} \times 6 \times 5 - \frac{1}{2} \times 3 \times 2.5\right) = 11.25 - 11.25 = 0$

$\frac{-9.29 - 13.04}{2} = -11.$

$Sf_{bet^n C \& D} = -9.29 - \frac{1}{2} \times 1.5 \times \frac{5}{4} = -10.22$

Bending moment calculation

$Bm_A = 0$

$Bm_{LB} = 10.71 \times 2 = 21.42$ kNm

$Bm_{RB} = 21.42 + 10 = 31.42$ kNm

$Bm_C = 10.71 \times 4 + 10 - 20 \times 2 = 12.84$ kNm

$Bm_D = 10.71 \times 7 + 10 - 20 \times 5 - \frac{1}{2} \times 3 \times 2.5 \left(\frac{1}{3} \times 3\right) = -18.78$

(At $x=3$)

At $x = 1.5$, $Bm = 10.71 \times 5.5 + 10 - 20 \times 3.5 - \frac{1}{2} \times 1.5 \times \frac{5}{4} \left(\frac{1}{3} \times 1.5\right) = -1.56$

$\frac{12.84 - 18.78}{2} = -2.97$

$Bm_E = 10.71 \times 10 + 10 - 20 \times 8 - \frac{1}{2} \times 6 \times 5 \left(\frac{1}{3} \times 6\right) + 24.29 \times 3 = -0.03 \approx 0$

(At $x=6$)

$x = 4.5$, $Bm = -5.18$

$\frac{-18.78}{2} = -9.39$

portion CD

$Bm_x = 10.71(4+x) + 10 - 20(2+x) - \left(\frac{1}{2} \times x \times \frac{5x}{6}\right) \times \frac{x}{3}$

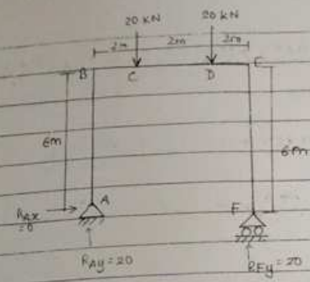
At x, $Bm_x = 0$

$\therefore 0 = 42.84 + 10.71x + 10 - 40 - 20x - \frac{5}{36}x^3$

or, $\frac{-5}{36}x^3 - 9.29x - 12.84 = 0$

Frame

3)



Solution

$\Sigma F_x = 0$ ($\xrightarrow{+ve}$)

$R_{Ax} = 0$

$\Sigma F_y = 0$ ($\uparrow$+ve)

$R_{Ay} + R_{Fy} - 20 - 20 = 0$

$R_{Ay} + R_{Fy} = 40$ ___ i)

$\Sigma M_{about\ A} = 0$ (↻+ve)

$20 \times 2 + 20 \times 4 - R_{Fy} \times 6 = 0$

$120 = R_{Fy} \times 6$

$R_{Fy} = 20$ kN $\qquad R_{Ay} = 20$ kN

Shear force calculation

$Sf_A = R_{Ax} = 0$

$Sf_B = 0$

$Sf_{LB} = 0$

$Sf_{RB} = R_{Ay} = 20$ kN

$Sf_{LC} = 20$ kN

$Sf_{RC} = 20 - 20 = 0$

$Sf_{LD} = 0$

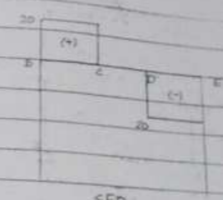
$Sf_{RD} = -20$ kN

$Sf_{LE} = -20$ kN

$Sf_{RE} = -20 + 20 = 0$

$Sf_F = 0$

$Sf_E = 0$



SFD

Bending moment calculation

$BM_A = 0$

$BM_B = -R_{Ax} \times 6 = 0$

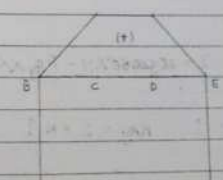
$BM_C = 20 \times 2 = 40$ kNm

$BM_D = 20 \times 4 - 20 \times 2 = 40$ kNm

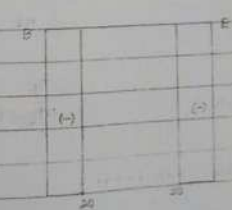
$BM_E = 20 \times 6 - 20 \times 4 - 20 \times 2 = 0$

From F: $BM_F = 0$

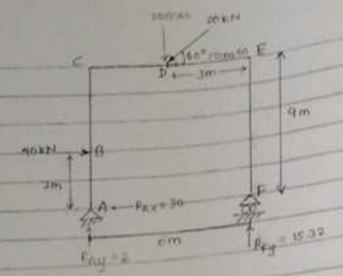
$BM_E = 0$



BMD



7)



Solution

$\Sigma F_x = 0$ ($\xrightarrow{+}$)

$R_{Ax} + 40 - 20\cos 60^\circ = 0$

$R_{Ax} = -30$ kN $\qquad R_{Ax} = 30$ kN $\leftarrow$

$\Sigma F_y = 0$ ($\uparrow +$)

$R_{Ay} + R_{Fy} - 20\sin 60^\circ = 0$

$R_{Ay} + R_{Fy} = 17.32$ ——— ①

$\Sigma M_A = 0$ ($\curvearrowright +$)

$40 \times 2 + 20\sin 60^\circ \times 3 - 20\cos 60^\circ \times 4 - R_{Fy} \times 6 = 0$

$91.96 = R_{Fy} \times 6$

$R_{Fy} = 15.32$ kN $\uparrow \qquad R_{Ay} = 2$ kN $\uparrow$

Shear force calculation

$Sf_A = R_{Ax} = -30$ kN

$Sf_B = 40 - 30 = 10$ kN

$Sf_C = 10$ kN

$Sf_{LC} = \cancel{10\,kN}\ 0$

$Sf_{RC} = R_{Ay} = 2$ kN

$Sf_{LD} = 2$ kN

$Sf_{RD} = 2 - 20\sin 60^\circ = -15.32$ kN

$Sf_{LE} = -15.32$ kN

$Sf_{RE} = -15.32 + R_{Fy} = 0$

$Sf_F = 0$

$Sf_E = 0$

Bending moment calculation

$BM_A = 0$

$BM_B = -R_{Ax} \times 2 = -(-30) \times 2 = 60$ kNm

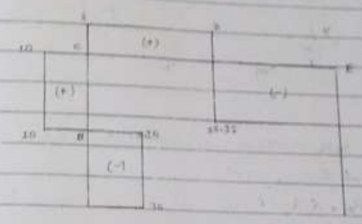
$BM_C = 30 \times 4 - 40 \times 2 = 40$ kNm

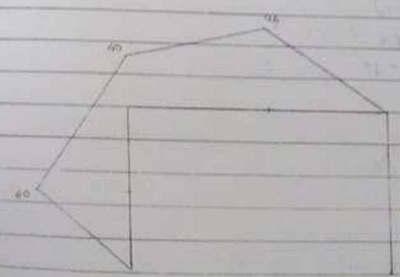
$BM_D = 2 \times 3 + 30 \times 4 - 40 \times 2 = 46$ kNm

$BM_E = 2 \times 6 + 30 \times 4 - 40 \times 2 - 20\sin 60^\circ \times 3 = 0.038$ kNm ≈ 0

From F: $BM_F = 0$

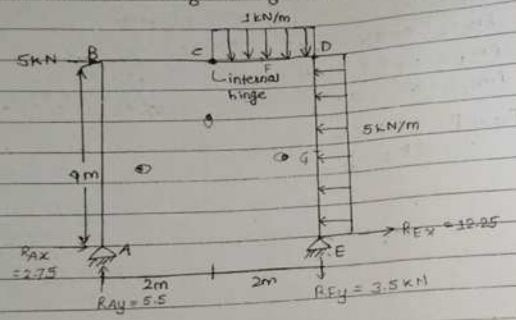
$BM_E = 0$





Q. Draw axial force diagram, bending moment diagram and shear force diagram of the following frame.

1)



Considering left portion of internal hinge and taking moment about C, (↻+ve)

$R_{Ay} \times 2 - R_{Ax} \times 4 = 0$ ——— ①

$\Sigma F_x = 0$ ($\xrightarrow{+}$)

$R_{Ax} + 5 - R_{Ex} - 5 \times 4 = 0$

$R_{Ax} - R_{Ex} = 15$ ——— ②

$\Sigma F_y = 0$ (↑+)

$R_{Ay} - 1 \times 2 + R_{Ey} = 0$

$R_{Ay} + R_{Ey} = 2$ ——— ③

Taking moment about A, (↻+ve)

$5 \times 4 + (1 \times 2) \times 3 - (5 \times 4) \times 2 - R_{Ey} \times 4 = 0$

$-14 = R_{Ey} \times 4$

$R_{Ey} = -3.5$ kN

$R_{Ey} = 3.5$ kN (↓)

From ③,

$R_{Ay} = 2 - R_{Ey} = 2 - (-3.5) = 5.5$ kN (↑)

Solving ① ~~& ②~~,

$2R_{Ay} = 4 \times R_{Ax}$

$2 \times 5.5 = 4 \times R_{Ax}$

$R_{Ax} = 2.75$ kN (→)

From ②,

$R_{Ax} - R_{Ex} = 15$

$2.75 - 15 = R_{Ex}$

$R_{Ex} = -12.25$ kN

$R_{Ex} = 12.25$ kN (→)

Shear force calculation

	L-R	R-L
	↑+	↓+
	↻+	↺+

column AB

$Sf_{DA} = 0$

$Sf_{UA} = -2.75$ kN

$Sf_{DB} = -2.75$ kN

$\boxed{Sf_{UB} = -2.75 - 5 = -7.75 \text{ kN}}$

beam ~~column~~ BD

$Sf_{LB} = 0$

$Sf_{RB} = 5.5$ kN

$Sf_{LC} = 5.5$ kN

$Sf_F = 5.5 - 1 \times 1 = 4.5$ kN

$Sf_{LD} = 5.5 - 1 \times 2 = 3.5$ kN

$Sf_{RD} = 3.5 - 3.5 = 0$ kN

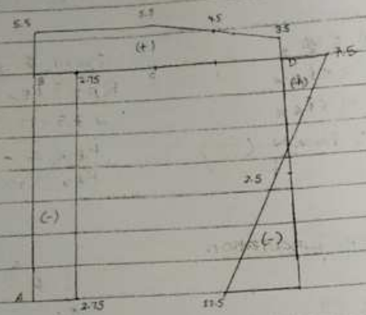
column ~~B~~ED (R→L)

$Sf_{DE} = 0$

$Sf_{UE} = -12.25$ kN

$Sf_G = -12.25 + 5 \times 2 = -2.5$ kN

$Sf_D = -12.25 + 5 \times 4 = 7.5$ kN



SFD

Bending moment calculation

column AB $BM_A = 0$

$BM_B = -2.75 \times 4 =$ ~~10 kNm~~ -11 kNm → $BM_B = -11$

beam BD $BM_C = 5.5 \times 2 - 2.75 \times 4 = 0$

B

Bending moment calculation

column AB

$BM_A = 0$

$BM_B = -2.75 \times 4 = -11$ kNm

beam BD

$BM_B = -11$ kNm

$BM_C = 5.5 \times 2 - 2.75 \times 4 = 0$

$BM_F = 5.5 \times 3 - 2.75 \times 4 - 1 \times 1 \times 0.5 = 5$ kNm

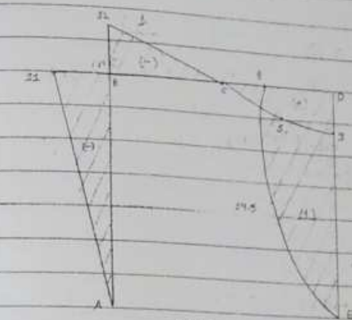
$BM_D = 5.5 \times 4 - 2.75 \times 4 - 1 \times 2 \times 1 = 9$ kNm

column ED (R-L)

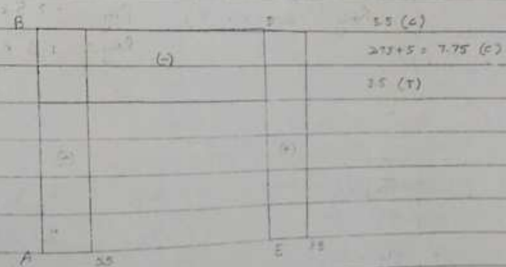
$BM_E = 0$

$BM_G = 12.25 \times 2 - 5 \times 2 \times 1 = 14.5$ kNm

$BM_D = 12.25 \times 4 - 5 \times 4 \times 2 = 8$ kNm

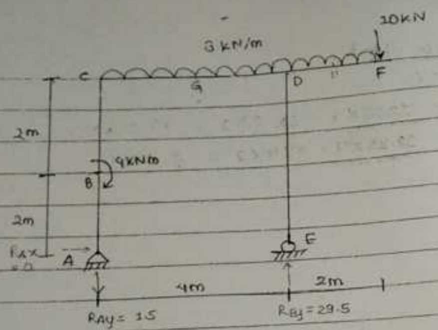


BMD



AFD

2)



Using eq^n of equilibrium,

$\Sigma F_x = 0$ $(\xrightarrow{+})$

$R_{Ax} = 0$

$\Sigma F_y = 0$ $(\uparrow +)$

$R_{Ay} + R_{Ey} - 3 \times 6 - 10 = 0$

$R_{Ay} + R_{Ey} = 28$ ——— (1)

$\Sigma M_A = 0$ $(\circlearrowright +)$

$4 + (3 \times 6) \times 3 + 10 \times 6 - R_{Ey} \times 4 = 0$

$118 = R_{Ey} \times 4$

$R_{Ey} = 29.5$ kN $(\uparrow)$ $\qquad$ $R_{Ay} = -1.5$ kN

$\qquad\qquad\qquad\qquad$ $R_{Ay} = 1.5$ kN $(\downarrow)$

⇒ Shear force calculation

column AC

$Sf_{DA} = 0$

$Sf_{UA} = 0$

$Sf_C = 0$

beam CD

$Sf_{LC} = 0$ $\qquad$ $Sf_{RC} = -1.5$ kN

$Sf_G = -3 \times 2 - 1.5 = -7.5$ kN

$Sf_{LD} = -1.5 - 3 \times 4 = -13.5$ kN

$Sf_{RD} = -13.5 + 29.5 = 16$ kN

~~beam FD~~

~~$Sf_{RF} = 0$~~

~~$Sf_{LF} = 10$ kN~~

~~$Sf_{RD} = 10$ kN~~

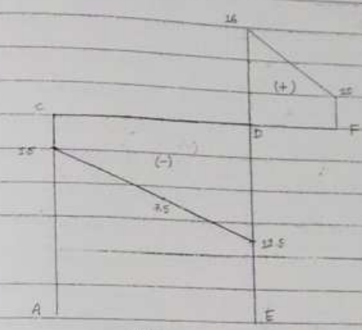
~~$Sf_{LD} = 10 - 29.5 = -19.5$ kN~~

column ED (L→R)

$Sf_{DE} = 0$

$Sf_{UE} = 0$

$Sf_D = 0$



SFD

beam FD (R→L)

$Sf_{RF} = 0$

$Sf_{LF} = 10$ kN

$Sf_{RD} = 10 + 3 \times 2 = 16$ kN

$Sf_{LD} = 16 - 29.5 = -13.5$ kN

⇒ Bending moment calculation

column AC

$BM_A = 0$

$BM_{LB} = 0$

$BM_{RB} = 4$ kNm

$BM_C = 4$ kNm

beam CD

$BM_C = 4$ kNm

$BM_G = -1.5 \times 2 + 4 - 3 \times 2 \times 1 = -5$ kNm

$BM_D = -1.5 \times 4 + 4 - 3 \times 4 \times 2 = -26$ kNm beam FD (R→L)

$BM_F = 0$

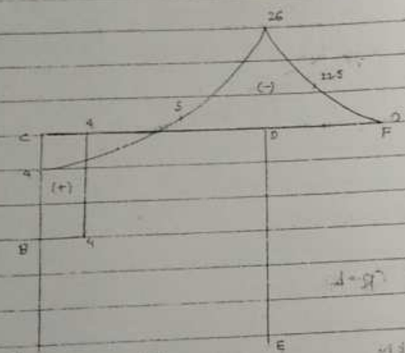
$BM_D = -10 \times 2 - 3 \times 2 \times 1 = -26$ kNm

$BM_H = -10 \times 1 - 3 \times 1 \times 0.5 = -11.5$ kNm

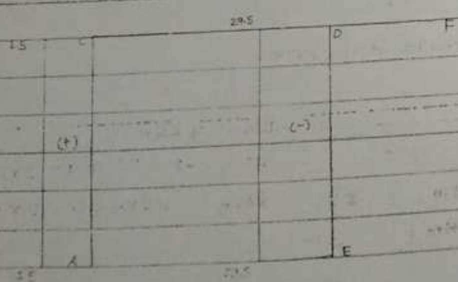
column ED (R→L)

$BM_E = 0$

$BM_D = 0$

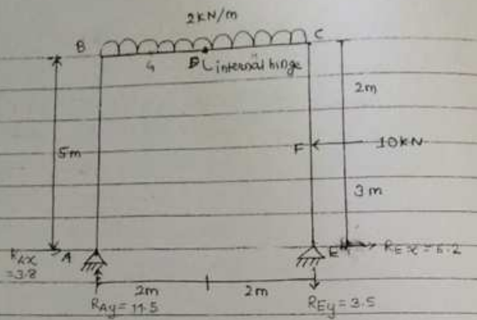


BMD



AFD

3)



Using eqn. of equilibrium,

$\Sigma F_x = 0$ (+→)

$R_{Ax} + R_{Ex} - 10 = 0$

$R_{Ax} + R_{Ex} = 10$ ——— ①

$\Sigma F_y = 0$ (↑+)

$R_{Ay} + R_{Ey} - 2\times4 = 0$

$R_{Ay} + R_{Ey} = 8$ ——— ②

$\Sigma M_A = 0$ (↻+)

$2\times4\times2 - 10\times3 - R_{Ey}\times4 = 0$

$-14 = R_{Ey}\times4$

$R_{Ey} = -3.5$ kN

⇒ $R_{Ey} = 3.5$ kN (↓)

$R_{Ay} - 3.5 = 8$

⇒ $R_{Ay} = 11.5$ kN (↑)

Considering left portion of internal hinge & taking moment about D, (↻+)

$R_{Ay}\times2 - R_{Ax}\times5 - 2\times2\times1 = 0$

$2R_{Ay} - 5R_{Ax} = 4$

$2\times11.5 - 4 = 5R_{Ax}$

⇒ $R_{Ax} = 3.8$ kN (→)

⇒ $R_{Ex} = 10 - 3.8 = 6.2$ kN (→)

⇒ Shear force calculation

column AB

$Sf_{LA} = 0$

$Sf_{RA} = -3.8$ kN

$Sf_B = -3.8$ kN

beam BC

$Sf_{LB} = 0$

$Sf_{RB} = 11.5$ kN

$Sf_D = 11.5 - 2\times2 = 7.5$ kN

$Sf_{LC} = 7.5 - 2\times2 = 3.5$ kN

$Sf_{RC} = 3.5 - 3.5 = 0$

column EC (R-L)

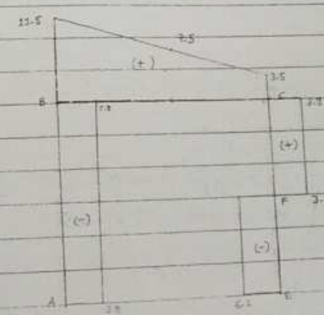
$Sf_{RE} = 0$

$Sf_{LE} = -6.2$ kN

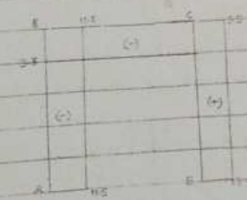
$Sf_{RF} = -6.2$ kN

$Sf_{LF} = -6.2 + 10 = 3.8$ kN

$Sf_C = 3.8$ kN



SFD



AFD

Bending moment calculation

column AB

$BM_A = 0$

$BM_B = -3.8 \times 5 = -19$ kNm

beam BC

$BM_G = -3.8\times5 + 11.5\times1 - 2\times1\times0.5 = -8.5$ kNm

$BM_H = -3.8\times5 + 11.5\times3 - 2\times3\times1.5 = 6.5$ kNm

$BM_B = -19$ kNm

$BM_D = -3.8\times5 + 11.5\times2 - 2\times2\times1 = 0$ (Internal hinge)

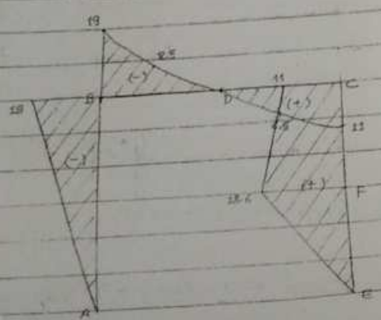
$BM_C = -3.8\times5 + 11.5\times4 - 2\times4\times2 = 11$ kNm

column EC (R→L)

$BM_E = 0$

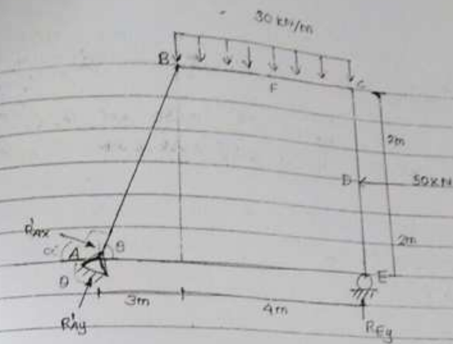
$BM_F = 6.2\times3 = 18.6$ kNm

$BM_C = 6.2\times5 - 10\times2 = 11$ kNm



BMD

4)



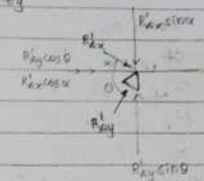
$\tan\theta = \frac{4}{3}$

$\theta = 53.14°$

$\alpha = 180 - 90 - \theta$

$= 90 - \theta$

$= 36.86°$



$\Sigma F_x = 0$ ($\xrightarrow{+}$)

$R'_{Ax}\cos\alpha + R'_{Ay}\cos\theta - 50 = 0$

$R'_{Ax}\cos 36.86° + R'_{Ay}\cos 53.14° = 50$ ——— ①

$\Sigma F_y = 0$ (↑+)

$R'_{Ay}\sin\theta - R'_{Ax}\sin\alpha - 30\times4 + R_{Ey} = 0$

$R'_{Ay}\sin 53.14° - R'_{Ax}\sin 36.86° + R_{Ey} = 120$ ——— ②

Taking moment about A (↻+)

$30\times4\times5 - 50\times2 - R_{Ey}\times7 = 0$

$R_{Ey} = 71.43$ kN

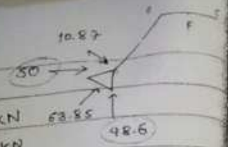
$-R'_{Ax}\sin 36.86° + R'_{Ay}\sin 53.14° = 48.57$ ——— ②'

solving ① & ②',

$R'_{Ax} = 10.87$ kN

$R'_{Ay} = 68.85$ kN

Shear force calculation



ΣF_x: $R'_{Ax}\cos 36.86° + R'_{Ay}\cos 53.14° = 50$ kN

$SF_1 = R'_{Ay}\sin 53.14° - R'_{Ax}\sin 36.86° = 48.6$ kN

column

$SF_{LA} = 0$

$SF_{RA} = -R'_{ax} = -10.87$ kN

$SF_B = -10.87$ kN

beam BC

$SF_{LB} = 0$

$SF_{RB} = 48.6$ kN

$SF_F = 48.6 - 30 \times 2 = -11.4$ kN

$SF_C = 48.6 - 30 \times 4 = -71.4$ kN

$SF_C = -71.43 + 71.43 = 0$

column EC (R→L)

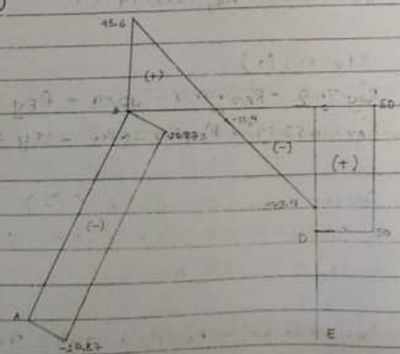
$SF_{RE} = 0$

$SF_{LE} = 0$

$SF_{RD} = 0$

$SF_{LD} = 50$ kN

$SF_C = 50$ kN



SFD

Bending moment calculation

$BM_A = 0$

$BM_B = -50 \times 4 + 48.6 \times 3 = -54.2$ kNm

beam BC

$BM_F = 48.6 \times 5 - 50 \times 4 - 30 \times 2 \times 1 = -17$ kNm $\quad BM_B = -59.2$ kNm

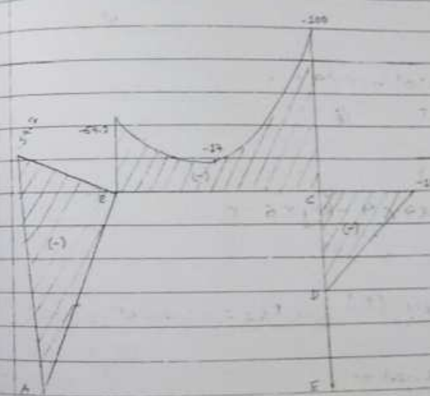
$BM_C = 48.6 \times 7 - 50 \times 4 - 30 \times 4 \times 2 = -99.8$ kNm ≈ -100 kNm

column EC (R→L)

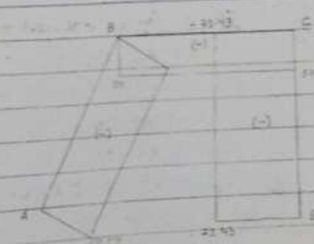
$BM_E = 0$

$BM_D = 0$

$BM_C = -50 \times 2 = -100$ kNm

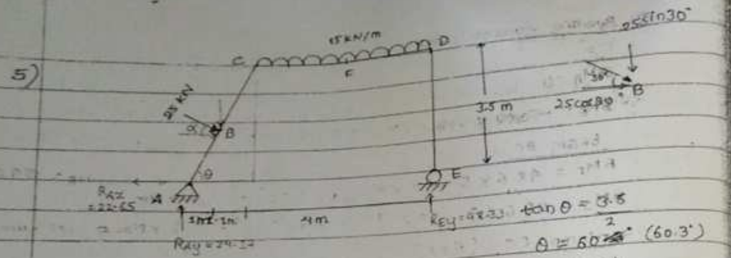


BMD



AFD

5)



$\tan\theta = \frac{3.5}{2}$

$\theta = 60°$ (60.3°)

$\alpha = 90° - \theta$

$= 30°$ (29.7°)

$\Sigma F_x = 0$ $(\xrightarrow{+})$

$R_{Ax} + 25\cos 30° = 0$

$R_{Ax} = -21.65$ kN $R_{Ax} = 21.65$ kN $(\leftarrow)$

AB

$\Sigma F_y = 0$ $(\uparrow +)$

$R_{Ay} + R_{Ey} - 25\sin 30° - 15 \times 4 = 0$

$R_{Ay} + R_{Ey} = 72.5$ ___ (2)

$\Sigma M_A = 0$ $(\curvearrowright +)$

$25 \times \frac{1}{\cos\theta}^{(AB)} + 15 \times 4 \times 4 - R_{Ey} \times 6 = 0$

$290 = 6R_{Ey}$

$R_{Ey} = 48.33$ kN $(\uparrow)$ $R_{Ay} = 24.17$ kN $(\uparrow)$

Shear force calculation

portion AC $Sf_{LA} = 0$

$Sf_{RA} = 21.65\sin 60° + 24.17\cos 60°$

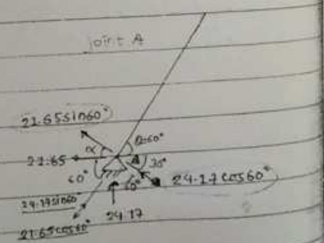
$= 30.83$ kN

$Sf_{LB} = 30.83$ kN

$Sf_{RB} = 30.83 - 25 = 5.83$ kN

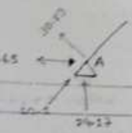
$Sf_C = 5.83$ kN

joint A



$\Sigma F_{\text{along plane}} = 24.17\sin 60° - 21.65\cos 60°$
$= 10.10$ kN

$\Sigma F_{\perp \text{ to plane}} = 21.65\sin 60° + 24.17\cos 60° = 30.8$



beam CD

$Sf_{LC} = 0$

$Sf_{RC} = 24.17 - 25\sin 30° = 11.67$ kN

$Sf_F = \cancel{24.17}\ 11.67 - 15 \times 2 = \cancel{-5.83\text{ kN}}\ -18.33$ kN

$Sf_{LD} = \cancel{24 - 5.83}\ -18.33 - 15 \times 2 = \cancel{-35.83\text{ kN}}\ -48.33$ kN

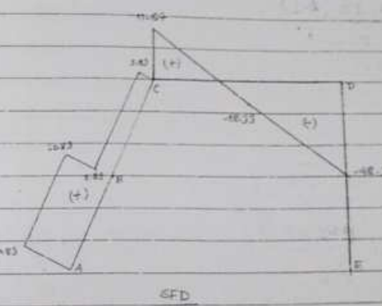
$Sf_{RD} = \cancel{-35.83 + 48.33 = 12.5\text{ kN}}\ -48.33 + 48.33 = 0$ kN

column ED $(R \to L)$

$Sf_{RE} = 0$

$Sf_{LE} = 0$

$Sf_D = 0$



SFD

Bending moment calculation

portion AC

$BM_A = 0$

$BM_B = \cancel{21.65\sin 60° \times}\ 30.83 \times \frac{1}{\cos 60°} = 61.66$ kNm

$BM_C = 30.83 \times \frac{2}{\cos 60°} - 25 \times \frac{1}{\cos 60°} = 73.32$ kNm

beam CD

$BM_C = 24.17 \times 2 + 21.65 \times 3.5 - 25\sin 30° \times 1 = 111.61$ kNm

$BM_F = 24.17 \times 4 + 21.65 \times 3.5 - 25\sin 30° \times 3 - 15 \times 2 \times 1 = 104.95$ kNm

$\cancel{BM_D = 24.17 \times 6 + 21.65 \times 3.5} - 25\sin 30^\circ \times 5 - 15 \times 4 \times 2 = 38.29$ kNm

beam CD

$BM_C = 73.32$ kNm

$BM_F = 24.17 \times 4 + 21.65 \times 3.5 - 25\sin 30^\circ \times 3 - 25\cos 30^\circ \times \frac{3.5}{2} - 15 \times 2 \times 1$

$= 67.06$ kNm (in middle -36.66)

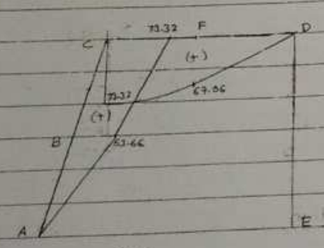
$BM_D = 24.17 \times 6 + 21.65 \times 3.5 - 25\sin 30^\circ \times 5 - 25\cos 30^\circ \times \frac{3.5}{2} - 15 \times 4 \times 2$

$= 0.40 \approx 0$

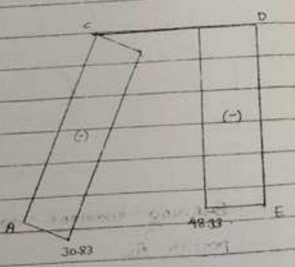
column ED (R-L)

$BM_E = 0$

$BM_D = 0$

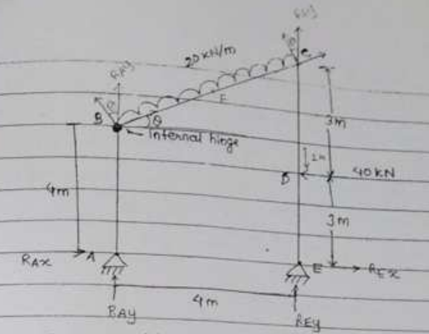


BMD



AFD

6)



Using eqn. of equilibrium,

$\Sigma F_x = 0$ ($\overset{+}{\rightarrow}$)

$R_{Ax} + R_{Ex} = 40$ ——— ①

$\tan\theta = \frac{2}{4} = \frac{1}{2}$

$\theta = 26.6^\circ$

$\Sigma F_y = 0$ ($\uparrow +$)

$R_{Ay} + R_{Ey}$

Consider left portion of internal hinge & taking moment about B,

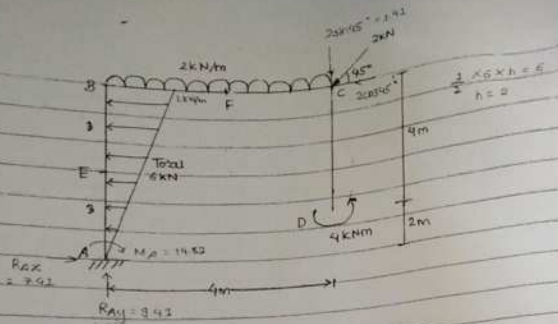
($\circlearrowleft +$) $-R_{Ax} \times 4 = 0$

$R_{Ax} = 0$

From ①, $R_{Ex} = 40$ kN

2011

7)



$\Sigma F_x = 0 \ (\xrightarrow{+})$

$R_{Ax} - 6 - 2\cos 45^\circ = 0$

$R_{Ax} = 7.41 \text{ kN} \ (\rightarrow)$

$\Sigma F_y = 0 \ (\uparrow +)$

$R_{Ay} - 2\sin45^\circ - 2\times4 = 0$

$R_{Ay} = 9.41 \text{ kN} \ (\uparrow)$

Taking moment about A, $(\circlearrowright +)$

$M_A - 6 \times \frac{2}{3} \times 6 + 2\times4\times2 + (2\sin45^\circ \times 4) - (2\cos45^\circ \times 6) - 4 = 0$

$M_A = 14.82 \text{ kNm}$

Shear force calculation — L-R $\uparrow +$ — R-L $\downarrow +$

column AB

$Sf_{LA} = 0$

$Sf_{RA} = -7.41 \text{ kN}$

$Sf_E = -7.41 + \frac{1}{2} \times 3 \times 1 = -5.91 \text{ kN}$

$Sf_B = -7.41 + \frac{1}{2} \times 6 \times 2 = -1.41 \text{ kN}$

beam BC

$Sf_{LB} = 0$

$Sf_{RB} = 9.41 \text{ kN}$

$Sf_F = 9.41 - 2\times2 = 5.41 \text{ kN}$

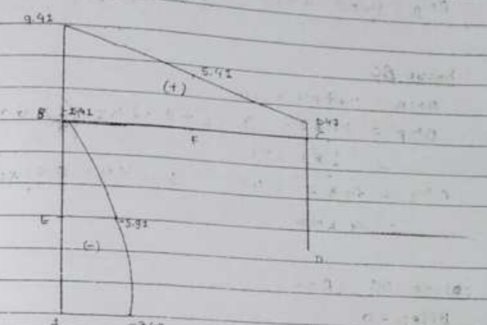
$Sf_{LC} = 9.41 - 2\times4 = 1.41 \text{ kN}$

$Sf_{RC} = 1.41 - 2\sin45^\circ = 0$

column DC (R-L)

$Sf_D = 0$

$Sf_C = 0$



SFD

Bending force calculation

column AB

$BM_{LA} = 0$

$BM_{RA} = 14.82 \text{ kNm}$

$BM_E = 14.82 - 7.41 \times 3 + \frac{1}{2} \times 3 \times 1 = -5.91 \text{ kNm}$

$BM_B = 14.82 - 7.41 \times 6 + \frac{1}{2} \times 6 \times 2 = -23.64 \text{ kNm}$

beam BC

Bending moment calculation

Column AB

$BM_{LA} = 0$ $\quad BM_{RA} = 14.82$ kNm

$BM_{E} = 14.82 - 7.41 \times 3 + \left(\frac{1}{2} \times 3 \times 1\right) \times \frac{1}{3} \times 3 = -5.91$ kNm

$BM_{B} = 14.82 - 7.41 \times 6 + \left(\frac{1}{2} \times 6 \times 2\right) \times \frac{1}{3} \times 6^2 = -17.64$ kNm

beam BC

$BM_B = -17.64$ kNm

$BM_F = 14.82 - 7.41 \times 6 + 9.41 \times 2 + \left(\frac{1}{2} \times 6 \times 2\right) \times \frac{6^2}{3} - 2 \times 2 \times 1$

$= -2.82$ kNm

$BM_C = 14.82 - 7.41 \times 6 + 9.41 \times 4 + 6 \times \frac{1}{3} \times 6^2 - 2 \times 4 \times 2$

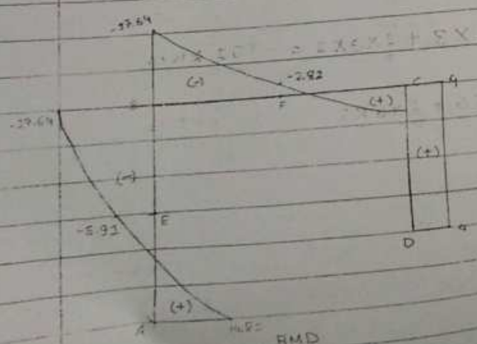
$= 4$ kNm

column DC (R → L)

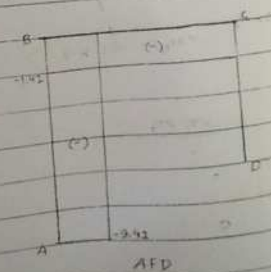
$BM_{RD} = 0$

$BM_{LD} = 4$ kNm

$BM_C = 4$ kNm

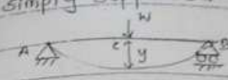


BMD



AFD

1) simply supported

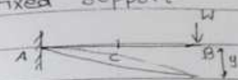


At supports - no deflection occurs

Rotation - ✓ $\quad$ At max deflection occuring position, no slopes (i.e. at C)

Max. B.M → at C

2) Fixed support



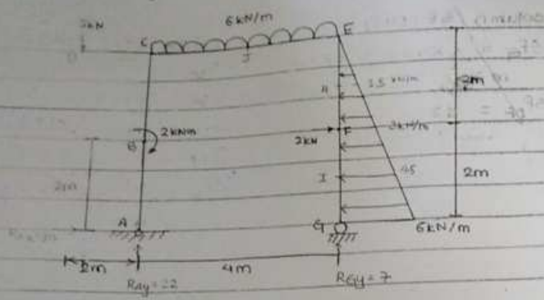
At A, no deflection & no rotation

$y = 0$ $\quad$ slope $\left(\frac{dy}{dx} = 0\right)$

Max B.M → at ~~B~~ A

→ hinge
→ fixed

8)



$\Sigma F_x = 0$ ($\xrightarrow{+}$)

$R_{Ax} + 2 - \frac{1}{2} \times 4 \times 6 = 0$

$R_{Ax} = 10$ kN ($\rightarrow$)

$\Sigma F_y = 0$ ($\uparrow +$)

$R_{Ay} + R_{Gy} - 5 - 6 \times 4 = 0$

$R_{Ay} + R_{Gy} = 29$ ——— (1)

$\Sigma M_A = 0$ ($\curvearrowright$)

$2 - 5 \times 2 + (6 \times 4) \times 2 + 2 \times 2 - \left(\frac{1}{2} \times 4 \times 6\right) \times \frac{1}{3} \times 4 - R_{Gy} \times 4 = 0$

$44 - 16 = 4 R_{Gy}$

$R_{Gy} = 7$ kN ($\uparrow$) $\qquad R_{Ay} = 22$ kN ($\uparrow$)

Shear force calculation

column AC

$Sf_A = 0$

$Sf_{UA} = -10$ kN

$Sf_C = -10$ kN

beam ED (R→L)

$Sf_{RE} = 0$

$Sf_{LE} = -7$ kN

$Sf_{RC} = -7 + (6 \times 4) = 17$ kN

$Sf_{LC} = 17 - 22 = -5$ kN

$Sf_{RD} = +5$ kN

$Sf_{LD} = -5 + 5 = 0$

column GE (R→L)

$Sf_{EG} = \frac{1}{2} \times 4 \times 6 = 12$ kN

$Sf_{DF} = 12$

$Sf_I = 9 - \left(\frac{1}{2} \times \ldots 4.5 - \frac{1}{2} \times \ldots \times 3\right) = 5.25$ (3 m) $\qquad Sf_H = 10 - \left(\frac{1}{2} \times 1 \times 1.5\right) = 9.25$ kN (3.5 m)

column EG (L→R)

$Sf_{UE} = 0$

$Sf_{DE} = 10$ kN

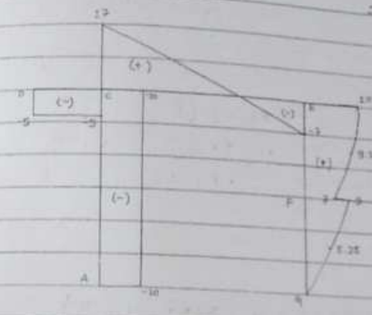
$Sf_{UF} = 10 - \left(\frac{1}{2} \times 2 \times 3\right) = 7$ kN

$Sf_{DF} = 7 + 2 = 9$ kN

$Sf_G = 9 - \left(\frac{1}{2} \times 4 \times 6 - \frac{1}{2} \times 2 \times 3\right)$

$= 9 - 9$

$= 0$



SFD

Bending moment calculation

column AC

$BM_A = 0$

$BM_{DB} = 2 - 10 \times 2 = -20$ kNm $\qquad BM_{UB} = -20 + 2 = -18$ kNm

$BM_C = 2 - 10 \times 4 = -38$ kNm

beam DE

$BM_D = 0$

$BM_C = -5 \times 2 = -10$ kNm

beam CE

$BM_C = 2 - 10 \times 4 - 5 \times 2 = -48$ kNm

$BM_J = 2 - 10 \times 4 + 22 \times 2 - 5 \times 4 - 6 \times 2 \times 1 = -26$

$BM_E = 2 - 10 \times 4 + 22 \times 4 - 5 \times 6 - 6 \times 4 \times 2 = -28$

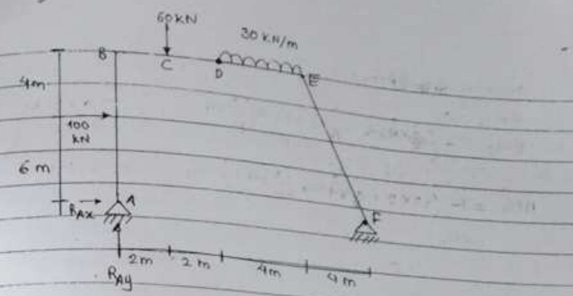
Column GE (R→L)

$BM_G = 0$

$BM_F = -\left(\frac{1}{2}\times\overset{2}{4}\times 6 - \frac{1}{2}\times 2\times 3\right)\times\left(\frac{2}{3}\times 4 - 2\right) = -6\ kN\,m$

$BM_E = -\left(-2\times 2 + \frac{1}{2}\times\overset{2}{4}\times 6\left(\frac{2}{3}\times 4\right)\right) = -28\ kNm$

9)



Chapter 6: Theory of Flexure

The bending moment developed at a section tends to bend or deflect the beam & the internal stress resist its bending. The process of bending stops (i.e. the structure fails) when every cross-section sets-up full resistance to the bending moment. The resistance offered by the internal stress to the bending is called bending stress, and the relevant theory is called theory of simple bending.

Assumptions

1) The material of beam is perfectly homogeneous (i.e. same kind throughout) and isotropic (equal elastic properties in all directions).

2) The beam material is stressed upto elastic limit i.e. it obeys Hooke's law

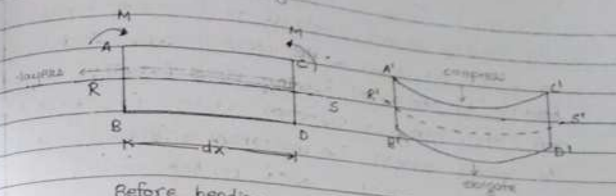
3) The transverse section which were plane before bending remains plane after bending.

4) Each layer of beam is free to expand or contract independently.

5) The value of E (Young's modulus) is same in tension & compression.

6) The beam is in equilibrium i.e. there is no resultant push or pull in beam section.

Theory of simple bending



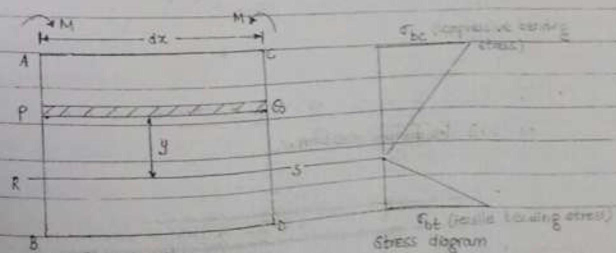
Before bending After bending

Top layer AC : suffered compression
reduced to A'C'

Bottom layer BD : suffered tension
elongates to B'D'

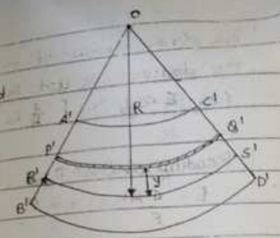
Layers above RS have been compressed and those below RS have been elongated. The amount by which the layer compress or elongates depends upon the position of individual layer w.r.t. RS. The layer RS which is neither compressed nor stretched is called neutral layer or neutral plane.

⇒ Layers near to R'S' are less compressed / elongated
Layers far from R'S' are more compressed / elongated



Before bending

Consider an elemental length of beam 'δx' which is subjected to bending moment 'M'. As a result of applied moment, the beam deflects forming curvature.



After bending

O - centre of curvature
R - radius of neutral layer
RS / R'S' - neutral layer
PQ - element considered at a distance 'y' from RS before bending
P'Q' - element at a distance 'y' from RS after bending

The length PQ is compressed to P'Q' after bending.
Change in length = PQ - P'Q'

$$\text{strain } (\varepsilon) = \frac{\text{change in length}}{\text{original length}} = \frac{PQ - P'Q'}{PQ} \quad \text{--- (1)}$$

From the geometry of curved beam,

$$\frac{P'Q'}{R-y} = \frac{R'S'}{R}$$

$$\frac{P'Q'}{R'S'} = \frac{R-y}{R}$$

$$1 - \frac{P'Q'}{R'S'} = 1 - \frac{R-y}{R}$$

$$\frac{R'S' - P'Q'}{R'S'} = \frac{R-R+y}{R}$$

no bending (neutral layer)

$$\frac{PQ - P'Q'}{PQ} = \frac{y}{R} \quad [\because R'S' = RS = PQ]$$

$$\Rightarrow \boxed{\varepsilon = \frac{y}{R}} \quad [\because \text{from (1)}] \quad \text{--- (2)}$$

The strain produced in any layer is proportional to the distance of that layer from neutral axis.
i.e. $\varepsilon \propto y$ $\left[\because \frac{1}{R} \text{ is constant}\right]$

Bending stress

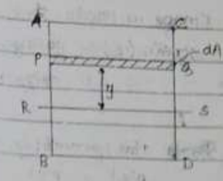
$$E = \frac{\sigma}{\varepsilon}$$

$$\text{stress } (\sigma) = \varepsilon E = \frac{y}{R} \cdot E \quad \left[\because \varepsilon = \frac{y}{R}\right]$$

$$\therefore \boxed{\sigma = \frac{E}{R} \cdot y} \quad \text{or} \quad \boxed{\frac{\sigma}{y} = \frac{E}{R}}$$

Imp

Moment of resistance



Here, one side of neutral layer tends to contract & that of another tends to elongate. Due to this, there is development of compressive and tensile forces. The tensile and compressive stresses thus developed, forms a couple which must be equal to the externally applied moment (M).

The moment due to couple which resist the externally applied moment is called moment of resistance.

Let PQ be the layer at a distance 'y' from neutral layer and elemental area 'dA'.

Bending stress at layer PQ

$\sigma = \frac{E}{R} y \Rightarrow \frac{\sigma}{y} = \frac{E}{R}$ —— ①

Total stress on layer PQ (force) $= \sigma \times dA$

$= \frac{E}{R} xy \times dA$

Moment of this total stress about neutral layer = force × y

$= \frac{E}{R} xy \times dA \times y$

$= \frac{E}{R} y^2 dA$

The algebric sum of moment due to individual layer must be equal to externally applied moment (M)

i.e. $M = \int \frac{E}{R} y^2 dA$

$= \frac{E}{R} \int y^2 dA$

$M = \frac{E}{R} I \Rightarrow \frac{M}{I} = \frac{E}{R}$ —— ②

Here, $I = I_{xx} = \int y^2 dA$ represents moment of inertia about neutral layer.

M = bending m.

From ① & ②,

$$\boxed{\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}}$$

This is known as eqn. of simple bending.

8. Determine the dimension of joist (beam) of a timber for span (length) 8m to carry a brick wall 200 mm thick & 5m high. If the density of brick work is 1850 kg/m³ and the max. permissible stress is limited to 7.5 MN/m². Given that the depth of joist is twice the width.

Soln.

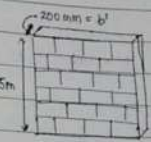
dimension of joist (b, d) = ? $l = 8$ m

density of brick (ρ) = 1850 kg/m³

permissible stress (σ) = 7.5 MN/m²

$= 7.5 \times 10^3$ kN/m²

Condn: $d = 2b$



Assume simply supported beam

wt. due to brick work /unit length (W) $= \gamma \times b'd'$

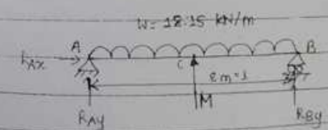
$= 1850 \times 9.81 \times 0.2 \times 5$

$= 18148.5$ N/m

$= 18.15$ kN/m

$I = \frac{bd^3}{12} = \frac{b(2b)^3}{12} = \frac{8b^4}{12} = \frac{2b^4}{3}$

$y = \frac{d}{2} = \frac{2b}{2} = b$



~~(↑+) RAx = 0~~

(→+) $R_{Ax} = 0$ (↑+) $R_{Ay} + R_{By} - 18.15 \times 8 = 0$

$R_{Ay} + R_{By} = 145.2$

$M_{about\ A} = 0$ ($\circlearrowright$+)

$18.15 \times 8 \times 4 - R_{By} \times 8 = 0$

$R_{By} = 72.6$ kN $\qquad R_{Ay} = 72.6$ kN

$BM_C = 72.6 \times 4 - 18.15 \times 4 \times 2 = 145.2$ kNm (max. bending moment)

or

$$BM_{max}\,(M) = \frac{wl^2}{8} \quad \frac{wl^2}{8} = \frac{18.15 \times 8^2}{8} = 145.2 \text{ kNm}$$

From eq^n of simple bending,

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\frac{145.2}{\frac{2}{3}b^4} = \frac{7.5 \times 10^3}{b}$$

$$\frac{435.6}{2b^3} = 7.5 \times 10^3$$

$b = 0.307$ m

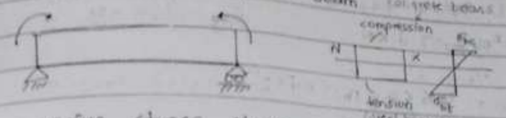
$d = 2b = 2 \times 0.307 = 0.614$ m

Dimensions of joist are breadth $(b) = 0.307$ m

depth $(d) = 0.614$ m //

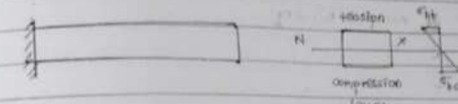
Distribution of bending stress across the section

a) In case of simply supported beam (concrete beams)



Compressive stress above neutral axis, and tensile stress below neutral axis

b) In case of cantilever beam



Tensile stress above neutral axis, and compressive stress below that

Modulus of section (section modulus)

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\Rightarrow M = \frac{\sigma}{y} \times I$$

Stress in fibre is directly proportional to the distance from c.g. If y_{max} is the distance from c.g. to extreme fiber, maximum stress would occur (upmost/lowermost fibre)

$$M = \frac{\sigma_{max}}{y_{max}} \times I = \sigma_{max} \times \frac{I}{y_{max}} = \sigma_{max} \times Z$$

where, $Z = \frac{I}{y_{max}}$ is called section modulus

If object is symmetrical, $y_{max} = d/2$

for rectangular section

$Z = \frac{I}{y} = \frac{bd^3}{12 \times d/2} = \frac{bd^2}{6}$

$\boxed{M = \sigma \times Z = \sigma \frac{I}{y}}$

For circular section

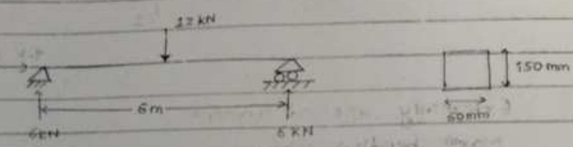
$Z = \frac{\pi d^4}{64 \times d/2} = \frac{\pi d^3}{32}$

8. A rectangular beam 60mm wide & 150 mm deep is simply supported over a span of 6m. If the beam is subjected to central point load of 12kN, find the maximum bending stress induced in the beam.

Soln.

width of beam (b) = 60 mm = 0.06 m

depth of beam (d) = 150 mm = 0.15 m



Maxm bending moment occurs at centre :

$M = 6 \times 3 = 18 \text{ kNm} = 18 \times 10^6 \text{ Nmm}$

$M = \frac{WL}{4} = \frac{12 \times 6}{4} = 18 \text{ kNm}$

$Z = \frac{bd^2}{6} = \frac{60 \times 150^2}{6} = 225000 \text{ mm}^3$

Strength of member $(M) = \sigma_{max} Z$

$\therefore \sigma_{max} = \frac{M}{Z} = \frac{18 \times 10^6}{225000} = 80 \text{ N/mm}^2$ //

MPa = N/mm²

9. A rectangular beam 300 mm deep is simply supported over a span of 4m. What uniformly distributed load the beam may carry, if the bending stress is not to exceed 120 MPa. Take $I = 225 \times 10^6 \text{ mm}^4$

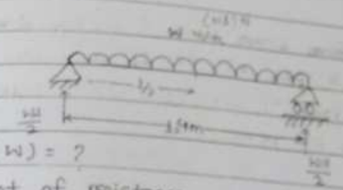
Soln

depth (d) = 300 mm = 0.3 m

length (l) = 4m

maxm bending stress $(\sigma_{max}) = 120 \text{ MPa} = 120 \times 10^6 \text{ N/m}^2 = 120 \text{ N/mm}^2$

$I = 225 \times 10^6 \text{ mm}^4$



UDL (W) = ?

Moment of resistance

$M = \sigma_{max} Z = \sigma_{max} \frac{I}{y} = 120 \times \frac{225 \times 10^6}{d/2} = \frac{120 \times 225 \times 10^6}{300/2}$

$= 1.8 \times 10^8 \text{ Nmm}$ ——— ①

Externally applied moment

Maxm bending moment

$M = \frac{Wl}{2} \times \frac{l}{2} - \frac{Wl}{2} \times \frac{1}{2} \cdot \frac{l}{2} = \frac{Wl^2}{4} - \frac{Wl^2}{8} = \frac{Wl^2}{8}$ ——— ②

Equating ① & ②,

$\frac{Wl^2}{8} = 1.8 \times 10^8$

$\frac{W \times (4 \times 10^3)^2}{8} = 1.8 \times 10^8$

$\Rightarrow W = 90 \text{ N/mm (UDL)}$ //

Imp.

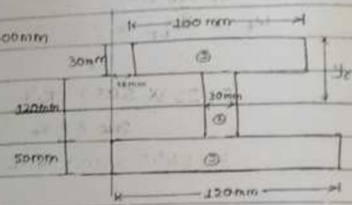
Q. The beam simply supported having cross-section as shown in fig. is loaded with the UDL over whole of its span. If the beam is 8m long, find the UDL. If max. permissible bending stresses in tension is limited to 30 MN/m² and compression to 45 MN/m². What are the actual maxm. bending stresses ~~str~~ set up in the section?

Soln.

Length of beam (l) = 8m = 8000mm

$\sigma_{bt} = 30\ MN/m^2$
$= 30\ N/mm^2$

$\sigma_{bc} = 45\ MN/m^2$
$= 45\ N/mm^2$



	Area (A_i)	x_i	y_i	$A_i x_i$	$A_i y_i$
1) rect ①	$100 \times 30 = 3000$	~~10+50~~ ~~= 60~~	$170 + \frac{30}{2} = 185$	~~180000~~	
2) rect ②	$30 \times 120 = 3600$		$50 + \frac{120}{2} = 110$		
3) rect ③	$120 \times 50 = 6000$		$\frac{50 \cancel{+120+30}}{2} = \cancel{185}\ 25$		

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3} = \frac{3000 \times 185 + 3600 \times 110 + 6000 \times 25}{3000 + 3600 + 6000}$$

$$= 87.38\ mm$$

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3$$

$$= \left[(I_{xx})_{CG} + A_1 (y_1 - \bar{y})^2\right] + \left[(I_{xx})_{CG} + A_2 (y_2 - \bar{y})^2\right] + \left[(I_{xx})_{CG} + A_3 (y_3 - \bar{y})^2\right]$$

$$= \left[\frac{100 \times 30^3}{12} + 3000 (185 - 87.38)^2\right] + \left[\frac{30 \times 120^3}{12} + 3600 (110 - 87.38)^2\right]$$

$y_t = \bar{y} = 87.38\ mm$
$y_c = (200 - y_t)\ mm$

$$+ \left[\frac{120 \times 50^3}{12} + \cancel{356}\ 6000 (25 - 87.38)^2\right]$$

$$= 58578571.44\ mm^4$$

$$I = 5857 \times 10^4\ mm^4$$

In tension side,

$$M_t = \sigma_{bt} \times \frac{I}{y_t}$$

$$= 30 \times \frac{5857 \times 10^4}{\cancel{200}\ 87.38}$$

$$= \underset{(smaller)}{2045 \times 10^4}\ Nmm$$

In compression side,

$$M_c = \sigma_{bc} \times \frac{I}{y_c}$$

$$= 45 \times \frac{5857 \times 10^4}{(200 - 87.38)}$$

$$= 2380 \times 10^4\ Nmm$$

Maxm bending moment (M) $= \frac{wl^2}{8} = \frac{w(8000)^2}{8} = 8 \times 10^6 w$

$\Rightarrow$ $M = M_t$

$8 \times 10^6 w = 2045 \times 10^4$

$w = 2.5562\ N/mm$

$\Rightarrow$ For $M_t = 2045 \times 10^4\ Nmm$

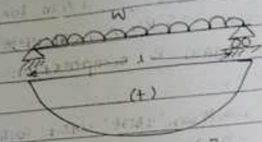
$$2045 \times 10^4 = \sigma_{bc} \times \frac{I}{y_c}$$

$$2045 \times 10^4 = \sigma_{bc} \times \frac{5857 \times 10^4}{(200 - 87.38)}$$

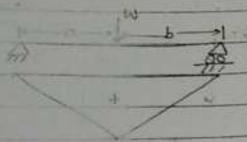
$$\therefore \sigma_{bc} = 38.66\ N/mm^2$$

Note:

$BM_{max} = \frac{Wl}{4}$



$BM_{max} = \frac{Wl^2}{8}$



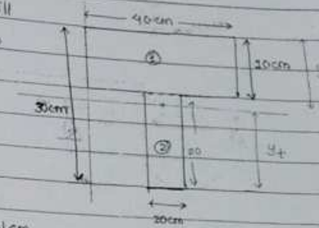
$BM_{max} = \frac{Wab}{l}$

→ $\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$ → bending / flexural equation

→ $M = \sigma \cdot Z = \sigma \cdot \frac{I}{y}$ $\quad Z = \frac{I}{y_{max}}$ is section modulus

6. A simply supported T-beam having cross section as shown in fig is 3.5m long. If it carries bending moment 25 kN m, calculate maximum bending stresses in tension & compression.

Also mention that what will be difference in stresses if T-beam is in inverted position.

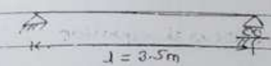


Soln.

$\sigma_{bt} = ?$ $\quad \sigma_{bc} = ?$

$M = 25 \text{ kNm} = 25 \times 10^5 \text{ Ncm}$

$l = 3.5 \text{ m}$



$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{(40 \times 10) \times (20+5) + (20 \times 20) \times (10)}{400 + 400}$$

$$= 17.5 \text{ cm}$$

$y_1 = 25 \text{ cm}$ $\qquad A_1 = 400 \text{ cm}^2$

$y_2 = 10 \text{ cm}$ $\qquad A_2 = 400 \text{ cm}^2$

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2$$

$$= \left[(I_{xx})_{cg} + A_1 (y_1 - \bar{y})^2\right] + \left[(I_{xx})_{cg} + A_2 (y_2 - \bar{y})^2\right]$$

$$= \left[\frac{40 \times 10^3}{12} + 400 (25 - 17.5)^2\right] + \left[\frac{20 \times 20^3}{12} + 400 (10 - 17.5)^2\right]$$

$$I = 61666.67 \text{ cm}^4$$

$y_t = \bar{y} = 17.5$ cm

$y_c = 30 - y_t = 12.5$ cm

⇒ In tension side,

$M = \sigma_{bt} \times \dfrac{I}{y_t}$

$25 \times 10^5 = \sigma_{bt} \times \dfrac{61666.67}{17.5}$

$\therefore \sigma_{bt} = 709.45$ N/cm²

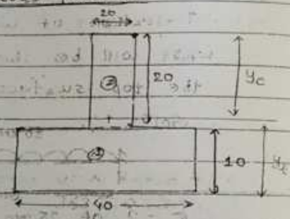
In compression side,

$M = \sigma_{bc} \times \dfrac{I}{y_c}$

$25 \times 10^5 = \sigma_{bc} \times \dfrac{61666.67}{12.5}$

$\sigma_{bc} = 506.75$ N/cm²

(not needed) Case II: When T beam is in inverted position



$\bar{y} = \dfrac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$

$= \dfrac{400 \times 5 + 400 \times (10+10)}{400+400}$

$= 12.5$ cm

$I_{xx} = (I_{xx})_1 + (I_{xx})_2$

$= \left[\dfrac{40 \times 10^3}{12} + 400(5-12.5)^2\right] + \left[\dfrac{20 \times 20^3}{12} + 400(20-12.5)^2\right]$

$I = 61666.67$ cm⁴

$y_t = 12.5$ cm $(\bar{y})$

$y_c = 30 - y_t = 17.5$ cm

⇒ In tension side;

$M = \sigma_{bt} \times \dfrac{I}{y_t}$

$25 \times 10^5 = \sigma_{bt} \times \dfrac{61666.67}{12.5}$

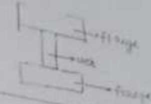
$\sigma_{bt} = 506.75$ N/cm²

In compression side,

$M = \sigma_{bc} \times \dfrac{I}{y_c}$

$25 \times 10^5 = \sigma_{bc} \times \dfrac{61666.67}{17.5}$

$\sigma_{bc} = 709.45$ N/cm²



⇒ Difference in stresses

$\Delta\sigma_{bt} = 709.45 - 506.75$

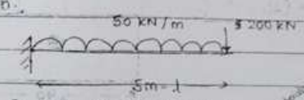
$= 202.7$ N/cm²   → no change in value, change in nature only

$\Delta\sigma_{bc} = 709.45 - 506.75$

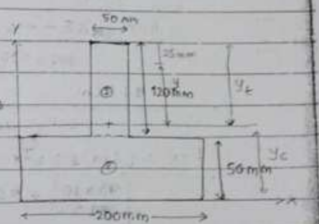
$= 202.7$ N/cm²

8. A cantilever beam of length 5 m is carrying a uniformly distributed load of intensity 50 kN/m along with a point load of 200 kN at the free end. If the cross section of the beam is inverted T-section of web 50 × 120 mm² and flange 50 × 200 mm², what will be the bending stress 25 mm below from the top surface of beam?

Solⁿ.



$\sigma = ?$ at 25 mm from top



$\bar{y} = \dfrac{A_1 y_1 + A_2 y_2}{A_1 + A_2}$

$= \dfrac{(50 \times 120)(50+60) + (200 \times 50)(25)}{6000 + 10000}$

$= 56.87$ mm

$y_1 = 110$

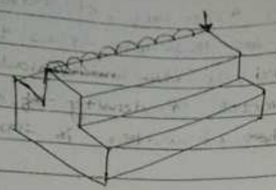
$y_2 = 25$

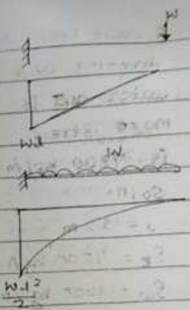
$I_{xx} = (I_{xx})_1 + (I_{xx})_2$

$= \left[\dfrac{50 \times 120^3}{12} + 6000(110 - 56.87)^2\right] + \left[\dfrac{200 \times 50^3}{12} + 10000(25 - 56.87)^2\right]$

$\therefore I = 36377083.73$ mm⁴

⇒ maxm BM in fixed support (cantilever beam)





NA ⇒ $y_c = \bar{y} = 56.87$ mm

$y_t = 170 - y_c = 113.13$ mm

$y = y_t - 25 = 113.13 - 25 = 88.13$ mm

- In cantilever beam, maxm BM is developed in support.

$M = 200 \times 5 + (50 \times 5) \times 2.5$

$= 1625$ kNm

$= 1625 \times 10^6$ Nmm

$\Rightarrow M = \sigma \frac{I}{y} \Rightarrow \sigma = \frac{My}{I} = \frac{1625 \times 10^6 \times 88.13}{36377083.73}$

$= 3936.85$ N/mm² //

The bending stress 25mm below the top surface of beam will be 3936.85 N/mm² //

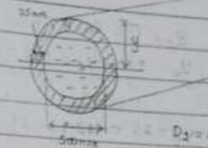
9. A cast iron water main 12 m long, of 500 mm inside diameter and 25 mm wall thickness runs full of water and is supported at its ends. Calculate the maxm stress in the metal if density of cast iron is 7200 kg/m³ and that of water is 1000 kg/m³.

Soln:

$l = 12$ m

$\rho_c = 7200$ kg/m³

$\rho_w = 1000$ kg/m³



$D_1 = 500 + (2 \times 25) = 550$ mm $= 0.55$ m

$D_2 = 500$ mm $= 0.5$ m

weight of cast iron (W_c)

= sp. unit weight × area

$= 7200 \times 9.81 \times \frac{\pi}{4}(0.55^2 - 0.5^2)$

$= 2912.39$ N/m

weight of water (W_ω)

= sp. weight × area

$= 1000 \times 9.81 \times \frac{\pi (0.5^2)}{4}$

$= 1926.18$ N/m

$W = W_c + W_\omega = 4838.57$ N/m

$M = \frac{Wl^2}{8} = \frac{4838.57 \times 12^2}{8} = 87094.26$ Nm

$I_x = \frac{\pi (D_1^4 - D_2^4)}{64} = \frac{\pi (0.55^4 - 0.5^4)}{64} = 1.42 \times 10^{-3} = I$

$y = \frac{0.5}{2} + 0.025 = \cancel{0.525}\ 0.275$ m

$\therefore$ Max bending moment

$M = \frac{\sigma I}{y} \Rightarrow \sigma = \frac{My}{I} = \frac{87094.26 \times \cancel{0.525}\ 0.275}{1.42 \times 10^{-3}}$

$\therefore \sigma_{max} = \cancel{32192848.24}\ 16866846.13$ N/m^2 //

Beam of heterogeneous material (flitched beam)

Total resisting moment of any section is the sum of resisting moment caused by individual material making the section.

$M = M_1 + M_2$

Modular ratio $\frac{E_1}{E_2} = m$

strain on I = strain on II

$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}$

2013F

7 marks

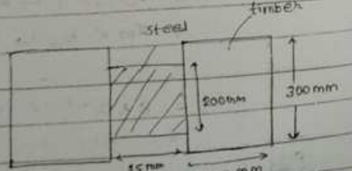
Q. A flitched timber beam consists of two joists 100mm wide and 300 mm deep. With a steel plate 200mm deep and 15mm thick placed symmetrically in both and clamped to them. Calculate the total moment of resistance of the section if allowable stress on joist is 9 MN/m^2.

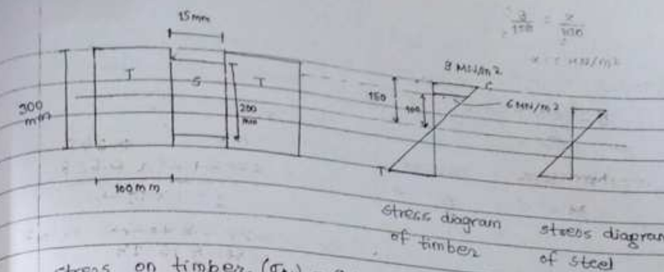
Given $E_s = 20 E_w$

Soln.

width (b) = 100 mm

depth (d) = 300 mm





stress on timber (σ_w) = 9 MN/m^2

stress on timber (σ_w) = 6 MN/m^2 at 100mm from neutral axis

$\Rightarrow$ Stress on steel at 100 mm from neutral layer

$\frac{\sigma_s}{E_s} = \frac{\sigma_w}{E_w}$ w = wood s = steel

$\Rightarrow \frac{\sigma_s}{\sigma_w} = \frac{E_s}{E_w} = 20$

$\Rightarrow \sigma_s = 20 \times \sigma_w$

$= 20 \times 6$

$= 120$ MN/m^2

$\Rightarrow$ M = moment of resistance due to timber + moment of resistance due to steel

$M = M_w + M_s$

$= \sigma_w z_w + \sigma_s z_s$

$= \sigma_w \times \frac{b_w d_w^2}{6} + \frac{\sigma_s \times b_s d_s^2}{6}$

$= 9 \times 10^6 \times \frac{0.2 \times 0.3^2}{6} + 120 \times 10^6 \times \frac{0.015 \times 0.2^2}{6}$

$= 39000$ Nm

2034S

Q. A timber beam 16cm wide & 20cm deep is to be reinforced by bolting on 2 steel flitches each 16cm & 1cm in section. Find the moment of resistance when

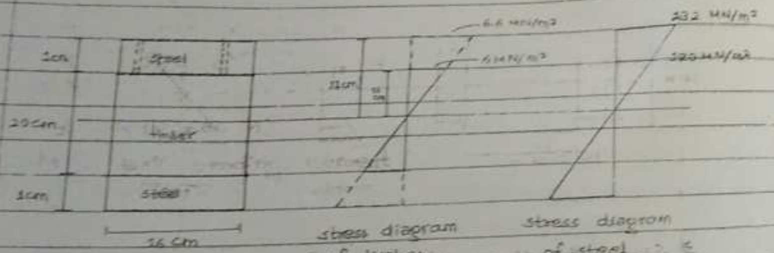
i) the flitches are attatched ~~at~~ symetrically at the top and bottom.

ii) the flitches are attatched symetrically at the sides.

Allowable stresses in timber is 6 MN/m². What is the max. stress in steel in each case? Take $E_s = 20E_w$ where E_s & E_w are Young's modulus for steel & timber.

Soln.

Case I : Flitches are attached symetrically at top & bottom



stress diagram of timber   stress diagram of steel.

⇒ Stress on steel at 11 cm from neutral layer

$$\frac{\sigma_s}{E_s} = \frac{\sigma_\omega}{E_\omega}$$

$$\sigma_s = \frac{E_s}{E_\omega} \cdot \sigma_\omega = 20 \times 6.6 = 132 \ \ MN/m^2$$

⇒ Moment of resistance

$b_{s1} = b_{s2} = 16cm = 0.16m$

$d_{s1} = 22cm = 0.22m$

$d_{s2} = 20cm = 0.2m$

$$M = M_\omega + M_s$$

$$= \sigma_\omega z_\omega + \sigma_s z_s$$

$$= 6 \times \frac{b_\omega d_\omega^2}{6} + \left[132 \times \frac{b_{s1} d_{s1}^2}{6} - 120 \times \frac{b_{s2} d_{s2}^2}{6}\right]$$

$$= 0.16 \times 0.2^2 + \left[132 \times \frac{0.16 \times 0.22^2}{6} - 120 \times \frac{0.16 \times 0.2^2}{6}\right]$$

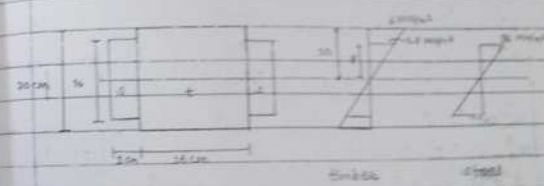
$$= 6.4 \times 10^{-3} + 0.042368$$

$$= 0.048768 \ MNm$$

$$= 48768 \ Nm$$

$d_{s1} = 22cm = 0.22m$

Case II : Fitches are attatched symmetrically at the sides



⇒ Stress on steel at 8 cm from neutral layer

$$\frac{\sigma_s}{\sigma_\omega} = \frac{E_s}{E_\omega} = 20$$

$$\sigma_s = 20 \times \sigma_\omega = 20 \times 4.8$$

$$\therefore \sigma_s = 96 \ \ MN/m^2$$

⇒ Moment of resistance

$$M = M_\omega + M_s$$

$$= \sigma_\omega z_\omega + \sigma_s z_s$$

$$= \cancel{6} \times \frac{b_w d_w^2}{\cancel{6}} + \left[\overset{16}{\cancel{96}} \times \frac{b_s d_s^2}{\cancel{6}} \right]$$

$$= 0.16 \times 0.2^2 + \left[16 \times 0.02 \times 0.16^2 \right]$$

$b_s = (0.01 + 0.01)\,m$
$= 0.02\,m$

$$= 0.014592 \text{ MNm}$$

$$= 14592 \text{ Nm} \;//$$

Imp

Relation between slope, deflection & bending moment (Deflection equation)

$$EI \frac{d^2y}{dx^2} = M \quad \text{(derive yourself)}$$

E = Young's modulus

I = moment of inertia

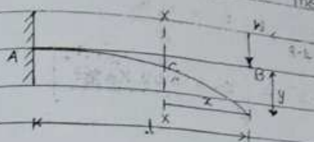
$\frac{dy}{dx}$ = slope (θ)

y = deflection (Δ)

M = Max^m bending moment

Deflection by double integration method

1)



In fixed support, no deflection no rotation

Bending moment at x

$$M = -Wx$$

&

$$EI \frac{d^2y}{dx^2} = -Wx$$

Integrating both sides, w.r.t. x,

$$EI \int \frac{d^2y}{dx^2} = -W \int x$$

$$EI \frac{dy}{dx} = -W \frac{x^2}{2} + C_1 \quad \text{—— (1)}$$

Using boundary cond^n,

At $x = l$, $\frac{dy}{dx} = 0$

$$\therefore \; \cancel{EI} \times 0 = \frac{-Wl^2}{2} + C_1$$

$$C_1 = \frac{Wl^2}{2}$$

Now, (1) is $\boxed{EI \frac{dy}{dx} = \frac{-Wx^2}{2} + \frac{Wl^2}{2}}$ —— (2)

Again, integrating (2) w.r.t. x,

$$EI \int \frac{dy}{dx} = \frac{-W}{2} \int x^2\,dx + \frac{W}{2} \int l^2\,dx$$

$$EIy = \frac{-W}{2} \cdot \frac{x^3}{3} + \frac{Wl^2}{2} x + C_2$$

$$EI \cdot y = \frac{-Wx^3}{6} + \frac{Wl^2 x}{2} + C_2 \quad \text{—— (3)}$$

Using boundary condn,

At $x = l$, $y = 0$

$$\therefore 0 = \frac{-Wl^3}{6} + \frac{Wl^3}{2} + c_2$$

$$c_2 = \frac{Wl^3}{6} - \frac{Wl^3}{2} = \frac{Wl^3 - 3Wl^3}{6} = \frac{-2Wl^3}{6} = \frac{-Wl^3}{3}$$

Now, ③ is

$$\boxed{EI \cdot y = \frac{-Wx^3}{6} + \frac{Wl^2 x}{2} - \frac{Wl^3}{3}} \quad ④$$

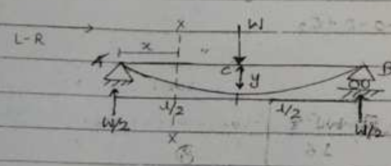
Max. deflection occurs at B i.e. $x = 0$

Above eqn becomes

$$EI \cdot y = -\frac{Wl^3}{3}$$

$$\boxed{y_B = -\frac{Wl^3}{3EI}}$$

2)



Bending moment at x

$$M_x = \frac{W}{2} x$$

$$EI \frac{d^2y}{dx^2} = -\frac{W}{2} x$$

Integrating both sides w.r.t x,

$$EI \int \frac{d^2y}{dx^2} = -\frac{W}{2} \int x$$

$$EI \frac{dy}{dx} = -\frac{W}{2} \cdot \frac{x^2}{2} + c_1 \quad ①$$

Boundary condn : At $x = l/2$, $\frac{dy}{dx} = 0$

$$0 = \frac{-Wl^2}{4 \cdot 4} + c_1$$

$$c_1 = \frac{-Wl^2}{16}$$

Slope at A

i.e. $x = 0$

$$EI\,\theta_A = -\frac{Wl^2}{16}$$

$$\therefore \boxed{EI \frac{dy}{dx} = \frac{+Wx^2}{4} \mp \frac{Wl^2}{16}} \quad ②$$

Again, integrating ②

$$EI \int \frac{dy}{dx} = \frac{+W}{4} \int x^2 dx \mp \frac{W}{16} \int l^2 dx$$

$$EI \cdot y = \frac{+W}{4} \cdot \frac{x^3}{3} \mp \frac{Wl^2}{16} \cdot x + c_2 \quad ③$$

B/I

Boundary condn : At $x = 0$, $y = 0$

$$0 = -0 - 0 + c_2$$

$$\therefore c_2 = 0$$

$$\therefore \boxed{EI \cdot y = \frac{+Wx^3}{12} \mp \frac{Wl^2 x}{16}} \quad ④$$

Max deflection occurs at centre i.e. $x = l/2$

$$EI \cdot y = \frac{+W(l/2)^3}{12} - \frac{Wl^2\, l/2}{16}$$

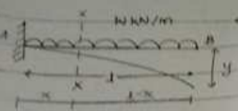
$$= \frac{+Wl^3}{12 \times 8} - \frac{Wl^3}{32}$$

$$= \frac{+Wl^3}{24} - \frac{Wl^3}{48}$$

$$\Rightarrow \boxed{y_C = \frac{-Wl^3}{48EI}}$$

-ve sign shows deflection is downward

3)



Consider a section xx at a distance x from fixed end A.

$$M_x = -\frac{w(l-x)^2}{2}$$

$$EI\frac{d^2y}{dx^2} = \frac{-w}{2}(l-x)^2$$

Integrating, we get

$$EI\frac{dy}{dx} = \frac{w}{6}(l-x)^3 + C_1$$

At A, $x=0$, $dy/dx = 0$.

$$C_1 = \frac{-wl^3}{6}$$

$$\boxed{EI\frac{dy}{dx} = \frac{w(l-x)^3}{6} - \frac{wl^3}{6}}$$ This is required slope eq^n. —— (1)

At B, $x = l$, we have

$$EI\theta_B = \frac{w}{6}(l-l)^3 - \frac{wl^3}{6} = \frac{-wl^3}{6}$$

$$\therefore \boxed{\theta_B = \frac{-wl^3}{6EI}}$$

To get deflection integrating (1)

$$EIy = -\frac{w}{24}(l-x)^4 - \frac{wl^3}{6}x + C_2$$

At A, $x=0$, $y=0$

$$0 = \frac{-wl^4}{24} + C_2$$

$$\therefore C_2 = \frac{wl^4}{24}$$

Hence,

$$\boxed{EIy = \frac{-w}{24}(l-x)^4 - \frac{wl^3}{6}x + \frac{wl^4}{24}}$$

This is the required deflection equation.

Deflection at B: put $x = l$

$$EIy_{max} = -\frac{w}{24}(l-l)^4 - \frac{wl^3}{6}\times l + \frac{wl^4}{24}$$

$$EIy_{max} = -\frac{wl^4}{6} + \frac{wl^4}{24} = -\frac{wl^4}{8}$$

$$\boxed{y_{max} = \frac{-wl^4}{8EI}}$$

⇒ Downward deflection at B is

$$\boxed{y_{max} = \frac{wl^4}{8EI}}$$

Radius of gyration

$r_{xx} = \sqrt{\frac{I_{xx}}{A}}$ $r_{yy} = \sqrt{\frac{I_{yy}}{A}}$

Chapter 9 : Buckling & Stability in columns

Column & Struts

A member of structure or bar which carries an axial compressive load is called strut. If the strut is vertical i.e. inclined at 90° to the horizontal, then it is k/s column, pillar, stanchion.

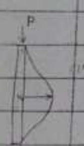
Generally, a member in any position other than vertical, subjected to a compressive load is called strut, and vertical ~~a~~ member subjected to a compressive load is called column.

Slenderness ratio (λ)

It is the ratio of effective unsupported length of column to the minimum radius of gyration of the cross-sectional ends of the column.

$$\lambda = \frac{l}{r_{min}} \qquad I_{min} = I_v = A r_{min}^2 \qquad r_{min} = \sqrt{\frac{I_{min}}{A}}$$

Bukling load (crippling load)



The maximum limiting load at which the column tends to have lateral displacement / tends to buckle is called buckling or crippling load. The buckling takes place about the axis having minimum radius of gyration or least moment of inertia.

Safe load

It is the load to which a column is actually subjected to and is well below the buckling load. It is obtained by dividing the buckling load by suitable factor of safety.

$$\text{Safe load} = \frac{\text{buckling load}}{\text{factor of safety}}$$

Euler's formula

Assumptions

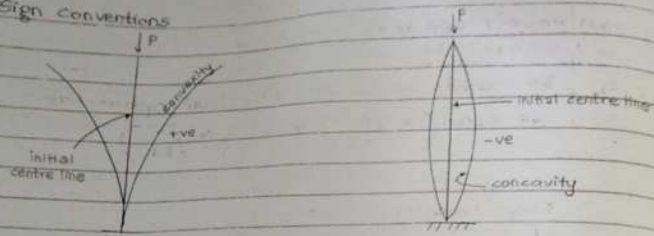
1. The column is initially straight and of uniform lateral dimension. (uniform cross-section)
2. The compressive load is exactly axial and it passes through the centroid of column section.
3. The material of column is perfectly homogeneous & isotropic.
4. Pin joints are frictionless and fixed ends are perfectly rigid.
5. The wt. of column itself is neglected.
6. The column fails by buckling alone.
7. Limit of proportionality is not exceeded. (obeys Hooke's law)

Note: D = lateral dimension

$\frac{l}{D} < 12$ short column

$\frac{l}{D} > 12$ long column

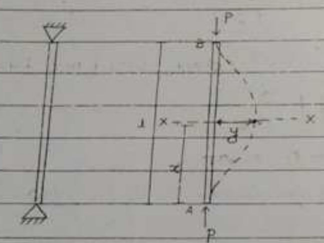
Sign conventions



Imp Derivation of Euler's formula for different end conditions

Case I

Exam a) When both ends are hinged / pinned



Consider a column having both ends hinged and subjected to axial compressive load P. Let 'l' be the length of column & 'y' be the deflection at a distance 'x' from end A.

Bending moment at the considered section x-x is given by

$M = -Py$

From differential eqn. of deflection,

$$EI\frac{d^2y}{dx^2} = -Py$$

$$EI\frac{d^2y}{dx^2} + Py = 0$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

The solution of above differential equation is

$$y = C_1\cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2\sin\left(x\sqrt{\frac{P}{EI}}\right)$$

Using boundary condn.,

at $x=0, y=0$ i.e. at hinged support (end A)

$0 = C_1$

Again,

at $x = l, y = 0$ (end B)

$$0 = C_2\sin\left(l\sqrt{\frac{P}{EI}}\right)$$

if C_2 is zero, the column could not buckle. Then,

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

$$\Rightarrow l\sqrt{\frac{P}{EI}} = 0, \pi, 2\pi \ldots$$

Consider practical value

$$l\sqrt{\frac{P}{EI}} = \pi$$

$$l^2\frac{P}{EI} = \pi^2 \Rightarrow \boxed{P_{cr} = \frac{\pi^2 EI}{l^2}}$$

which is Euler's formula for buckling load

Here, $I = I_{min}$ (Ixx or Iyy)

Summary

S.N.	End condition	(l_e) Effective length	Euler's buckling load (P_{cr})
1.	Both ends pinned/ hinged	$l_e = l$	$P_{cr} = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 EI}{l^2}$
2.	One end fixed, other end free	$l_e = 2l$	$P_{cr} = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 EI}{4l^2}$
3.	One end fixed, other end hinged / pinned	$l_e = \frac{l}{\sqrt{2}}$	$P_{cr} = \frac{\pi^2 EI}{l_e^2} = \frac{2\pi^2 EI}{l^2}$
4.	Both ends fixed	$l_e = \frac{l}{2}$	$P_{cr} = \frac{\pi^2 EI}{l_e^2} = \frac{4\pi^2 EI}{l^2}$

Q. A solid round bar 60 mm in diameter and 2.5 m long is used as a strut, one end of strut is fixed while its other end is hinged. Find the safe compressive load for this strut using Euler's formula. Assume E = 200 GN/m² and factor of safety 3.

Soln.

$d = 60\,mm = 0.06\,m$ $\qquad E = 200\,GN/m^2 = 200 \times 10^9\,N/m^2$

$l = 2.5\,m$

factor of safety = 3 $\qquad l_e = \frac{l}{\sqrt{2}} = \frac{2.5}{\sqrt{2}} = 1.77$

We have,

$$\text{Bukling load } (P_{cr}) = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 \times 200 \times 10^9 \times \frac{\pi \times (0.06)^4}{64}}{1.77^2}$$

$= \cancel{497.07\ MN}$

$= 400828.05\ N$

$= 400.8\ kN$

$\Rightarrow$ Safe compressive load $= \frac{\text{buckling load}}{\text{factor of safety}}$

$= \frac{\cancel{497.07}\ 400828.05}{3}$

$= \cancel{65.69\ MN}\ 133609.35\ N$

$= 133.6\ kN$ //

Q. A slender pin connected aluminium column 1.8 m long of circular cross section is to have an outside diameter of 50 mm. Calculate the necessary internal diameter to prevent failure by buckling if the actual load applied is 13.6 kN and the critical load applied is twice the actual load. Take E for aluminium as 70 GN/m².

Soln

$l = 1.8\,m$

outside diameter (D) = 50 mm = 0.05 m $\qquad E = 70\,GN/m^2 = 70 \times 10^9\,N/m^2$

internal diameter (d) = ?

actual load applied (P) = 13.6 kN

(P_{cr}) critical load = 2 × actual load

$= 2 \times 13.6$

$= 27.2\,kN = 27.2 \times 10^3\,N$

Moment of inertia $(I) = \frac{\pi (D^4 - d^4)}{64}$

$= \frac{\pi (0.05^4 - d^4)}{64}$

End condn. : both ends pinned

Effective length $l_e = l = 1.8\,m$

Buckling load $(P_{cr}) = \dfrac{\pi^2 EI}{l_e^2}$

$$27.2 \times 10^3 = \frac{\pi^2 \times 70 \times 10^9 \times \dfrac{\pi(0.05^4 - d^4)}{64}}{1.8^2}$$

$$564019.2 = 13565246.05 - 2.17 \times 10^{12} d^4$$

$$2.17 \times 10^{12} d^4 = 13001226.85$$

$$\Rightarrow d = 0.049 \text{ m} \,//$$

Q. An I section joist 400 mm * 200mm * 20 mm and 6m long is used as a strut with both ends fixed. What is Euler's crippling load for columns? Take Young's modulus of joist as 200 GPa.

All dimensions are in mm.

Soln.

$\bar{x} = 100$ mm

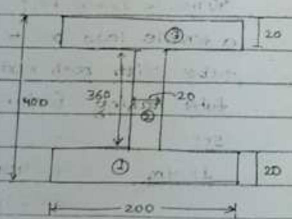
$\bar{Y} = 200$ mm

$x_1 = 100$ mm $\quad x_2 = 100$ mm

$y_1 = 10$ mm $\quad y_2 = 200$ mm

$x_3 = 100$ mm

$y_3 = 390$ mm



$$P_{cr} = \frac{\pi^2 EI}{l_e^2}$$ — minimum of I_{xx} & I_{yy}

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3$$

$$= \left[\frac{200 \times 20^3}{12} + 4000(10 - 200)^2\right] + \left[\frac{20 \times 360^3}{12} + 7200(\overset{0}{\cancel{200}} - 200)^2\right]$$

$$+ \left[\frac{200 \times 20^3}{12} + 4000(390 - 200)^2\right] = 366826666.7 \text{ mm}^4$$

$$I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3$$

$$= \left[\frac{20 \times 200^3}{12} + 4000(\overset{0}{\cancel{100 - 100}})^2\right] + \left[\frac{360 \times 20^3}{12} + 0\right] + \left[\frac{20 \times 200^3}{12} + 0\right]$$

$$= 26906666.67 \text{ mm}^4$$

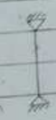
$I_{yy} < I_{xx}$ so, $I = 26906666.67$ mm^4

$$\Rightarrow P_{cr} = \frac{\pi^2 EI}{l_e^2} \qquad l_e = \frac{l}{2} = \frac{6}{2} = 3\text{m}$$

$$= \frac{\pi^2 \times 200 \times 10^9 \times 26906666.67 \times 10^{-12}}{3^2}$$

$$= 5901.29 \text{ kN} \,//$$

Q. A hollow tube 4m long with external & internal diameter 40 mm & 25mm resp. was found to extend 4.8 mm under a tensile load of 60 KN. Find the buckling load for the tube with both end pinned. Also, find the safe load on the tube taking factor of safety as 5.



Soln

$l = 4$m $\quad D = 40$mm $= 0.04$ m $\quad$ FOS $= 5$

$d = 25$mm $= 0.025$ m $\quad \Delta l = 4.8$ mm $= 4.8 \times 10^{-3}$ m

Tensile load $= 60$ kN $= 60 \times 10^3$ N

Buckling load $= ?$

Both ends are pinned so, $l_e = l = 4$ m

$$P_{cr} = \frac{\pi^2 EI}{l_e^2} \quad \text{—(1)}$$

$$I = \frac{\pi(D^4 - d^4)}{64} = \frac{\pi(0.04^4 - 0.025^4)}{64} = 1.06 \times 10^{-7}$$

$$E = \frac{Fl}{A\,\Delta l} = \frac{60\times10^3 \times 4}{\frac{\pi(D^2-d^2)}{4} \times 4.8\times10^{-3}} = \frac{16\times 60\times10^3}{\pi(0.04^2 - 0.025^2)\times 4.8\times10^{-3}}$$

$$= 6.53\times10^{10}\ N/m^2$$

⇒ From ①, buckling load

$$P_{cr} = \frac{\pi^2 \times 6.53\times10^{10} \times 1.06\times10^{-7}}{4^2}$$

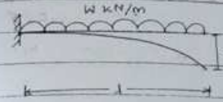
$$= 4270\ N$$

⇒ Safe load = $\frac{\text{buckling load}}{FOS} = \frac{4270}{5} = 854\ N$ //

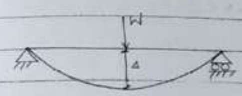
Deflection on different beam

1) 

$$\Delta = \frac{Wl^3}{3EI}$$

2) 

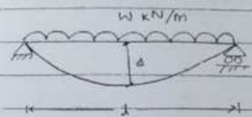
$$\Delta = \frac{wl^4}{8EI}$$

3) 

$$\Delta = \frac{Wl^3}{48EI}$$

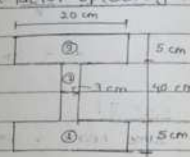
In case of beam,
$I = I_{xx}$

In case of column,
$I = I_{min}\ (I_{xx} / I_{yy})$

4) 

$$\Delta = \frac{5wl^4}{384EI}$$

Q. A simply supported beam acting 40 kN/cm of UDL through out of its span. Determine the length of the beam if deflection on centre by above load is 1 cm. Also find safe load if same beam is used as column with both end fixed. Take $E = 2100$ tones/cm^2 & factor of safety = 3



Soln.

$\bar{x} = 10$ cm

$\bar{y} = 25$ cm

$x_1 = 10$ cm $\quad y_1 = 2.5$ cm

$x_2 = 10$ cm $\quad y_2 = 25$ cm

$x_3 = 10$ cm $\quad y_3 = 47.5$ cm

Case I:

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2 + (I_{xx})_3$$

$$= \left[\frac{20 \times 5^3}{12} + 100(2.5-25)^2\right] + \left[\frac{3 \times 40^3}{12} + 120(25-25)^2\right] + \left[\frac{20 \times 5^3}{12} + 100(47.5-25)^2\right]$$

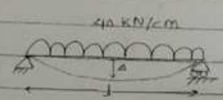
$$= 117666.67 \text{ cm}^4$$

$$I_{yy} = (I_{yy})_1 + (I_{yy})_2 + (I_{yy})_3$$

$$= \left[\frac{5 \times 20^3}{12} + 100(10-10)^2\right] + \left[\frac{40 \times 3^3}{12} + 120 \times 0\right] + \left[\frac{5 \times 20^3}{12} + 100 \times 0\right]$$

$$= 6756.67 \text{ cm}^4$$

$I = I_{xx} = \cancel{6756.67}$ 117666.67 cm^4



⇒ Given: deflection $(\Delta) = 1$ cm

$w = 40$ kN/cm

$l = ?$ $\quad E = 2100$ tones/cm^2 $= 21 \times 10^5$ kg/cm^2

$= 21 \times 10^5 \times \frac{9.81}{1000}$ kN/cm^2

$= 20601$ kN/cm^2

We know,

⇒ Deflection $\Delta = \frac{5}{384} \frac{wl^4}{EI}$

or, $\frac{1 \times 384}{5} = \frac{40 \times l^4}{20601 \times 117666.67}$

$\therefore l = 261.2$ cm

Case II:

Same beam is used as column with both ends fixed

$$l_e = \frac{l}{2} = \frac{261.2}{2} = 130.6 \text{ cm}$$

Euler's buckling load

$$P_{cr} = \frac{\pi^2 EI}{l_e^2}$$

Here, I is minimum of I_{xx} & I_{yx}

$\therefore I = I_{yy} = 6756.67$ cm^4

$\therefore P_{cr} = \frac{\pi^2 \times 20601 \times 6756.67}{130.6^2}$ $\quad$ FOS = 3

$= 80544.22$ kN

⇒ Safe load $= \frac{\text{Crippling load}}{\text{FOS}} = \frac{80544.22}{3}$

$= 26848.07$ kN //

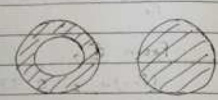
8. Determine the ratio of buckling strength for of 2 columns of circular cross-section one hollow & other solid when both are made of same material, have same length, same cross-sectional area & same end conditions. The internal diameter of hollow column is half of its external diameter.

Soln.

$$\frac{(P_{cr})_{hollow}}{(P_{cr})_{solid}} = ?$$

Given :

hollow		solid
E_H	=	E_S
l_H	=	l_S
A_H	=	A_S



$$d_H = \frac{1}{2} D_H$$

$$(P_{cr})_{hollow} = \frac{\pi^2 E_H I_H}{l_e^2} \quad \text{——— (1)}$$

$$(P_{cr})_{solid} = \frac{\pi^2 E_S I_S}{l_e^2} \quad \text{——— (2)}$$

$$\Rightarrow \frac{(P_{cr})_{hollow}}{(P_{cr})_{solid}} = \frac{I_h}{I_s} = \frac{\pi (D_H^4 - d_H^4)/64}{\pi D_S^4/64} = \frac{D_H^4 - d_H^4}{D_S^4}$$

$$= \frac{D_H^4 - (0.5 D_H)^4}{D_S^4}$$

$$= \frac{0.938 D_H^4}{D_S^4} \quad \text{—— (1)}$$

Since cross-sectional area of both columns is same;

$$A_S = A_H$$

or, $\frac{\pi D_S^2}{4} = \frac{\pi}{4}(D_H^2 - d_H^2)$

or, $D_S^2 = D_H^2 - (0.5 D_H)^2$

or, $D_S^2 = 0.75 D_H^2$

or, $\left(\frac{D_H}{D_S}\right)^2 = 1.3333$

$\therefore \frac{D_H}{D_S} = 1.1547$

From (1),

$$\frac{(P_{cr})_H}{(P_{cr})_S} = 0.938 (1.1547)^4 = 1.66 \,//$$

→ Hollow column can bear 1.66 times more strength than the solid one of with given conditions.

Exam. 8. Calculate the max value of slenderness ratio of mild steel column for which Euler's formula is valid, considering both ends are pin connected.

Take $\sigma_c = 330\ MN/m^2$ (allowable compressive stress)

$E = 210\ GN/m^2$

Soln.

Slenderness ratio $(\lambda) = \frac{l}{K_{min}}$

For both end pin connected,

$l_e = l$

Crippling load $(P_{cr}) = \frac{\pi^2 E I}{l^2}$

$K = \sqrt{\frac{I}{A}}$

$\Rightarrow I = AK^2$

$P_{cr} = \frac{\pi^2 E A K^2}{l^2}$

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{(l/k)^2}$$

$E = 210 \times 10^9 \, N/m^2$

$\sigma_c = 330 \times 10^6 \, N/m^2$

$\therefore$ σ_{cr} (critical stress) $= \dfrac{\pi^2 E}{(l/k)^2}$

The colume will be safe if

$$\sigma_{cr} \le \sigma_c$$

$$\frac{\pi^2 E}{(l/k)^2} \le 330 \times 10^6$$

k = least radius of gyration

$$\frac{\pi^2 \times 210 \times 10^9}{330 \times 10^6} \le \left(\frac{l}{k}\right)^2$$

$$\left(\frac{l}{k}\right)^2 \ge 6280.66$$

$$\frac{l}{k} \ge 79.25$$

$\therefore$ For Euler's formula to be valid, slenderness ratio should be ≥ 79.25 //

Case II

When one end fixed & other end free

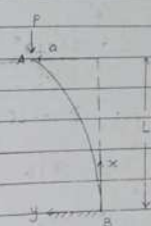
Here, column is fixed at B and free at A.

Let the maximum deflection at A be 'a'.

Taking B as the origin of co-ordinates, the bending moment at any point D is given by $P(a-y)$ and is hogging.

Thus, $M = P(a-y)$

$$EI \frac{d^2y}{dx^2} = P(a-y)$$



$$\frac{d^2y}{dx^2} + \frac{P}{EI} y = \frac{Pa}{EI}$$

$$\frac{d^2y}{dx^2} + \mu^2 y = \mu^2 a \quad \text{--- (1)}$$

Its sol^n is given by

$$y = a + c_1 \sin \mu x + c_2 \cos \mu x$$

where, $\mu = \sqrt{\dfrac{P}{EI}}$

Using boundary condition,

$x = 0$, $y = 0$ & $\dfrac{dy}{dx} = 0$

~~$x = L$, $y = a$~~

$\therefore$ $\dfrac{dy}{dx} = c_1 \mu \cos \mu x - c_2 \mu \sin \mu x$

$c_1 = 0$

Thus, the deflected shape is given by

$$y = a + c_2 \cos \mu x$$

Using boundary $cond^n$ at B, $x = L$, $y = a$

$$0 = a + c_2$$

$$c_2 = -a$$

Also, $a = a - a\cos \mu L$

i.e. $a \cos \mu L = 0$

If $a = 0$, the column will not buckle & remains straight, therefore μL should an odd multiple of $\pi/2$.

$$\therefore \boxed{P = \frac{\pi^2 EI}{4l^2}}$$

Hence, the effective length of column is double of its actual length.

Case III

When one end of the column is fixed and other end pinned / hinged

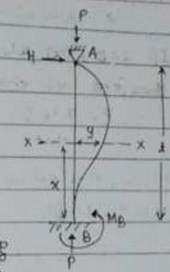


Fig. shows a column AB of length l, whose upper end A is hinged while its lower end is fixed.

Let P be the crippling load. Studying the nature of bending, we realize that there will be a restraint moment M_B at the lower fixed end. The existence of restraint moment therefore justifies the need for a horizontal force also at the top end A without which no BM can occur at B.

Hence, the hinge at A must exert a horizontal force H at A.

Consider any section XX at a distance x from the lower fixed end B. The BM at the section is given by

$$EI \frac{d^2y}{dx^2} = -Py + H(l-x)$$

$$EI \frac{d^2y}{dx^2} + Py = H(l-x)$$

The solution of above differential eq^n is

$$y = c_1 \cos\left(x\sqrt{P/EI}\right) + c_2 \sin\left(x\sqrt{P/EI}\right) + H/P(l-x) \quad \text{①}$$

The slope at any section is given by

$$\frac{dy}{dx} = -c_1\sqrt{P/EI}\,\sin\left(x\sqrt{P/EI}\right) + c_2\sqrt{P/EI}\,\cos\left(x\sqrt{P/EI}\right) - H/P \quad \text{--- (2)}$$

At B, the deflection is zero

$\therefore$ At $x=0,\ y=0$

From (1), $\therefore\ 0 = c_1 + H/P\, l$

$c_1 = -H/P\, l$

At B, the slope is zero

$\therefore$ At $x=0,\ dy/dx = 0$

From (2), $0 = c_2\sqrt{P/EI} - H/P$

$c_2 = H/P\sqrt{EI/P}$

At A, the deflection is zero

$\therefore$ At $x=l,\ y=0$

From (1), $0 = -\frac{H}{P}\, l\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right)$.

Simplifying, we get

$$\tan\left(l\sqrt{\frac{P}{EI}}\right) = l\sqrt{\frac{P}{EI}}$$

or, $l\sqrt{\frac{P}{EI}} = 4.5$ radians

or, $\frac{l^2 P}{EI} = (4.5)^2 = 20.25$

or, $P = \frac{20.25\,EI}{l^2}$

$\therefore$ $\boxed{P = \frac{2\pi^2 EI}{l^2}}$ [Approximately $20.25 = 2\pi^2$]

Case IV

When both ends of the column are fixed

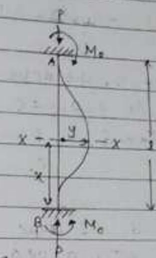


Fig. shows a column AB of length l whose both ends A & B are fixed. Obviously there will be restraint moment say M_0 at each end. Let P = crippling load

Considering any section XX at a distance x from the lower end B. The BM at the section XX is given by

$$EI\frac{d^2y}{dx^2} = M_0 - Py$$

$$EI\frac{d^2y}{dx^2} + Py = M_0$$

$$\therefore\ \frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_0}{EI}$$

The solution of above diff eqn is

$$y = c_1\cos\left(x\sqrt{\frac{P}{EI}}\right) + c_2\sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P} \quad \text{--- (1)}$$

The slope of any section is given by

$$\frac{dy}{dx} = -c_1\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + c_2\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) \quad \text{--- (2)}$$

$\rightarrow$ At B, the deflection is zero

$\therefore$ At $x=0,\ y=0$

From ①,

$$0 = C_1 + \frac{M_0}{P} \Rightarrow C_1 = \frac{-M_0}{P}$$

→ At B, the slope is zero

$\therefore$ At $x = 0$, $\frac{dy}{dx} = 0$

From ②,

$$0 = C_2 \sqrt{\frac{P}{EI}} \Rightarrow C_2 = 0$$

→ At A, deflection is zero

$\therefore$ At $x = l$, $y = 0$

From ③,

$$0 = -\frac{M_0}{P} \cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

$$\frac{M_0}{P}\left[1 - \cos\left(l\sqrt{\frac{P}{EI}}\right)\right] = 0$$

$$\cos\left(l\sqrt{\frac{P}{EI}}\right) = 1$$

$$\therefore \; l \times \sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi, 6\pi, \ldots$$

Considering the first practical value,

$$l\sqrt{\frac{P}{EI}} = 2\pi$$

$$\frac{l^2 P}{EI} = 4\pi^2$$

$$\therefore \boxed{P = \frac{4\pi^2 EI}{l^2}}$$

Case II

When one end of the column is fixed and other end free

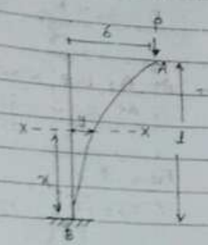


Fig. shows a column AB of length l whose lower end B is fixed, the upper end A being free. Let due to crippling load P, the column just buckle. Let δ be the deflection at the top end.

At any section xx distant x from the fixed end B, the BM is given by

$$EI\frac{d^2y}{dx^2} = P(\delta - y)$$

$$EI\frac{d^2y}{dx^2} + Py = P\delta$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{P\delta}{EI}$$

The solution of above diff. eqn. is

$$y = C_1\cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2\sin\left(x\sqrt{\frac{P}{EI}}\right) + \delta \quad \text{—— ①}$$

→ At B, the deflection is zero

$\therefore$ At $x = 0$, $y = 0$

$$0 = C_1 + \delta$$

$$\Rightarrow C_1 = -\delta$$

The slope at any section is given by

$$\frac{dy}{dx} = -c_1\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + c_2\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) \quad \text{--- (2)}$$

→ At B, the slope is zero

$\therefore$ At $x=0, \frac{dy}{dx}=0$

$$0 = c_2\sqrt{\frac{P}{EI}} \Rightarrow c_2 = 0$$

→ At A, the deflection is δ

$\therefore$ At $x=l, y=\delta$

From (1),

$$\delta = -\delta\cos\left(l\sqrt{\frac{P}{EI}}\right) + \delta$$

$$\cos\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

$$\therefore l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

Considering the first practical value,

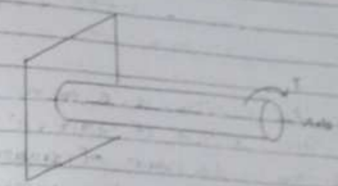
$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$$

$$\frac{l^2 P}{EI} = \frac{\pi^2}{4}$$

$$\therefore \boxed{P = \frac{\pi^2 EI}{4l^2}}$$

Chapter 7: Torsion

Torsion is a twisting moment (T) or couple or torque which tends to rotate the plane perpendicular to longitudinal axis.

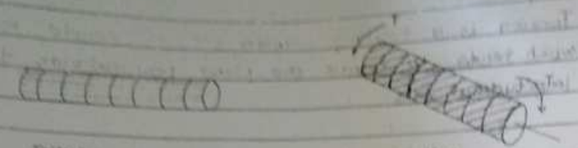


Imp

Derivation of Torsional equation (circular shaft):

Assumptions:

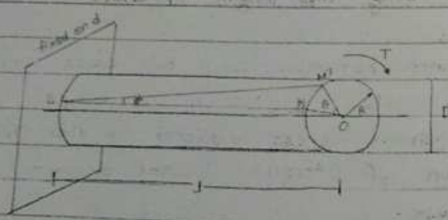
1) Material of shaft is uniform throughout
2) The shaft circular in section remains circular after bending/twisting.
3) The plane section of shaft normal to its axis before loading remains plane after the torque have been applied.
4) The twist along the length of shaft is uniform throughout.
5) The distance between any two cross section remains the same after application of torque.
6) Maximum shear stress induced in the shaft due to application of torque doesnot exceed its elastic limit value.

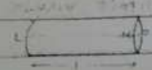


Initially   After twisting

To transmit energy by rotation, it is necessary to apply a turning force. In case of shaft, if force is applied tangentially and in the plane of transverse cross-section to the torque or twisting moment may be calculated by multiplying the force with radius of shaft. If the shaft is subjected to 2 opposite turning moments, it is said to be in pure torsion, and it will exhibit the tendency of shearing off at every cross section which is perpendicular to longitudinal axis.

e.g. Power transmission shaft.
Axles
Ship propellar shaft, etc.





Let T = maximum twisting torque / torsional strength
D = diameter of shaft
I_p = polar moment of inertia
τ = maximum shear stress
C = shear modulus / modulus of rigidity (G)
θ = Angle of twist in radian
l = length of shaft

Let the shaft be fixed at a point and a torque or twisting moment (T) is applied at another and If a line LM is drawn on a shaft before application of torque, the fiber will be twisted by ϕ. The cross-section will be twisted by angle θ and surface by ϕ.

Shear strain $\phi = \frac{MM'}{aL}$

$$\phi = \frac{MM'}{l} \quad \text{———} ①$$

Also,

Shear modulus $(C) = \frac{\text{Shear stress}}{\text{Shear strain}}$

$$C = \frac{\tau}{\phi}$$

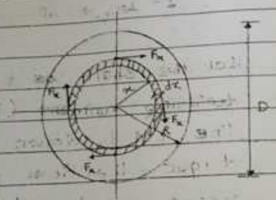
$$\Rightarrow \phi = \frac{\tau}{C} \quad \text{———} ②$$

From ① & ②,

$$\frac{MM'}{l} = \frac{\tau}{C} \qquad \theta = \frac{l}{R} = \frac{MM'}{R}$$

$$\frac{R\theta}{l} = \frac{\tau}{G} \qquad \Rightarrow MM' = R\theta$$

$$\boxed{\frac{C\theta}{l} = \frac{\tau}{R}} \quad ③$$



Let us consider an elementary area having width 'dx' at a distance 'x' from centre.

Let τ_x be the stress developed at a distance of x. τ be the maximum stress developed at distance R.

Then,

$$\frac{\tau}{R} = \frac{\tau_x}{x} \Rightarrow \tau_x = \frac{\tau x}{R}$$

The turning force on elementary ring

$$= \underset{\text{stress}}{\tau_x} \cdot \underset{\text{area}}{2\pi x\,dx}$$

Turning moment due to this turning force

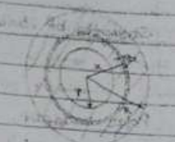
$$dT = \tau_x \cdot 2\pi x\,dx \cdot x$$

To get total turning moment, integrating above moment,

$$\int dT = \int_0^R \tau_x\, 2\pi x^2\,dx$$

$$= 2\pi \int_0^R \frac{\tau x}{R} \cdot x^2\,dx$$

\# In case of hollow shaft, integrate from r to R



$$= \frac{2\pi\tau}{R} \int_0^R x^3\,dx$$

$$= \frac{2\pi\tau}{R}\left[\frac{x^4}{4}\right]_0^R$$

$$= \frac{2\pi\tau}{R} \cdot \frac{R^4}{4}$$

$$= \frac{\pi\tau R^3}{2} = \frac{\pi\tau D^3}{16}$$

$$\therefore \boxed{T = \frac{\pi\tau D^3}{16}}$$

T = strength of shaft

Polar moment of inertia

$$I_p = \frac{\pi D^4}{32} = \frac{\pi D^3}{16} \cdot \frac{D}{2}$$

$$I_p = I_{xx} + I_{yy} = \frac{\pi D^4}{64} + \frac{\pi D^4}{64} = \frac{\pi D^4}{32}$$

$$I_p = \frac{T}{\tau} \cdot \frac{D}{2}$$

$$I_p = \frac{T}{\tau} \cdot R$$

$$\therefore \boxed{\frac{\tau}{R} = \frac{T}{I_p}} \quad ④$$

From ③ & ④ / combining ③ & ④,

$$\boxed{\frac{\tau}{R} = \frac{C\theta}{l} = \frac{T}{I_p}}$$

This is called torsional equation.

Note:

Strength of solid shaft $T_S = \frac{\pi}{16} \tau D^3$

Strength of hollow shaft $T_H = \frac{\pi}{16} \tau \left[\frac{D^4 - d^4}{D}\right]$

Polar moment of inertia of solid shaft

$$I_P = \frac{\pi D^4}{32}$$

Polar moment of inertia of hollow shaft

$$I_P = \frac{\pi}{32}\left[D^4 - d^4\right]$$

Power transmitted by shaft (P)

$$P = \frac{2\pi N}{60} T \text{ (Watt)}$$

P = power in watt

T = mean torque (Nm)

N = no. of revolution per minute (rpm)

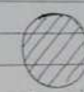
Imp Q. A solid cylindrical shaft is to transmit 300 kW at 100 rpm.

i) If the shear stress is not to exceed 80 MN/m², find its diameter.

ii) What percentage saving in weight would be obtained if this shaft is replaced by a hollow one whose internal diameter equals 0.6 of the external diameter, the length, the material & maximum shear stress being same?

Soln.

$P = 300 \text{ kW} = 300 \times 10^3 \text{ W}$



D_s

$N = 100$ rpm

$\tau = 80 \text{ MN/m}^2$ (max. shear stress)

$= 80 \times 10^6 \text{ N/m}^2$

i) ⇒ diameter (D) = ?

Power transmitted by shaft

$$P = \frac{2\pi N}{60} T$$

or, $300 \times 10^3 = \frac{2\pi \times 100}{60} \times T$

$\therefore T_{mean} = 28647.89$ Nm

Assuming $T_{mean} = T_{max}$ (τ)

Strength of solid shaft $T = \frac{\pi}{16} \tau D_s^3$

or, $28647.89 = \frac{\pi}{16} \times 80 \times 10^6 \times D_s^3$

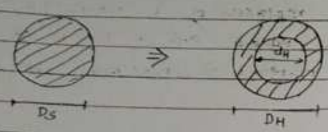
$\therefore D_s = 0.122$ m

$= 122$ mm //

ii) ⇒ ~~internal diameter $(d_i) = 0.6 \times D_H$~~

~~$= 0.6 \times 0.122$~~

~~$= 0.0732\ m$~~



$d_H = 0.6 D_H$

$J_S = J_H$

$C_S = C_H$

$\tau_S = \tau_H = \tau$

if power transmitted by both shafts are equal,

$P_S = P_H$

$\frac{2\pi N\, T_S}{60} = \frac{2\pi N\, T_H}{60}$

$T_S = T_H$

$\frac{\pi}{16} \tau_S D_S^3 = \frac{\pi}{16} \tau_H \left[\frac{D_H^4 - d_H^4}{D_H} \right]$

$D_S^3 = \frac{D_H^4 - (0.6 D_H)^4}{D_H}$

$D_S^3 = \frac{0.8704\, D_H^{4\,3}}{D_H}$

$D_S = 0.955\, D_H$

$0.122 = 0.955\, D_H$

⇒ $D_H = 0.128\ m$

⇒ $d_H = 0.0768\ m$ $\boxed{\because d_H = 0.6 D_H}$

Percentage saving in weight

$= \left(\frac{W_S - W_H}{W_S} \right) \times 100\%$

$= \left(1 - \frac{W_H}{W_S} \right) \times 100\%$

$= \left(1 - \frac{\gamma_H V_H}{\gamma_S V_S} \right) \times 100\%$ $\boxed{\text{same material } \gamma_H = \gamma_S}$

$= \left(1 - \frac{A_H l_H}{A_S l_S} \right) \times 100\%$

$= \left(1 - \frac{\frac{\pi}{4}(D_H^2 - d_H^2)}{\frac{\pi}{4} D_S^2} \right) \times 100\%$

$= \left(1 - \frac{0.128^2 - 0.0768^2}{0.122^2} \right) \times 100\%$

$= 29.6\ \%$ //

2010 Q. A solid shaft of 150 mm diameter is proposed to be replaced by a hollow shaft of external diameter D. If the same power is to be transmitted at the same speed & same level of shear stress, determine external diameter D. Take internal diameter as half of the external.

Soln.

$D_S = 150\ mm$

$D_H = D = ?$ (external diameter)

$d_H = \frac{1}{2} D$

Same power is transmitted by both shafts

$P_S = P_H$

$\frac{2\pi N_s}{60} T_s = \frac{2\pi N_H}{60} T_H$ (same speed i.e. $N_s = N_H$)

$\frac{\pi}{16} T_S D_S^3 = \frac{\pi}{16} T_H \left[\frac{D_H^4 - d_H^4}{D_H}\right]$ (same shear stress i.e. $T_S = T_H$)

$D_S^3 = \frac{D^4 - (0.5D)^4}{D}$

$150^3 = 0.9375\, D^3$

$\cancel{6750000}$ 3600000 $= D^3$

$\therefore D = \cancel{188.99}$ 153.3 mm is the required external diameter //

$d_H = \frac{1}{2} D = 76.65$ mm //

2013 Q8. Determine the diameter of solid shaft which will transmit 440kW at 280 rpm. The angle of twist must not exceed one degree per meter length and max. torsional shear stress is to be limited to 40 N/mm². Assume G = 84 kN/mm².

Soln

diameter of solid shaft (D_s) = ?

$P = 440$ kW $= 440 \times 10^3$ W

$N = 280$ rpm

max. torsional shear stress (τ) $= 40$ N/mm² $= 40 \times 10^6$ N/m²

(G) $G = 84$ kN/mm² $= 84 \times 10^9$ N/m²

length (l) = 1m

Strength of solid shaft

$T = \frac{60P}{2\pi N}$ $\left(\because P = \frac{2\pi N T}{60}\right)$

$= \frac{60 \times 440 \times 10^3}{2\pi \times 280}$

$= 15006$ Nm

Angle of twist $(\theta) = 1° = \left(\frac{\pi}{180} \times 1\right)$ rad $= 0.0175$ rad

Case I:

Shear stress is limited to 40 N/mm²

$T = \frac{\pi}{16} \tau D^3$

$15006 = \frac{\pi}{16} \times 40 \times 10^6 \times D^3$

$\therefore D = 0.124$ m (124 mm)

Case II:

Angle of twist is limited to 1° per meter

We have,

$\frac{T}{I_p} = \frac{C\theta}{l}$ [torsional eqⁿ]

$\frac{15006}{I_p} = 84 \times 10^9 \times 0.0175$

$I_p = 1.021 \times 10^{-5}$

$\frac{\pi D^4}{32} = 1.021 \times 10^{-5}$

$D_4 = 0.100$ m (100 mm)

Hence, adopt maxm of above two diameters.
i.e. D = 124 mm //

010
Q. A hollow circular shaft 10 m long is required to transmit 130 kW power. When running at a speed of 800 rpm. If the max. ~~stress~~ shearing stress allowed in the shaft is 100 N/mm² and the ratio of inner to outer diameter is 0.5, find the dimension of the shaft and also the angle of twist. Take G = 80 kN/mm²

Soln.

Length $(l) = 10m$

Power $(P) = 130 \times 10^3$ W

$N = 300$ rpm

max. shearing stress $(\tau) = 100 \times 10^6 \ N/m^2$

$\frac{\text{inner diameter } (d)}{\text{outer diameter } (D)} = 0.5$

dimension of the shaft = ? (D, d)

angle of twist (θ) = ?

$G = 80 \times 10^9 \ N/m^2$ (C)

Strength of shaft is

$$T_H = \frac{60P}{2\pi N} = \frac{60 \times 130 \times 10^3}{2\pi \times 300}$$

$$= 4138.03 \ Nm$$

$$T_H = \frac{\pi \tau}{16}\left[\frac{D^4 - d^4}{D}\right]$$

$$4138.03 = \frac{\pi}{16} \times \cancel{4138.03}^{100 \times 10^6} \left[\frac{D^4 - (0.5D)^4}{D}\right]$$

$$4138.03 = \frac{\cancel{9362126.2}\ \cancel{812.5}}{812.5} \times 0.9375 \frac{D^{\cancel{4}3}}{\cancel{D}}$$

D = ·

$$T_H = \frac{\pi}{16}\tau\left[\frac{D^4 - (0.5D)^4}{D}\right]$$

$$4138.03 = \frac{\pi}{16} \times 100 \times 10^6 \times 0.9375 \frac{D^{\cancel{4}3}}{\cancel{D}}$$

$\Rightarrow D = 0.0608$ m $= 60.8$ mm

$d = 0.5D = 0.0304$ m $= 30.4$ mm

Now,

$$\frac{T}{I_P} = \frac{C\theta}{l}$$

$$\frac{4138.03}{\frac{\pi}{32}[D^4 - d^4]} = \frac{80 \times 10^9 \times \theta}{10}$$

$$\frac{4138.03 \times 32}{\pi(0.0608^4 - 0.0304^4)} = \frac{80 \times 10^9 \times \theta}{10}$$

$\Rightarrow \theta = 0.4112$ rad

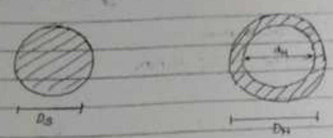
$$\theta = \frac{180}{\pi} \times 0.4112 \qquad \pi^c = 180^\circ$$

$$= 23.564^\circ \ //$$

Comparison of solid & hollow shaft

a) Comparison by strength

Cond^n : same cross sectional area



Consider a solid shaft of diameter 'D_S' and a hollow shaft having internal diameter d_H & external diameter 'D_H'.

In this case, it is assumed that both the shafts have same length, material, same weight & hence the same max shear stress.

Let D_S = diameter of solid shaft

D_H = diameter of hollow shaft (external)

d_H = internal diameter of hollow shaft

A_S = cross-sectional area of solid shaft

A_H = cross sectional area of hollow shaft

T_S = Torque transmitted by the solid shaft

T_H = Torque transmitted by the hollow shaft

Now,

$$T_S = \frac{\pi}{16} \tau D_S^3 \qquad T_H = \frac{\pi}{16} \tau \left[\frac{D_H^4 - d_H^4}{D_H} \right]$$

$$\frac{\text{strength of hollow shaft}}{\text{strength of solid shaft}} = \frac{T_H}{T_S} = \frac{\frac{\pi}{16} \tau \left\{ \frac{D_H^4 - d_H^4}{D_H} \right\}}{\frac{\pi}{16} \tau D_S^3}$$

$$\frac{T_H}{T_S} = \frac{D_H^4 - d_H^4}{D_H \cdot D_S^3} \quad \text{——} ①$$

Let $\frac{D_H}{d_H} = x$

$$D_H = x d_H$$

Substituting it in ①, we get

$$\frac{T_H}{T_S} = \frac{x^4 d_H^4 - d_H^4}{x d_H \cdot D_S^3} = \frac{d_H^4 (x^4 - 1)}{x d_H D_S^3} = \frac{d_H^3 (x^4 - 1)}{x D_S^3} \quad \text{——} ②$$

As the weight, length & material of both the shafts are same,

$$A_S = A_H$$

$$\frac{\pi D_S^2}{4} = \frac{\pi (D_H^2 - d_H^2)}{4}$$

$$D_S^2 = D_H^2 - d_H^2$$

$$D_S = \sqrt{D_H^2 - d_H^2}$$

$$D_S^3 = (D_H^2 - d_H^2)\sqrt{D_H^2 - d_H^2}$$

$$D_S^3 = (x^2 d_H^2 - d_H^2)\sqrt{x^2 d_H^2 - d_H^2}$$

$$D_S^3 = d_H^3 (x^2 - 1)\sqrt{x^2 - 1} \quad \text{——} ③$$

Substituting this value in ②, we get

$$\frac{T_H}{T_S} = \frac{d_H^3 (x^4 - 1)}{x\, d_H^3 (x^2 - 1)\sqrt{x^2 - 1}}$$

$$= \frac{(x^2 - 1)(x^2 + 1)}{x (x^2 - 1)\sqrt{x^2 - 1}}$$

$$= \frac{x^2 + 1}{x\sqrt{x^2 - 1}}$$

We have, $D_H > d_H$ & $\frac{D_H}{d_H} > x$

It's sure that $x > 1$

Let $x = 2$

Then, $\frac{T_H}{T_S} = \frac{2^2+1}{2\sqrt{4-1}} = 1.44$

$\therefore$ This shows that the torque produced by hollow shaft is greater than solid shaft. Hence, the hollow shaft is stronger than the solid shaft.

b) comparison by weight

Case: strength of shafts are equal

In this case, it is assumed that both the shafts have same length & material. Now, if the torque applied to both the shafts is same, then the maximum shear stress will also be same in both the cases.

Now,

$$\frac{\text{Weight of hollow shaft}}{\text{Weight of solid shaft}} = \frac{W_H}{W_S} = \frac{A_H}{A_S}$$

$$= \frac{\frac{\pi}{4}(D_H^2 - d_H^2)}{\frac{\pi}{4} D_S^2}$$

$$\frac{W_H}{W_S} = \frac{D_H^2 - d_H^2}{D_S^2} \quad \text{——} ①$$

Let $\frac{D_H}{d_H} = x$

$D_H = x d_H$

Substituting this value in ①, we get

$$\frac{W_H}{W_S} = \frac{x^2 d_H^2 - d_H^2}{D_S^2} = \frac{d_H^2(x^2-1)}{D_S^2} \quad \text{——} ②$$

$T_S = T_H$

$$\frac{\pi}{16}\tau D_S^3 = \frac{\pi}{16}\tau\left[\frac{D_H^4 - d_H^4}{D_H}\right]$$

$$D_S^3 = \frac{D_H^4 - d_H^4}{D_H}$$

$$= \frac{x^4 d_H^4 - d_H^4}{x d_H}$$

$$= \frac{d_H^{\cancel{4}3}(x^4-1)}{x \cancel{d_H}}$$

$$D_S = d_H\left[\frac{x^4-1}{x}\right]^{1/3}$$

$$D_S^2 = d_H^2\left[\frac{x^4-1}{x}\right]^{2/3} \quad \text{——} ③$$

Substituting the value of D_S^2 in ③, we have

$$\frac{W_H}{W_S} = \frac{\cancel{d_H^2}(x^2-1)}{\cancel{d_H^2}\left[\frac{x^4-1}{x}\right]^{2/3}} = \frac{(x^2-1)x^{2/3}}{(x^4-1)^{2/3}} \quad \text{——} ③$$

if $x = 2$, then

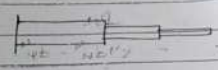
$$\frac{W_H}{W_S} = \frac{(2^2-1)\times 2^{2/3}}{(2^4-1)^{2/3}} = 0.7829$$

$\therefore$ Weight of hollow shaft is lesser than solid shaft. So, hollow shafts are economical compared to solid shafts as regards torsion.

Shafts in series

In such cases, each shaft transmits the same torque. The angle of twist is sum of angle of twist of two shafts connected in series.

- Total angle of twist $\theta = \theta_1 + \theta_2 + \theta_3 + \dots + \theta_n$
- Since two shafts transmits same torque $T_1 = T_2$



- From torsional equation,

$$\frac{T}{I_p} = \frac{\tau}{R} = \frac{C\theta}{l}$$

$$\Rightarrow \theta = \frac{Tl}{CI_p}$$

In case of shaft in series

$$\theta = \theta_1 + \theta_2 = \frac{T_1 l_1}{C_1 I_{p_1}} + \frac{T_2 l_2}{C_2 I_{p_2}}$$

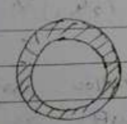
$$\theta = T\left[\frac{l_1}{C_1 I_{p_1}} + \frac{l_2}{C_2 I_{p_2}}\right] \quad \text{as } T_1 = T_2 = T$$

Shafts in parallel

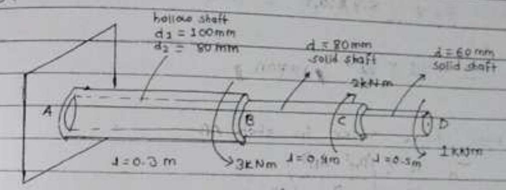
Angle of twist is same for each shaft, but applied torque is divided between the two shafts.

i.e. $\theta_1 = \theta_2$

$T = T_1 + T_2$

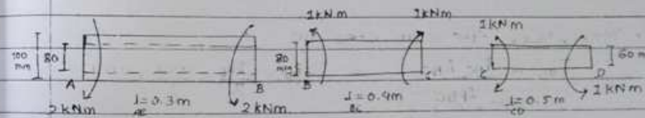


Q. A stepped shaft is subjected to torque as shown in fig. Determine the angle of twist at the free end. Take $G = 80\ kN/mm^2$. Also find max. shear stress in any step.



Soln.

Free body diagram



Since, shafts are in series, angle of twist at free end

$$\theta = \theta_{AB} - \theta_{BC} + \theta_{CD} \quad (\text{Angle of twist is -ve for clockwise torque})$$

$$= \left[\frac{T_{AB} l_{AB}}{C I_{PAB}} - \frac{T_{BC} l_{BC}}{C I_{PBC}} + \frac{T_{CD} l_{CD}}{C I_{PCD}}\right]$$

$$= \frac{1}{C}\left[\frac{T_{AB} l_{AB}}{I_{PAB}} - \frac{T_{BC} l_{BC}}{I_{PBC}} + \frac{T_{CD} l_{CD}}{I_{PCD}}\right] \quad \text{--- (1)}$$

$$I_{PAB} = \frac{\pi}{32}\left[D_H^4 - d_H^4\right]_{AB} = \frac{\pi}{32}\left[0.100^4 - 0.08^4\right] = 5.7 \times 10^{-6}\ m^4$$

$$I_{PBC} = \frac{\pi}{32}\left[D_S^4\right]_{BC} = \frac{\pi}{32} \times 0.08^4 = 4.02 \times 10^{-6}\ m^4$$

$$I_{PCD} = \frac{\pi}{32}\left[D_S^4\right]_{CD} = \frac{\pi}{32} \times 0.06^4 = 1.27 \times 10^{-6}\ m^4$$

$C = G = 80\ kN/mm^2 = 8 \times 10^7\ kN/m^2$

Putting these values in ①,

$$\theta = \frac{1}{8 \times 10^7}\left[\frac{2 \times 0.3}{5.7 \times 10^{-6}} + \frac{1 \times 0.4}{4.02 \times 10^{-6}} + \frac{1 \times 0.5}{1.27 \times 10^{-6}}\right]$$

$$= 4.99 \times 10^{-3} \text{ radian}$$

- Max. shear stress in step AB

$$T_H = \frac{\pi}{16} \tau \left(\frac{D_H^4 - d_H^4}{D_H}\right)_{AB}$$

$$\tau = \frac{16 T_H D_H}{\pi (D_H^4 - d_H^4)} = \frac{16 \times 2 \times 0.1}{\pi (0.1^4 - 0.08^4)} = 17252.56\ kN/m^2 = 17.25\ N/mm^2$$

- Max. shear stress in step BC

$$\tau_{BC} = \frac{T_{BC} R_{BC}}{I_{PBC}} \qquad R_{BC} = \frac{D_{BC}}{2} = \frac{0.08}{2} = 0.04$$

$$= \frac{1 \times 0.04}{4.02 \times 10^{-6}}$$

$$= 9950.24\ kN/m^2$$

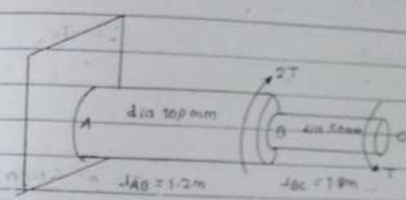
Max. shear stress in step CD

$$\tau_{CD} = \frac{T_{CD} R_{CD}}{I_{PCD}}$$

$$= \frac{1 \times 0.03}{1.27 \times 10^{-6}}$$

$$\tau_{max} = 23622.04\ kN/m^2$$

6. The stepped steel shaft shown in fig is subjected to torque T at the free end and torque 2T in the opposite direction at the junction of 2 sizes, what is the total angle of twist at the free end, if the max. shear stress in the shaft is limited to 70 MN/m² ? Assume the modulus of rigidity to be 84 GN/m².



Soln.

Since shafts are in series, total angle of twist at free end is

$$\theta = \theta_{AB} - \theta_{BC} \quad ①$$ [Angle of twist is -ve for anticlockwise torque]

Max. shear stress in shaft (τ) = 70 MN/m² = $70 \times 10^6\ N/m^2$

$$\Rightarrow \tau_{AB} = \frac{T_{AB} R_{AB}}{I_{PAB}} \qquad I_{PAB} = \frac{\pi D_{AB}^4}{32} = \frac{\pi \times 0.1^4}{32} = 9.81 \times 10^{-6}$$

$$70 \times 10^6 = \frac{T_{AB} \times 0.05}{9.81 \times 10^{-6}}$$

$$= 13734\ Nm$$

$$\Rightarrow \tau_{BC} = \frac{T_{BC} R_{BC}}{I_{PBC}}$$

$$I_{PBC} = \frac{\pi}{32} D_{BC}^4 = \frac{\pi}{32} \times 0.05^4 = 6.13 \times 10^{-7}$$

$$70 \times 10^6 = \frac{T_{BC} \times 0.025}{6.13 \times 10^{-7}}$$

$$T_{BC} = 1716.4 \text{ Nm}$$

If $T_{AB} > T_{BC}$ so,

$$T_{AB} = T = 13734 \text{ Nm}$$

If $T_{BC} > T_{AB}$,

$$T_{BC} = T$$

From ①,

$$\theta = \frac{T_{AB} l_{AB}}{C I_{PAB}} - \frac{T_{BC} l_{BC}}{C I_{PBC}}$$

$$C = 84 \text{ GN/m}^2 = 84 \times 10^9 \text{ N/m}^2$$

$$= \frac{1}{84 \times 10^9} \left[\frac{13734 \times 1.2}{9.81 \times 10^{-6}} - \frac{13739 \times 1.8}{6.13 \times 10^{-7}} \right]$$

$$= -0.46 \text{ rad}$$

$$= -26.36^\circ$$ //

Assuming $T_{AB} = T$

$$\theta = \left[\frac{13739 \times 1.2}{9.81 \times 10^{-6}} - \frac{13734 \times 1.8}{6.13 \times 10^{-7}} \right] \cdot \frac{1}{84 \times 10^9}$$

$$= -0.46 \text{ rad}$$

$$= -26.36^\circ$$

Assuming $T_{BC} = T$

$$\theta = \left[\frac{1716.4 \times 1.2}{9.81 \times 10^{-6}} - \frac{1716.9 \times 1.8}{6.13 \times 10^{-7}} \right] \cdot \frac{1}{84 \times 10^9}$$

$$= -0.05 \text{ rad}$$

$$= -3.29^\circ$$

Shafts in parallel

$$\theta_1 = \theta_2$$

$$T = T_1 + T_2$$

8. A composite shaft has an aluminium tube of external diameter 60 mm and internal diameter 40 mm closely fitted to a steel rod of 40 mm. If the permissible stress is 60 N/mm² in aluminium and 100 N/mm² in steel, find the maximum torque the composite section can take. Given $G_A = 27$ kN/mm² & $G_S = 80$ kN/mm²

Solⁿ

$D_A = 60$ mm

$d_A = 40$ mm

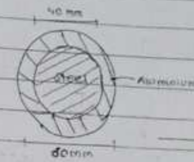
$D_S = 40$ mm

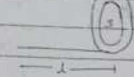
$\tau_A = 60$ N/mm²

$\tau_S = 100$ N/mm²

$G_A = 27$ kN/mm²

$G_S = 80$ kN/mm²





Since shafts are in parallel,

Angle of twist of aluminium = angle of twist of steel

$$\theta_A = \theta_S$$

$$\frac{T_A l_A}{G_A I_{PA}} = \frac{T_S l_S}{G_S I_{PS}}$$

$$I_{PA} = \frac{\pi}{32}\left[D_A^4 - d_A^4\right] = \frac{\pi}{32}\left[60^4 - 40^4\right] = 1021017.61 \text{ mm}^4$$

$$\frac{T_A}{27 \times 10^3 \times 1021017.61} = \frac{T_S}{80 \times 10^3 \times 251327.41}$$

$$\Rightarrow T_A = 1.37\, T_S \quad ①$$

$$I_{PS} = \frac{\pi D_S^4}{32} = \frac{\pi \times 40^4}{32} = 251327.41 \text{ mm}^4$$

Case I

If strength of shaft is governed by Aluminium,

$$\frac{T_A}{I_{PA}} = \frac{\tau_A}{R_A}$$

$$\frac{T_A}{1021017.61} = \frac{60}{30}^{2}$$

$T_A = 2042035.22$ Nmm

From ②,

$T_S = 1480536.6$

Torque developed $T = T_A + T_S$

$= 2042035.22 + 1490536.6$

$= 3532571.82$ Nmm

Case II

If strength of shaft is governed by steel,

$$\frac{T_S}{I_{PS}} = \frac{\tau_S}{R_S}$$

$$\frac{T_S}{251327.41} = \frac{100^5}{20}$$

$T_S = \cancel{1256862.05}$ Nmm

1256637.05

From ①,

$T_A = 1.37\, T_S$

$= \cancel{1721901.008}$ Nmm

1721592.759

Torque developed $T = T_A + T_S$

$= 2978229.8$ Nmm (min)

⇒ From above two values of T, strength of shaft is governed by steel. Hence,

Max. torque the composite section can take $T = 2978229.8$ Nmm

Chapter 8: Thin Walled Structures

Thin cylindrical shells (water supply pipes, [illegible])

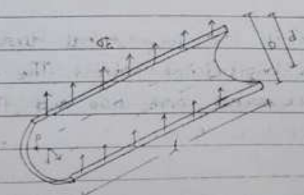
A cylindrical vessel or shells may be thin or thick depending upon thickness of plate in relation to internal diameter of cylinder.

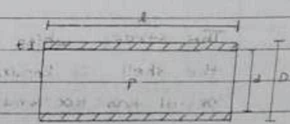
→ The ratio of $\frac{t}{d} = \frac{1}{20}$ can be considered as a suitable line of demarcation between thin & thick shells.

→ When thin cylinders are subjected to internal pressure, following stresses are developed.

1) Hoop or circumferential stress
2) Longitudinal stress
3) Radial stress

1) Hoop or circumferential stress





Those stress which act tangential direction to the circumference of shells is known as hoop stress and is denoted by σ_c. They are maximum at the inner circumference and minimum at the outer circumference. Since the thickness of shell is minimum, both of them are treated as same.

Let d = internal diameter of cylinder

t = thickness of cylinder

P = internal pressure developed due to flowing fluid

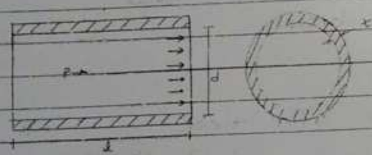
σ_c = circumferential / hoop stress

Here, Bursting pressure = Resisting strength

$$P \times d \times \cancel{l} = \sigma_c (l \times t + l \times t)$$

$$\boxed{\sigma_c = \frac{Pd}{2t}}$$

2) Longitudinal stress



The stress which acts in the longitudinal direction of the shell is known as longitudinal stress. The cylindrical vessel has two ends covered with two end flat plates rigidly connected to them.

Pressure at the end = Resisting force developed in the material in longitudinal direction

$$P \times \frac{\pi d^2}{4} = \sigma_l \times \underbrace{\pi d \times t}_{\text{(circumference × thickness)}}$$

$$\boxed{\sigma_l = \frac{Pd}{4t}} = \frac{\sigma_c}{2} \Rightarrow \sigma_c = 2\sigma_l$$

3) Radial stress

It is the stress acting in radial direction. Since its value is minimum and that can be neglected.

Relationship betn Hoop stress & Longitudinal stress

$$\boxed{\sigma_c = 2\sigma_l}$$

Therefore, the hoop/circumferential stress (σ_c) is twice as greater as the longitudinal stress (σ_l) and in no case, should the hoop stress is greater than permissible stress in material of the cylinder.

Q. A pipe ~~water~~ main 20cm diameter contains water at a pressure head of 100 m. If the wt density of water is 9810 N/m^3, find ~~and~~ the thickness of the metal required for water main. Given the longitudinal stress as 20 kN/mm^2

Soln

diameter (d) = 20 cm = 200 mm = 0.2 m

pressure head (h) = 100 m

wt. density of water (γ) = 9810 N/m^3

thickness of metal (t) = ?

longitudinal stress (σ_l) = 20 kN/mm^2 = 2×10^{10} N/m^2

$$\sigma_l = \frac{Pd}{4t} = \frac{\gamma h d}{4t} = \frac{9810 \times 100 \times 0.2}{4 \times t}$$

or, $2 \times 10^{10} = \frac{49050}{t}$

$\therefore$ $t = 2.45 \times 10^{-6}$ m //

Maximum shear stress

In a cylindrical shell, at any point there is set of two mutually perpendicular stresses σ_c & σ_l which are principal stresses & such plane in which these two stresses act is known as principal plane.

Maximum shear stress $(\tau_{max}) = \frac{\sigma_c - \sigma_l}{2}$ (proved in chapter 3)

$$= \left(\frac{Pd}{2t} - \frac{Pd}{4t}\right) / 2$$

$$\boxed{\tau_{max} = \frac{Pd}{8t}}$$

where,

σ_c = maximum principal stress

σ_l = minimum principal stress

Strain

Hoop strain: Circumferential or hoop strain shows the unit deformation of shell along the circumference.

$$\varepsilon_c = \frac{\sigma_c}{E} - \frac{\nu \sigma_l}{E} \qquad \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E} - \frac{\nu \sigma_z}{E}$$

$$= \frac{2\sigma_l}{E} - \frac{\nu \sigma_l}{E} \qquad \boxed{\because \sigma_c = 2\sigma_l}$$

$$= \frac{\sigma_l}{E}(2-\nu) \qquad \varepsilon_c = \frac{\delta c}{c} = \frac{\delta d}{d}$$

$$\boxed{\varepsilon_c = \frac{Pd}{4tE}(2-\nu)}$$

Longitudinal strain: This strain shows the unit deformation along the length of shell.

Volumetric strain $(\varepsilon_v) = \frac{Pd}{4tE}(5-4\nu)$

$$= \frac{25\times 800\,(5-4\times 0.25)}{4\times 6.67\times 2\times 10^5}$$

$$= 1.5\times 10^{-3}$$

$\cancel{\varepsilon_v = \frac{\partial d}{d}}$

⇒ ~~change in diameter~~ $\cancel{(\partial d) = d\times \varepsilon_v = 800\times 1.5\times 10^{-3}}$

$\cancel{= 1.2\ mm}$ //

$\varepsilon_v = \frac{\Delta V}{V}$

⇒ change in volume $(\Delta V) = V\times \varepsilon_v = \frac{\pi d^2}{4}\times l\times \varepsilon_v$

$$= \frac{\pi\times 800^2}{4}\times 3000\times 1.5\times 10^{-3}$$

$$= 2261946.7\ mm^3\ //$$

hoop strain $(\varepsilon_c) = \frac{Pd}{4tE}(2-\nu)$

$$= \frac{2.5\times 800\,(2-0.25)}{4\times 6.67\times 2\times 10^5}$$

$$= 6.56\times 10^{-4}$$

⇒ $\varepsilon_c = \frac{\partial d}{d}$

⇒ change in diameter $(\partial d) = d\times \varepsilon_c = 800\times 6.56\times 10^{-4}$

$= 0.525\ mm$ //

OR

$\varepsilon_v = 2\varepsilon_c + \varepsilon_l$

$\frac{\Delta V}{V} = 2\varepsilon_c + \varepsilon_l$ $\Delta V = ?$

iii) Max. intensity of shear stress developed

$$\tau_{max} = \frac{\sigma_c - \sigma_l}{2}$$

$$= \frac{\sigma_c - \sigma_c/2}{2}$$

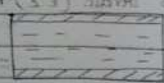
$$= \frac{150 - 150/2}{2}$$

$$= 37.5 \ N/mm^2$$

8. A cylindrical shell 90cm long and 20cm internal diameter having thickness of metal as 8mm is filled with fluid at atmospheric pressure. If an additional 20cm³ fluid is pumped into the cylinder, find
i) the pressure exerted by the fluid on the cylinder
ii) the hoop stress induced
Take $\nu = 0.3$ and $E = 200 \ GN/m^2$

Soln.

$1 \ atm = 1.01 \times 10^5 \ Pa \ (N/m^2) = 10.1 \ N/cm^2$



$l = 90 cm$

$d = 20 cm$

$t = 8 mm = 0.8 cm$

additional volume of fluid $(\delta V) = 20 \ cm^3$

$$E = 200 \ GN/m^2 = \frac{200 \times 10^9}{100 \times 100} = 2 \times 10^7 \ N/cm^2$$

i) $$\varepsilon_v = \frac{\delta V}{V} = \frac{20}{\frac{\pi \times 20^2}{4} \times 90} = 7.07 \times 10^{-4}$$

$$\varepsilon_v = \frac{Pd}{4tE}(5 - 4\nu)$$

$$7.07 \times 10^{-4} = \frac{P \times 20}{4 \times 0.8 \times 2 \times 10^7}(5 - 4 \times 0.3)$$

$$45248 = P \times 76$$

$$\Rightarrow P = 595.37 \ N/cm^2$$

ii) the hoop stress induced

$$\sigma_c = \frac{Pd}{2t} = \frac{595.37 \times 20}{2 \times 0.8} = 7442.13 \ N/cm^2$$

Chapter 4:

Degree of static indeterminacy

Indeterminacy
- Static
- Kinetic (Dynamic)

Static indeterminacy (D_s)

The number of additional equations necessary for the analysis of structure is k/a degree of static indeterminacy or the degree of redundancy of the structure. The total degree of static indeterminacy is the sum of following two ~~indeterm~~ indeterminacies.

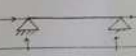
1) External static indeterminacy (D_{se})
2) Internal static indeterminacy (D_{si})

$D_s = D_{se} + D_{si}$

1) External indeterminacy → indeterminant due to support

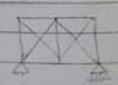
$D_{se} = r - 3$ (for $\overset{2D}{\text{plane}}$ structure)

$D_{se} = r - 6$ (for $\overset{3D}{\text{space}}$ structure)



2) Internal indeterminacy → indeterminant due to member

For pin jointed frame (truss) pin



a) For $\overset{3D}{\text{space}}$ pin jointed frame

$D_{si} = m - (3j - 6)$ → $D_s = (m+r) - 3j$

b) For $\overset{2D}{\text{plane}}$ pin jointed frame

$D_{si} = m - (2j - 3)$ → $D_s = (m+r) - 2j$

For rigid jointed frame

a) Rigid jointed space frame

$D_{si} = 6C$ → $D_s = (6m+r) - 6j$

b) Rigid jointed plane frame

$D_{si} = 3C$ → $D_s = (3m+r) - 3j$

Hybrid structure

a) Plane frame with hybrid joints

$D_{si} = 3C - \Sigma(m-1)$

no. of members emerging from pin joint

b) Space frame with hybrid joints

$D_{si} = 3C - \Sigma 3(m-1)$

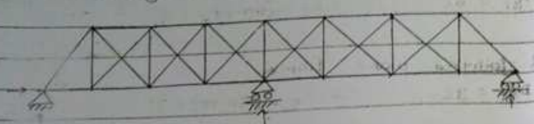
m = no. of members

r = no. of reactions

j = no. of joints

c = no. of closed loops

3. Determine the internal, external & total static indeterminacy of the following pin jointed truss.

1) 

Soln

$m = 35$ (no. of members)

$r = 4$

$j = 16$

External indeterminacy

$D_{se} = r - 3 = 4 - 3 = 1$

Internal indeterminacy

$D_{si} = m - (2j - 3)$

$= 35 - (2 \times 16 - 3)$

$= 6$

Total static indeterminacy

$D_s = D_{se} + D_{si}$

$= 1 + 6$

$= 7$

$D_s = (m + r) - 2j$

$= (35 + 4) - 2 \times 16$

$= 7$

Continue...

Chapter 10: Springs

Springs are elastic members which distort under load and regain their original shape when they are removed. They are used in railway carriers, motor cars, scooters, motorcycles, rickshaws, etc.

impact energy लाई absorb गर्ने काम गर्छ

Helical spring

A helical spring is a length of wire or bar wound into a helix. It may be of two types:

i) Close coiled

ii) Open coiled

Close-coiled helical spring with 'Axial load' Circular section

Let R = radius of coil

d = diameter of wire of coil

δ = deflection of coil under load W

C = modulus of rigidity

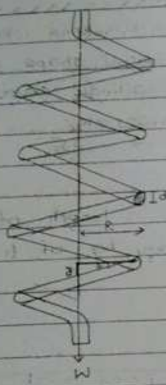
n = number of coils or turns

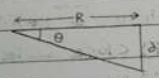
θ = angle of twist

l = length of wire $= 2\pi R n$

τ = shear stress

I_p = polar moment of inertia $= \frac{\pi d^4}{32}$





$\theta = \frac{\delta}{R}$

From torsional relation;

$$\frac{T}{I_p} = \frac{C\theta}{l} = \frac{\tau}{R'} \leftarrow \text{radius of wire}$$

$$\Rightarrow \frac{T}{I_p} = \frac{\tau}{R'}$$

$$T = \frac{\tau I_p}{R'} = \frac{\tau \pi d^4}{\frac{d}{2} \times 32} = \frac{\pi \tau d^3}{16}$$

$$\boxed{T = \frac{\pi \tau d^3}{16}}$$

$$\tau = \frac{16T}{\pi d^3} = \frac{16WR}{\pi d^3} \qquad (T = WR)$$

$$\Rightarrow \boxed{\tau = \frac{16WR}{\pi d^3}}$$ Max^m. shear stress

Deflection (θ)

$$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$\theta = \frac{Tl}{I_p C} = \frac{WR \times 2\pi R n}{\frac{\pi d^4}{32} \times C}$$

$$\Rightarrow \boxed{\theta = \frac{64WR^2 n}{Cd^4}}$$

Deflection of coil (δ)

$$\delta = \theta R$$

$$\Rightarrow \boxed{\delta = \frac{64WR^3 n}{Cd^4}}$$

Stiffness (K) $\Rightarrow$ Spring constant

$$K = \frac{W}{\delta} = \frac{W Cd^4}{64 \cancel{W} R^3 n}$$

$$\Rightarrow \boxed{K = \frac{Cd^4}{64R^3 n}}$$

8. A closely coiled helical spring is used to carry a load of 500 N. Its mean coiled diameter is to be 10 times that of wire diameter. Calculate these diameters if the max. stress shear stress in the material of the spring is to be 80 MN/m².

Solⁿ

load (W) = 500 N

mean coiled diameter (D) = 10 × wire diamter

$2R = 10d$

$R = 5d$

τ (max shear stress) = 80 MN/m² = 80×10^6 N/m²

We have,

$$\tau = \frac{16WR}{\pi d^3}$$

$$80 \times 10^6 = \frac{16 \times 500 \times 5\cancel{d}}{\pi d^{\cancel{3}\,2}}$$

$$d^2 = \frac{16 \times 500 \times 5}{\pi \times 80 \times 10^6}$$

$\therefore$ $d = 0.0126$ m $= 12.6$ mm

$D = 2R = 2 \times 5d = 10 \times 12.6 = 126$ mm //

Q. A helical spring is made of 12 mm diameter steel wire wound on a 120 mm diameter mandrel. If there are 10 active coils, what is spring constant? Take C = 82GN/m². What force must be applied to the spring to elongate it by 40 mm?

Soln.

diameter of wire (d) = 12 mm   n = 10

diameter of coil (D = 2R) = 120 mm   K = ?

deflection (δ) = 40 mm   R = 60 mm   C = 82 GN/m² = 82000 N/mm²

$$\text{Spring constant } (K) = \frac{Cd^4}{64R^3n}$$

$$= \frac{82000 \times 12^4}{64 \times 60^3 \times 10}$$

$= 12.3$ N/mm

$= 12300$ N/m

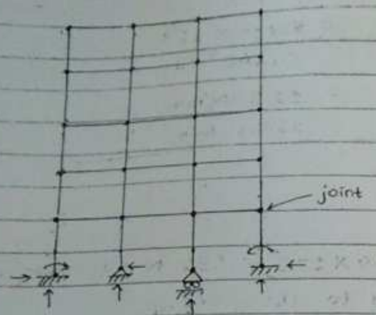
Again,

$$K = \frac{W}{\delta}$$

$\Rightarrow$ $W = \delta \times K = 40 \times 12.3 = 492$ N is the required force to be applied to the spring. //

Continue...

2) Rigid jointed plane frame



Sol^n.

$m = 35$

$r = 9$

$j = \cancel{20}\ 24$

$c = 12$

External indeterminacy

$D_{se} = r - 3 = 9 - 3 = 6$

Internal indeterminacy

$D_{si} = 3c = 3 \times 12 = 36$

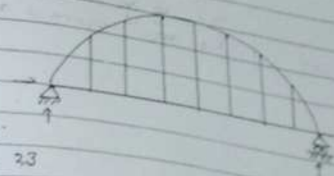
Total static indeterminacy

$D_s = D_{se} + D_{si}$ $\qquad$ $D_s = (3m + r) - 3j$

$= 6 + 36$ $\qquad$ $= (3 \times 35 + 9) - 3 \times 24$

$= 42$ $\qquad$ $= 42$ //

3) Rigid jointed plane frame



Sol^n.

$m = 23$

$r = 3$

$j = 16$

$c = 8$

→ $D_{se} = r - 3 = 3 - 3 = 0$

→ $D_{si} = 3c = 3 \times 8 = 24$

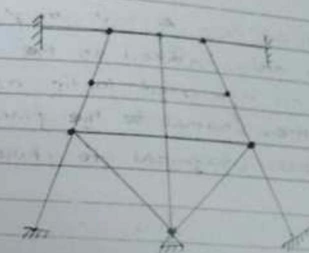
→ Total indeterminacy (static)

$D_s = D_{se} + D_{si}$ $\qquad$ $D_s = (3m + r) - 3j$

$= 0 + 24$ $\qquad$ $= (3 \times 23 + 3) - 3 \times 16$

$= 24$ $\qquad$ $= 24$ //

4) Hybrid structure



Sol^n.

$m = 16$

$r = 14$

$j = 14$

$c = 6$

→ $D_{se} = r - 3 = 14 - 3 = 11$

→ $D_{si} = 3c - \Sigma(m-1)$

$= 3 \times 6 - \{(3-1) + (3-1) + (4-1) + (3-1) + (3-1) + (4-1) + (3-1)\}$

$= 18 - 16$

$= 2$

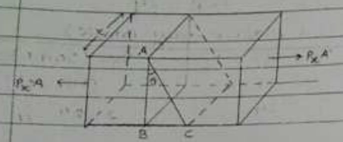
→ $D_s = D_{se} + D_{si} = 11 + 2 = 13$ //

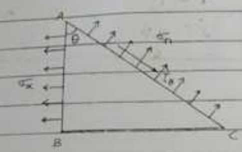
Chapter 5: Transformation of Stress & Strain
(15 marks)

1) Stress on an inclined plane under uniaxial loading

Consider a bar subjected to an axial force P_x along x-axis. Let A be the cross sectional area of the bar.

Value of average stress $(\sigma_x) = \frac{P_x}{A}$





If the bar is cut along AC, with an angle θ with line AB, the stresses induced in the inclined portion of the bar are represented in fig as

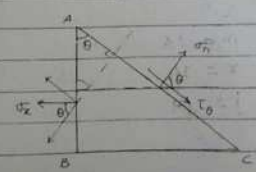
σ_n = normal stress (normal to the plane)

τ_θ = shear stress (tangential to plane)

Let A_θ be the cross sectional area of inclined plane.

$\cos\theta = \frac{A}{A_\theta}$

$\Rightarrow A = A_\theta \cos\theta$



(i) Σ Forces perpendicular to considered plane (normal to plane) $= 0$

$\sigma_n (AC \times t) - \sigma_x \cos\theta (AB \times t) = 0$

$\sigma_n = \sigma_x \cos\theta \cdot \frac{AB}{AC}$

$\sigma_n = \sigma_x \cos\theta \cdot \cos\theta$ $\left[\because \cos\theta = \frac{AB}{AC}\right]$

$\boxed{\sigma_n = \sigma_x \cos^2\theta}$ —— ①

(ii) Σ Forces along the considered plane (tangential direction) $= 0$

$\tau_\theta (AC \times t) - \sigma_x \sin\theta (AB \times t) = 0$

$\tau_\theta = \sigma_x \sin\theta \cdot \frac{AB}{AC}$

$= \sigma_x \sin\theta \cdot \cos\theta \times \frac{2}{2}$

$\boxed{\tau_\theta = \frac{\sigma_x}{2} \sin 2\theta}$ —— ②

→ For max^m normal stress (from ①)

$\sigma_{max} = \sigma_n = \sigma_x$ $(\theta = 0°)$

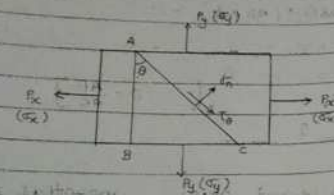
$\sigma_{min} = 0$ $(\theta = 90°)$

→ For max^m shear stress (from ②)

$\tau_{max} = \tau_\theta = \frac{\sigma_x}{2}$ $(\theta = 45°)$

$\tau_{min} = 0$ $(\theta = 0°)$

2) Stresses on an inclined plane subjected to biaxial loading

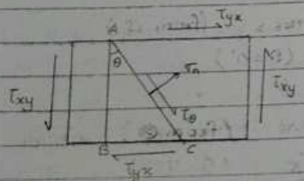


Normal stress

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta$$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta$$

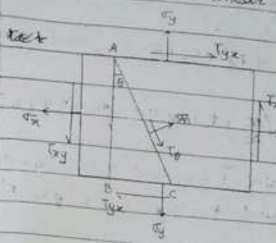
3) Stresses on an inclined plane due to pure shear

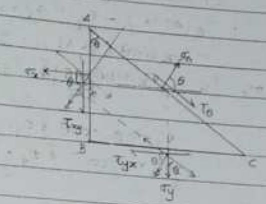


$$\sigma_n = \tau_{xy} \sin 2\theta$$

$$\tau_\theta = \tau_{xy} \cos 2\theta$$

4) Stresses on an inclined plane of an element subjected to both axial & shear force.





Let us consider an elemental inclined area and subjected to both normal & shear stresses on both directions.

Let θ be an angle subtended by an inclined plane with respect to vertical.

Σ Forces normal to plane $= 0$ (↗+)

$$\sigma_n (AC \times t) - \sigma_x \cos\theta (AB \times t) - \sigma_y \sin\theta (BC \times t)$$
$$- \tau_{xy} \sin\theta (AB \times t) - \tau_{yx} \cos\theta (BC \times t) = 0$$

or, $\sigma_n = \sigma_x \cos\theta \left(\frac{AB}{AC}\right) + \sigma_y \sin\theta \left(\frac{BC}{AC}\right) + \tau_{xy} \sin\theta \left(\frac{AB}{AC}\right) + \tau_{yx} \cos\theta \left(\frac{BC}{AC}\right)$

or, $\sigma_n = \sigma_x \cos\theta \cdot \cos\theta + \sigma_y \sin\theta \sin\theta + \tau_{xy} \sin\theta \cos\theta + \tau_{yx} \cos\theta \sin\theta$

or, $\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2 \tau_{xy} \sin\theta \cos\theta$ $\quad [\tau_{xy} = \tau_{yx}]$

$$= \frac{\sigma_x}{2} 2\cos^2\theta + \frac{\sigma_y}{2} 2\sin^2\theta + \tau_{xy} \sin 2\theta$$

$$= \frac{\sigma_x}{2} (1 - \sin^2\theta + \cos^2\theta) + \frac{\sigma_y}{2} (1 - \cos^2\theta + \sin^2\theta) + \tau_{xy} \sin 2\theta$$

$$\therefore \boxed{\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + \tau_{xy} \sin 2\theta} \quad (1)$$

~~Σ Forces perpendicular to plane = 0~~ (↗+)

Σ Forces along the plane = 0 (↘+)

$\tau_\theta (AC \times t) - \sigma_x \sin\theta (AB \times t) + \sigma_y \cos\theta (BC \times t) + \tau_{xy} \cos\theta (AB \times t)$

$- \tau_{yx} \sin\theta (BC \times t) = 0$

or, $\tau_\theta = \sigma_x \sin\theta \left(\frac{AB}{AC}\right) \mp \sigma_y \cos\theta \left(\frac{BC}{AC}\right) \mp \tau_{xy} \cos\theta \left(\frac{AB}{AC}\right) + \tau_{yx} \sin\theta \left(\frac{BC}{AC}\right)$

$= \sigma_x \sin\theta . \cos\theta \mp \sigma_y \cos\theta \sin\theta \mp \tau_{xy} \cos\theta \cos\theta + \tau_{yx} \sin\theta \sin\theta$

$= \frac{2\sigma_x}{2} \sin\theta \cos\theta \mp \sigma_y \frac{2}{2} \sin\theta \cos\theta \mp \tau_{xy} \cos^2\theta + \tau_{xy} \sin^2\theta$

$= \frac{\sigma_x}{2} \sin 2\theta - \frac{\sigma_y}{2} \sin 2\theta - \tau_{xy} (\cos^2\theta - \sin^2\theta)$

$$\therefore \boxed{\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta - \tau_{xy} \cos 2\theta} \quad \text{——— ②}$$

~~(Maximum principal stress)~~

To obtain maxm value of normal stress

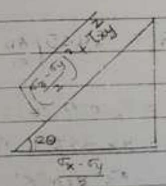
$\frac{\partial \sigma_n}{d\theta} = 0 - 2 \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta + 2 \tau_{xy} \cos 2\theta$

For maxm value,

$\frac{\partial \sigma_n}{d\theta} = 0$

$\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta \mp \tau_{xy} \cos 2\theta = 0$

$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}$



⇒ Substituting value of θ in ①,

$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cdot \frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} + \tau_{xy} \cdot \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$

$= \frac{\sigma_x + \sigma_y}{2} + \left[\frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}\right]$

$$\boxed{\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

This gives maxm principal stress.

To obtain min. value of normal stress
(Minimum principal stress)

$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$

$\sigma_2 = (\sigma_x + \sigma_y) - \sigma_1$

$= (\sigma_x + \sigma_y) - \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$$\boxed{\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

This gives min. principal stress.

$$\Rightarrow \boxed{\sigma_{\substack{max \\ min}} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

Normal stress

$$\sigma_n = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

Shear stress

$$\tau_\theta = \left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Maximum & minimum normal stresses

$$\sigma_{\substack{max\\min}} = \left(\frac{\sigma_x+\sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

Maximum & minimum shear stresses

For maximum shear stress

$$\frac{d\tau_\theta}{d\theta} = 0 \Rightarrow \left(\frac{\sigma_x-\sigma_y}{2}\right)\cdot 2\cos 2\theta + 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta = \frac{-\left(\frac{\sigma_x-\sigma_y}{2}\right)}{\tau_{xy}}$$

We have,

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)} \quad \text{and} \quad \tan 2\theta_s = \frac{-\left(\frac{\sigma_x-\sigma_y}{2}\right)}{\tau_{xy}} = \frac{-1}{\tan 2\theta_p}$$

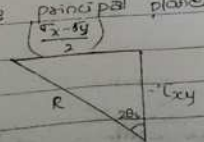
$$\tan(\theta \pm 90) = -\cot\theta$$

$$2\theta_p \pm 90 = 2\theta_s$$

$$\Rightarrow \boxed{\theta_s = \theta_p \pm 45^\circ}$$

dirn of shear stress   dirn of principal stress

From above, it can be concluded that max shear occurs at 45° to the principal planes.



$$\sin 2\theta = \left(\frac{\sigma_x-\sigma_y}{2}\right)/R$$

$$\cos 2\theta = -\tau_{xy}/R$$

Substituting value of $\sin 2\theta$ & $\cos 2\theta$ on eqn. of shear stress (2)

$$\tau_{max} = \left(\frac{\sigma_x-\sigma_y}{2}\right)\left(\frac{\sigma_x-\sigma_y}{2R}\right) + \tau_{xy} \times \frac{\tau_{xy}}{R}$$

$$\tau_{max} = \frac{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}{R}$$

$$\tau_{max} = \frac{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}} \qquad R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\therefore \boxed{\tau_{max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\boxed{\tau_{\substack{max\\min}} = \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

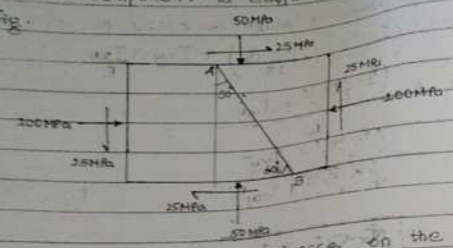
$$\sigma_1 - \sigma_2 = 2\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\Rightarrow \sigma_1 - \sigma_2 = 2\tau_{max}$$

$$\Rightarrow \boxed{\tau_{max} = \frac{\sigma_1-\sigma_2}{2}}$$

σ_1 = max. principal stress
σ_2 = min. principal stress

Q. A machine component is subjected to stresses as shown in fig.



Find the normal & shearing stresses on the section AB inclined at an angle of 60° with x-x axis. Also find the resultant stress on the section.

Soln.

$\sigma_x = -100\text{ MPa} = -100\text{ N/mm}^2$

$\sigma_y = -50\text{ MPa} = -50\text{ N/mm}^2$ $\qquad \theta = 30°$

~~$\tau_{xy} = 100\text{ MPa} = 100$~~

$\tau_{yx} = \tau_{xy} = 25\text{ MPa} = 25\text{ N/mm}^2$

Normal stress

$$\sigma_n = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \left(\frac{-100-50}{2}\right) + \left(\frac{-100+50}{2}\right)\cos 60° + \overset{25}{\cancel{\tau_{xy}}}\sin 60°$$

$$= -65.84\text{ N/mm}^2$$

Shear stress

$$\tau_\theta = \left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \left(\frac{-100+50}{2}\right)\sin 60° - 25\cos 60°$$

$$= -34.25\text{ N/mm}^2$$

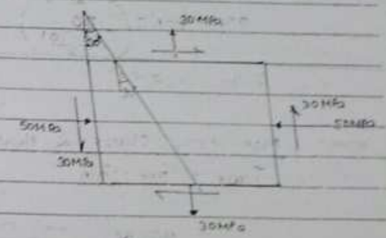
Resultant stress

$$= \sqrt{\sigma_n^2 + \tau_\theta^2}$$

$$= \sqrt{(65.84)^2 + (-34.15)^2}$$

$$= 74.17\text{ N/mm}^2$$

Q. For an infinitesimal elemental area, normal & shearing stresses in the 2 mutually perpendicular planes are given below. Determine the normal & shearing stresses on the inclined plane at an angle of 30° with vertical. Also, calculate principal stress, their planes (θ), maxm shear stress and their planes.



Soln.

$\sigma_x = -50\text{ N/mm}^2$ (C)

$\sigma_y = 30\text{ N/mm}^2$ (T)

$\tau_{xy} = 30\text{ MPa} = 30\text{ N/mm}^2$

$\theta = 30°$

Normal stress

$$\sigma_n = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \left(\frac{-50+30}{2}\right) + \left(\frac{-50-30}{2}\right)\cos 60° + 30\sin 60°$$

$$= -4.02\text{ N/mm}^2$$

Shear stress

$$\tau_\theta = \left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$= \left(\frac{-50-30}{2}\right)\sin 60° - 30\cos 60°$$

$$= -49.64\text{ N/mm}^2$$

Principal stress & their planes

$$\sigma_{\max/\min} = \left(\frac{\sigma_x+\sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{\max} = \left(\frac{-50+30}{2}\right) + \sqrt{\left(\frac{-50-30}{2}\right)^2 + 30^2}$$

$$= -10 + 50$$

$$= 40 \ N/mm^2$$

$$\sigma_{\min} = -10 - 50 = -60 \ N/mm^2$$

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)}$$

$$2\theta_p = \tan^{-1}\left(\frac{30}{\left(\frac{-50-30}{2}\right)}\right)$$

$\Rightarrow \theta_p = -18.43°$

$\theta_{P_1} = -18.43 + 90 = 71.57°$

$\theta_{P_2} = 71.57 + 90 = 161.57°$

Max. shear stress & their planes

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{40+60}{2} = 50 \ N/mm^2$$

$$\tau_{\min} = -\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = -\sqrt{\left(\frac{-50-30}{2}\right)^2 + 30^2} = -50 \ N/mm^2$$

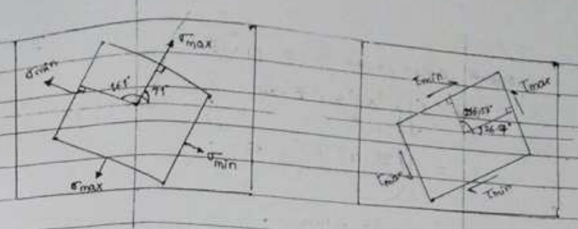
$$\tan 2\theta_s = \frac{-\left(\frac{\sigma_x-\sigma_y}{2}\right)}{\tau_{xy}}$$

$$2\theta_s = \tan^{-1}\left[\frac{-\left(\frac{-50-30}{2}\right)}{30}\right]$$

$\Rightarrow \theta_s = 26.57°$

$\theta_{s1} = 26.57°$

$\theta_{s2} = 26.57 + 90 = 116.57°$

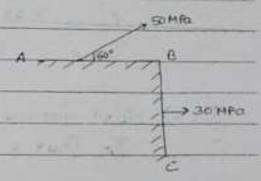


3. An element is found in a stress condition as shown in fig.

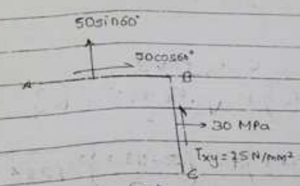
a) Find resultant stress on plane BC

b) Max^m shear stress & its plane

c) Max^m normal stress & its plane



Sol^n.



$\sigma_x = 30 \ N/mm^2$ (T)

$\sigma_y = 50\sin 60° = \cancel{40.45}$ 43.30 N/mm^2 (T)

$\tau_{xy} = 50\cos 60° = 25 \ N/mm^2$

a) Resultant stress on plane BC

$$R = \sqrt{\tau_{xy}^2 + \sigma_x^2}$$

$$= \sqrt{25^2 + 30^2}$$

$$= 39.05 \ MPa$$

b) Max^m shear stress & its plane

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{30-43.30}{2}\right)^2 + 25^2}$$

$$= 25.87 \text{ N/mm}^2$$

$$\tan 2\theta_s = \frac{-\left(\frac{\sigma_x-\sigma_y}{2}\right)}{\tau_{xy}}$$

$$2\theta_s = \tan^{-1}\left[\frac{-\left(\frac{30-43.30}{2}\right)}{25}\right]$$

$$\theta_{s1} = 7.45^\circ$$

$$\theta_{s2} = 7.45^\circ + 90^\circ = 97.45^\circ \quad //$$

c) Max^m normal stress & its plane

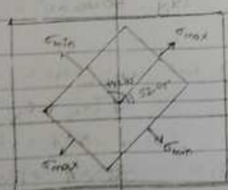
$$\sigma_n \; \sigma_{max} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \left(\frac{30+43.3}{2}\right) + \sqrt{\left(\frac{30-43.3}{2}\right)^2 + 25^2}$$

$$= 62.52 \text{ N/mm}^2$$

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)}$$

$$2\theta_p = \tan^{-1}\left[\frac{25}{\left(\frac{30-43.3}{2}\right)}\right]$$

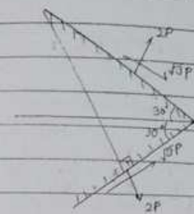


$$\theta_p = -37.55^\circ$$

$$\theta_{p1} = -37.55^\circ + 90^\circ = 52.45^\circ$$

$$\theta_{p2} = 52.45^\circ + 90^\circ = 142.45^\circ \quad //$$

Q8. Magnitude & directions of the stresses on 2 planes intersecting at a point as shown in fig. Determine the direction & magnitude of the principal stresses at this point. Sketch the result on the element.



Soln.

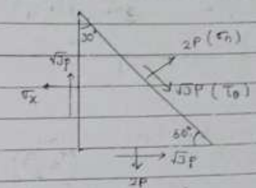
$\sigma_x = \sigma_x$ (T)

$\sigma_y = 2P$ (T)

$\tau_{xy} = -\sqrt{3}P$

$\theta = 30^\circ$

$\sigma_n = 2P$



We have,

$$\sigma_n = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

or, $2P = \left(\frac{\sigma_x+2P}{2}\right) + \left(\frac{\sigma_x-2P}{2}\right)\cos 60^\circ + -\sqrt{3}P\sin 60^\circ$

or, $4P = \sigma_x + 2P + (\sigma_x - 2P)\cos 60^\circ - 2\sqrt{3}\sin 60^\circ P$

or, $2P = \sigma_x + 0.5\sigma_x - P - 3P$

or, $6P = 1.5\sigma_x$

$\therefore \sigma_x = 4P$

Principal stresses & their planes

τ.

Also,

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

or, $\sqrt{3}P = \left(\frac{\sigma_x - 2P}{2}\right)\sin 60^\circ + \sqrt{3}P\cos 60^\circ$

or, $2\sqrt{3}P = 0.866\,\sigma_x - \cancel{1.732P} + \cancel{1.732P}$

$\therefore$ $\sigma_x = 4P$

Principal stresses & their planes

$$\sigma_{\max/\min} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

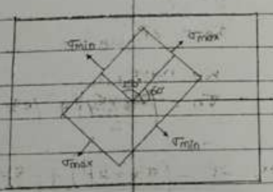
$\sigma_{max} = 3P + \sqrt{P^2 + 3P^2}$ $\qquad$ $\sigma_{min} = 3P - 2P$

$= 3P + 2P$ $\qquad$ $= P$

$= 5P$

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}$$

$$2\theta_p = \tan^{-1}\left[\frac{-\sqrt{3}P \times 2}{4P - 2P}\right]$$



$2\theta_p = -60^\circ$

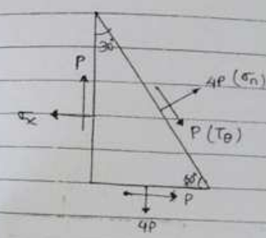
$\theta_{p1} = -30^\circ$

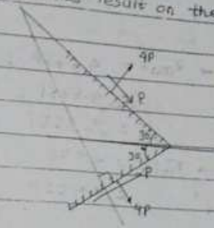
$\theta_{p1} = -30^\circ + 90^\circ = \cancel{120}^\circ\ 60^\circ$

$\theta_{p2} = 60^\circ + 90^\circ = 150^\circ$ //

Q. The magnitude & direction of the stresses on 2 planes intersecting at a point are as shown. Determine direction & magnitude of principal stresses at this point. Sketch the result on the element.

Soln.





$\sigma_x = \sigma_x$ $\qquad$ $\sigma_n = 4P$

$\sigma_y = 4P$ $\qquad$ $\tau_\theta = P$

$\tau_{xy} = -P$ $\qquad$ $\theta = 30^\circ$.

We have,

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$4P = \frac{\sigma_x + 4P}{2} + \left(\frac{\sigma_x - 4P}{2}\right)\cos 60^\circ - P\sin 60^\circ$$

$8P = \sigma_x + 4P + (\sigma_x - 4P)\cos 60^\circ - 2P\sin 60^\circ$

$4P = \sigma_x + 0.5\sigma_x - 2P - 1.73P$

$7.73P = 1.5\sigma_x$

$\therefore$ $\sigma_x = 5.15P$

Also,

$$\tau_\theta = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

$$P = \left(\frac{\sigma_x - 4P}{2}\right)\sin 60^\circ + P\cos 60^\circ$$

$2P = \sigma_x \sin 60^\circ - 4\sin 60^\circ P + 2\cos 60^\circ P$

$4.46P = 0.866\,\sigma_x$

$\therefore$ $\sigma_x = 5.15P$

Principal stresses & their planes

$$\sigma_{\substack{max\\min}} = \left(\frac{\sigma_x+\sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

→ $\sigma_{max} = 4.575P \pm \sqrt{0.33P^2 + P^2}$

$= 4.575P \pm 1.153P$

$= 5.728P$

→ $\sigma_{min} = 4.575P - 1.153P$

$= 3.422P$

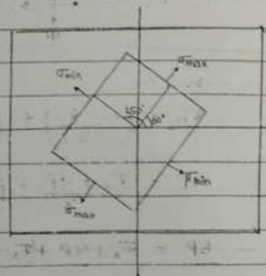
$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x-\sigma_y}{2}\right)}$$

$$2\theta_p = \tan^{-1}\left[\frac{-P\times 2}{5.15P-4P}\right]$$

$\theta_p = -30^\circ$

→ $\theta_{p1} = -30^\circ + 90^\circ = 60^\circ$

$\theta_{p2} = 60^\circ + 90^\circ = 150^\circ$



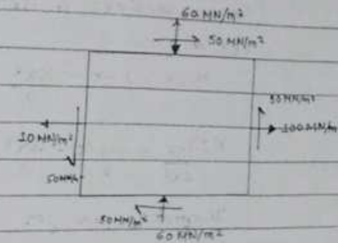
Q. In a material at a certain point, tensile stress of 100 MN/m² and compressive stress of 60 MN/m² are acting horizontally and vertically respectively. At the same point, shear stress of 50 MN/m² is also acting on these planes. Find the values and plane of max. shear stress and draw these planes.

Soln.

$\sigma_x = 100\ MN/m^2$ (T)

$\sigma_y = -60\ MN/m^2$ (C)

$\tau_{xy} = 50\ MN/m^2$



⇒ Max^m shear stress & its plane

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{100+60}{2}\right)^2 + 50^2}$$

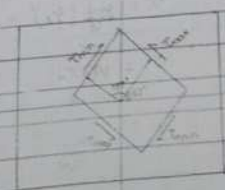
$= 94.34\ MN/m^2$

$$\tan 2\theta_s = \frac{-\left(\frac{\sigma_x-\sigma_y}{2}\right)}{\tau_{xy}}$$

$$\tan 2\theta_s = \frac{-(100+60)}{50\times 2}$$

$2\theta_s = -57.99$

$\theta_s = -28.99$



$\theta_{s1} = -28.99 + 90^\circ = 61^\circ$

$\theta_{s2} = 61^\circ + 90^\circ = 151^\circ$

4)

$R_{Ay} = \frac{wl}{2}$ $R_{By} = \frac{wl}{2}$

Let us consider a section XX at a distance x from end A.

$$M_x = \frac{wl}{2}\cdot x - \frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2}$$

Integrating, we get

$$EI\frac{dy}{dx} = \frac{wl}{4}x^2 - \frac{wx^3}{6} + c_1 \quad \text{——— (1)}$$

As the loading is symmetrical, the maximum deflection will occur at mid span and hence the slope at mid span = 0

$$x = \frac{l}{2}, \quad \frac{dy}{dx} = 0$$

From (1),

$$0 = \frac{wl}{4}(l/2)^2 - \frac{w(l/2)^3}{6} + c_1$$

$$0 = \frac{wl \times l^2}{16} - \frac{wl^3}{8\times 6} + c_1$$

$$c_1 = \frac{-wl^3}{24}$$

$$\therefore \boxed{EI\frac{dy}{dx} = \frac{wl}{4}x^2 - \frac{wx^3}{6} - \frac{wl^3}{24}} \quad \text{——— (2)}$$

This is the required slope equation.

At A, put $x = 0$, we get

$$EI\theta_A = \frac{-wl^3}{24}$$

$$\therefore \boxed{\theta_A = \frac{-wl^3}{24EI}}$$

Integrating the slope equation (2) we get

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3}{24}x + c_2$$

When $x = 0, y = 0$

$\therefore c_2 = 0$

$$\boxed{EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}}$$

Deflection at mid span (y_{max})

Put $x = l/2$, we get

$$EIy_{max} = \frac{wl}{12}(l/2)^3 - \frac{w(l/2)^4}{24} - \frac{wl^3(l/2)}{24}$$

$$\boxed{y_{max} = \frac{-5wl^4}{384EI}}$$

⇒ Maximum downward deflection $= \frac{5wl^4}{384EI}$

Tutorial IV

Q3

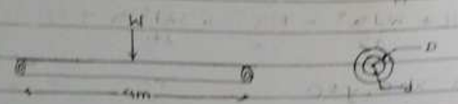
external diameter of pipe $(D) = 100$ mm

internal diameter of pipe $(d) = 80$ mm

span of beam $(l) = 4\text{m} = 4\times10^3$ mm

concentrated load $(W) = ?$

permissible stress in beam $(\sigma) = 120\ \text{N/mm}^2$



Max bending moment occurs at centre. So,

$$M = \frac{Wl}{4} = \frac{W\times4\times10^3}{4} = 10^3W \Rightarrow M = 10^3W \quad \text{--- (1)}$$

Moment of resistance

$$M = \sigma.Z = \sigma.\frac{I}{y} = \frac{\sigma}{D/2}\times\frac{\pi}{64}(D^4-d^4)$$

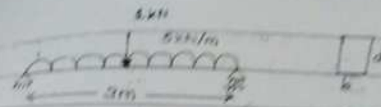
or, $10^3W = \frac{120\times2}{100}\times\frac{\pi}{64}(100^4-80^4)$

or, $10^3W = 6955486.1$

$\therefore W = 6955.4$ N

≈ 6960 N //

Q.4. Soln:



$l = 3$m $\qquad \sigma = 100\ \text{N/mm}^2$

$d = 2b$

Max. BM occurs at centre so,

$$M = \frac{Wl}{4} + \frac{W'l^2}{8} \qquad W = 1\text{kN} \qquad W' = 5\text{kN/m}$$

$$= \frac{1\times3}{4} + \frac{5\times3^2}{8}$$

$$= 6.375\ \text{kNm} = 6.375\times10^6\ \text{Nmm}$$

$$I = \frac{bd^3}{12} = \frac{b\times(2b)^3}{12} = \frac{8b^4}{12} = \frac{2}{3}b^4 \qquad y = \frac{d}{2} = \frac{2b}{2} = b$$

Moment of resistance

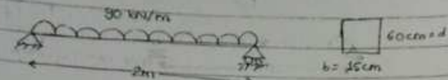
$$M = \sigma.\frac{I}{y} = 100\times\frac{\frac{2}{3}b^4}{b}$$

or, $6.375\times10^6 = \frac{200}{3}b^3$

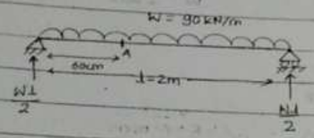
$\therefore b = 45.73$ mm

$d = 2b = 91.46$ mm //

Q.5. Soln.

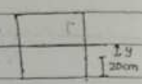


bending stress at a point 20cm above the bottom of beam = ? (σ)
& 60 cm from the left support



Moment at 60 cm from left support

$M_A = \frac{Wl}{2} \times 0.6 - (W \times 0.6) \times 0.3$



$= \frac{90 \times \cancel{2} \times 0.6}{\cancel{2}} - (90 \times 0.6 \times 0.3)$

$= 37.8$ KNm $= 37.8 \times 10^6$ Nmm

$y = 30 - 20 = 10$ cm $=$ ~~0.1 m~~ 100 mm

$I = \frac{bd^3}{12} = \frac{150 \times 600^3}{12} = 27 \times 10^8$ mm^4

$M_A = \sigma \frac{I}{y}$

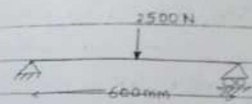
or, $37.8 \times 10^6 = \sigma \times \frac{27 \times 10^8}{100}$

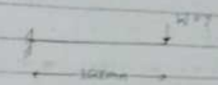
$\therefore \sigma = 1.4$ N/mm^2 //

Q.6 Soln

max central load $(W) = 2500$ N
length of beam $(l) = 600$ mm
size " " $= 25$ mm $\times$ 25 mm } simply supported

For cantilever beam of same material
length $(l) = 1500$ mm
size $= 25$ mm $\times$ 100 mm





Case I : simply supported

Max. BM occurs at centre

$M = \frac{Wl}{4} = \frac{2500 \times 600}{4}$

$= 375000$ Nmm

$I = \frac{25 \times 25^3}{12} = 32552.08$ mm^4

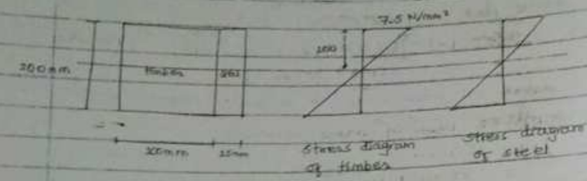
$y = \frac{d}{2} = \frac{25}{2} = 12.5$ mm

$M = \sigma \frac{I}{y}$

or, $375000 = \sigma \times \frac{32552.08}{12.5}$

$\therefore \sigma = 144$ N/mm^2

Q.7 Soln



Stress on timber (σ_t) = 7.5 N/mm², at 100 mm from neutral axis

⇒ Stress on steel at 100 mm from neutral axis

$$\frac{\sigma_s}{E_s} = \frac{\sigma_t}{E_t} \qquad \frac{E_s}{E_t} = 20$$

$$\sigma_s = \frac{E_s \times \sigma_t}{E_t}$$

$$= 20 \times 7.5$$

$$= 150 \text{ N/mm}^2$$

⇒ Total moment of resistance of the beam

M = moment of resistance (due to timber + steel)

$$= M_t + M_s$$

$$= \sigma_t Z_t + \sigma_s Z_s$$

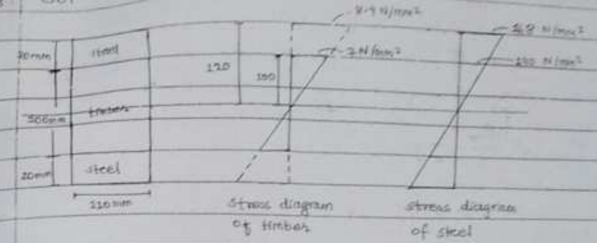
$$= \cancel{7.5 \times} \ \sigma_t \times \frac{b_t d_t^2}{6} + \sigma_s \times \frac{b_s d_s^2}{6}$$

$$= 7.5 \times \frac{100 \times 200^2}{6} + 150 \times \frac{15 \times 200^2}{6}$$

$$= 2 \times 10^7 \text{ N mm}$$

$$= 2 \times 10^4 \text{ N m} \quad //$$

Q.8 Soln.



Allowable stress on timber (σ_t) = 7 N/mm² $\quad E_s = 20E_t$

Stress on timber at 120 mm above neutral axis

(σ_t) = 8.4 N/mm²

Stress on steel at 120 mm above neutral axis

$$\frac{\sigma_s}{E_s} = \frac{\sigma_t}{E_t} \qquad b_{s1} = b_{s2} = 110\text{mm}$$

$$\sigma_s = \frac{E_s}{E_t} \times \sigma_t \qquad d_{s1} = 120+120 = 240\text{ mm}$$

$$d_{s2} = 100+100 = 200\text{ mm}$$

$$= 20 \times 8.4$$

$$\sigma_{s1} = 168 \text{ N/mm}^2 \qquad \sigma_{s2} = 7 \times 20 = 140 \text{ N/mm}^2$$

⇒ Moment of resistance of the section

$$M = M_t + M_s$$

$$= \sigma_t Z_t + \sigma_s Z_s$$

$$= \sigma_t \times \frac{b_t d_t^2}{6} + \left[\sigma_{s1} \times \frac{b_{s1} \times d_{s1}^2}{6} - \sigma_{s2} \times \frac{b_{s2} \times d_{s2}^2}{6}\right]$$

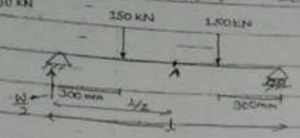
$$= 7 \times \frac{110 \times 200^2}{6} + \left[168 \times \frac{110 \times 240^2}{6} - 140 \times \frac{110 \times 200^2}{6}\right]$$

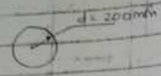
$$= 79874666.67 \text{ Nmm}$$

$$= 79874.67 \text{ Nm} \quad //$$

Q.12 Sol^n

diameter of beam (d) = 200 mm





max. bending stress (σ_{max}) = ?

⇒ Max. ~~bendenine~~ BM occurs at centre of beam (i.e. at A)

$$M = \frac{W}{2}\times\frac{l}{2} - 150\times\left(\frac{l}{2}-300\right)$$

$$= \frac{300\,l}{4} - 75\,l + 45000$$

$$= 75\,l - 75\,l + 45000$$

$$= 45000 \text{ kN mm} = 45\times10^{6} \text{ N/mm}$$

~~Max~~ $I = \frac{\pi d^4}{64} = \frac{\pi\times200^4}{64} = 7.85\times10^{7} \text{ mm}^4$

$y = \frac{d}{2} = 100 \text{ mm}$

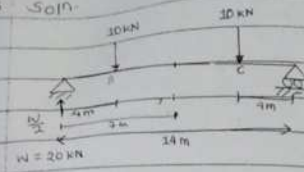
⇒ Moment of resistance

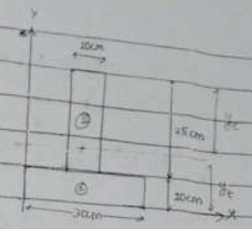
$$M = \frac{\sigma I}{y}$$

or, $45\times10^{6} = \sigma \times \frac{7.85\times10^{7}}{100}$

∴ Max bending stress (σ) = 57.325 N/mm^2 //

Q.13 Sol^n





$A_1 = 300 \text{ cm}^2 = 0.03 \text{ m}^2$

$y_1 = 5 \text{ cm} = 0.05 \text{ m}$

$A_2 = 250 \text{ cm}^2 = 0.025 \text{ m}^2$

$y_2 = \left(10+\frac{25}{2}\right) \text{cm} = 22.5 \text{ cm} = 0.225 \text{ m}$

$$\bar{y} = \frac{A_1y_1 + A_2y_2}{A_1+A_2} = \frac{0.03\times0.05 + 0.025\times0.225}{0.03+0.025}$$

$$= \cancel{0.20625}\ 0.13 \text{ m}$$

$$I_{xx} = (I_{xx})_1 + (I_{xx})_2$$

$$= \left[(I_{G})_1 + A_1(y_1-\bar{y})^2\right] + \left[(I_{G})_2 + A_2(y_2-\bar{y})^2\right]$$

$$= \left[\frac{0.3\times0.1^3}{12} + 0.03(0.05-0.13)^2\right] + \left[\frac{0.1\times0.25^3}{12} + 0.025(0.225-0.13)^2\right]$$

∴ $I = 5.728\times10^{-4} \text{ m}^4$

$$M = \frac{W}{2}\times7 - 10\times(7-4) = \frac{20}{2}\times7 - 10\times3 = 40 \text{ kNm}$$
$$= 40\times10^{3} \text{ Nm}$$

$y_t = \bar{y} = 0.13 \text{ m}$

$y_c = 0.35 - y_t = 0.22 \text{ m}$

⇒ Max. tensile stress

$M = \sigma_{bt}\times\frac{I}{y_t}$ ⇒ $40\times10^{3} = \sigma_{bt}\times\frac{5.728\times10^{-4}}{0.13}$

$\sigma_{bt} = 9.078\times10^{6} \text{ N/m}^2$

∴ $\sigma_{bt} = 9.078 \text{ N/mm}^2$ //

⇒ Max. compressive stress

$$M = \sigma_{bc} \times \frac{I}{y_c}$$

or, $40 \times 10^3 = \sigma_{bc} \times \frac{5.728 \times 10^{-4}}{0.22}$

∴ $\sigma_{bc} = 15.363 \times 10^6 \ N/m^2$

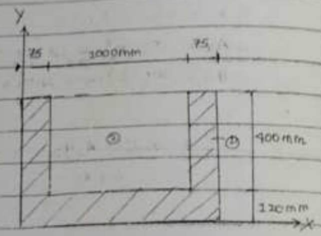
$= 15.363 \ N/mm^2$ //

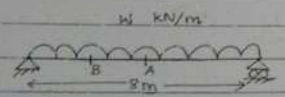
Q.14. Soln

$\gamma_c = 25 \ kN/m^3 = 25 \times 10^{-6} \ N/mm^3$

$\gamma_\omega = 10 \ kN/m^3 = 10 \times 10^{-6} \ N/mm^3$

Length of channel $(l) = 8m$
$= 8000 \ mm$





Wt. of concrete / unit length (W_1) = $\gamma_c \times \underbrace{b \times d}$ ~~cross-sectional area~~

$= 25 \times 10^{-6} \times (1150 \times 520 - 1000 \times 400)$

$= 4.95 \times \cancel{10^6} \ \cancel{k}N/mm$

Wt. of water / unit length (W_2) = $\gamma_\omega \times$ cross-sectional area

$= 10 \times 10^{-6} \times (1000 \times 400)$

$= 4 \times \cancel{10^6} \ \cancel{k}N/mm$

$W = W_1 + W_2 = 8.95 \times \cancel{10^6} \ \cancel{k}N/mm$

⇒ Max. bending moment at A (mid-span)

$$M_A = \frac{Wl^2}{8} = \frac{8.95 \times \cancel{10^6} \times 8000^2}{8}$$

~~$= 7.16 \times 10^7 \ kNm$~~

$= 7.16 \times 10^7 \ Nmm$

Bending ~~stresses at mid-span~~

$A_1 = 1150 \times 520 = 598000 \ mm^2$

$y_1 = \frac{520}{2} = 260 \ mm$

$A_2 = 1000 \times 400 = 400000 \ mm^2$

$y_2 = 120 + \frac{400}{2} = 320 \ mm$

$$\bar{y} = \frac{A_1 y_1 - A_2 y_2}{A_1 - A_2} = \frac{598000 \times 260 - 400000 \times 320}{598000 - 400000}$$

$= 138.78 \ mm$

$I_{xx} = (I_{xx})_1 - (I_{xx})_2$

$$= \left[\frac{1150 \times 520^3}{12} + 598000(260 - 138.78)^2\right] - \left[\frac{1000 \times 400^3}{12} + 400000(320 - 138.78)^2\right]$$

$y_t = \bar{y} = 138.78 \ mm$

$y_c = 520 - y_t$
$= 381.22 \ mm$

∴ $I = \cancel{4.073 \times 10^{10}}$ mm^4
3.79×10^9

⇒ Bending stresses at mid-span

$$M_A = \sigma_{bt} \times \frac{I}{y_t}$$

or, $7.16 \times 10^7 = \sigma_{bt} \times \frac{\cancel{4.073 \times 10^{10}}\ 3.79 \times 10^9}{138.78}$

∴ $\sigma_{bt} = 2.62 \ \cancel{\times 10^6} \ N/mm^2$

$$M_A = \sigma_{bc} \times \frac{I}{y_c}$$

or, $7.16 \times 10^7 = \sigma_{bc} \times \frac{3.79 \times 10^9}{381.22}$

∴ $\sigma_b = 7.20 \ \cancel{\times 10^6} \ N/mm^2$ //

⇒ Bending moment at B (i.e. quarter span)

$$M_B = \frac{W l}{2}\frac{l}{4} - \frac{W l}{4} \times \frac{l}{8}$$

$$= \frac{W l^2}{8} - \frac{W l^2}{32} = \frac{3 W l^2}{32} = \frac{3 \times 8.95 \times \cancel{10^3} \times 8000^2}{32}$$

$$= \cancel{5.37 \times 10^7 \text{ kNm}}$$

$$= 5.37 \times 10^{7} \text{ Nmm}$$

⇒ Bending stresses at quarter span

$$M_B = \sigma_{bt} \times \frac{I}{y_t}$$

or, $5.37 \times 10^{7} = \sigma_{bt} \times \frac{3.79 \times 10^9}{138.78}$

$\therefore \sigma_{bt} = \cancel{1966.355}\ 1.96 \times \cancel{10^6}$ N/mm²

$$M_B = \sigma_{bc} \times \frac{I}{y_c}$$

or, $5.37 \times 10^{7} = \sigma_{bc} \times \frac{3.79 \times 10^9}{381.22}$

$\therefore \sigma_{bc} = 5.4 \times \cancel{10^6}$ N/mm² //

Q.10. Solⁿ

3 beams have same length, same allowable stress & same BM

For square beam

$$M = \sigma \times \frac{I}{y}$$

For rectangular beam