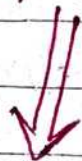


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Method - I

Newton - Raphson Method



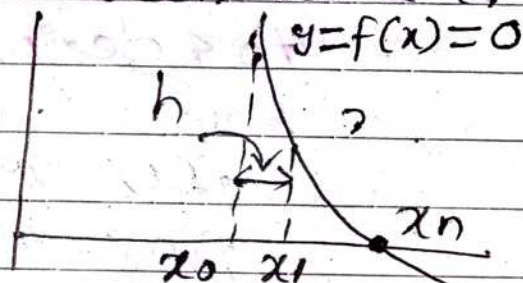
Let x_0 be an initial approximation to the root of the equation $y = f(x) = 0$. If $x_1 = x_0 + h$ be the exact root then $f(x) = 0$ where h be difference between x_0 & x_1 which is very small.

By using Taylor's series

$$f(x_1) = 0$$

$$f(x_0 + h) = 0$$

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) = 0$$



Since h is very small, neglecting h^2 & higher power of h , we get

$$f(x_0) + hf'(x_0) = 0$$

$$\therefore h = - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly starting with x_1 , better approximation $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

This process is continued to get the root of desired accuracy

In general $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ This is called N-R formula.

Find root of eqⁿ $x^3 - 6x + 4 = 0$ correct to 4 decimal places.

Here

$$f(x) = x^3 - 6x + 4$$

$$f(0) = 4 \text{ (ve)}$$

$$f(1) = -1 \text{ (-ve)}$$

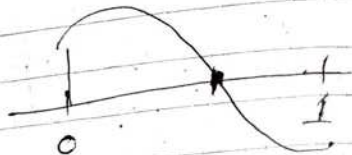
Since $f(0)$ & $f(1)$ have opposite sign, a root lies between 0 & 1. Let initial approximation $x_0 = 0.5$

$$\text{Now } f(x) = x^3 - 6x + 4$$

$$f'(x) = 3x^2 - 6$$

We know N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



The successive approximations are

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.5^3 - 6(0.5) + 4}{3(0.5)^2 - 6} = 0.71429$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.71429 - \frac{0.71429^3 - 6(0.71429) + 4}{3(0.71429)^2 - 6} = 0.73190$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.73190 - \frac{0.73190^3 - 6(0.73190) + 4}{3(0.73190)^2 - 6} = 0.73205$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.73205 - \frac{0.73205^3 - 6(0.73205) + 4}{3(0.73205)^2 - 6} = 0.73205$$

Since $x_3 = x_4$

The required root = 0.73205 Ans

2) Using N-R method to 3 decimal places $3x = \cos x + 1$

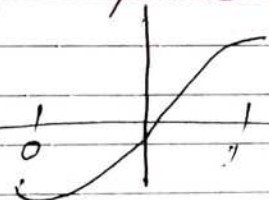
so/n:

$$\text{Here } f(x) = 3x - \cos x - 1$$

$$f(0) = -2 \text{ (-ve)}$$

$$f(1) = 1.4597 \text{ (+ve)}$$

Then,



Exercises

Let $x_0 = 0.4$

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

By formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The successive approximations are

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.6127$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6071$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071$$

Since $x_2 = x_3$

So Root = 0.6071 //

14) Compute a root of each

9) $x^2 - 5x + 6 = 0$ $x_0 = 5$ $\frac{d}{dx} x \log_{10} x$

Solving

Let $f(x) = x^2 - 5x + 6 = 0$ $x \frac{d \log_{10} x + \log_{10} x}{dx}$

$$f(0) = 6 (+ve)$$

$$f(1) = 2 (+ve)$$

$$f(2) = 0$$

$$f(3) = 0$$

$$f(4) = 2$$

$$f(5) = 4$$

$$x \cdot \frac{0.4343}{x} + \log_{10} x$$

$$0.4343 + \log_{10} x$$

shift calc ↓

Q.No.3 Compute a real root.

$x \log x = 1.2$ correct to 4 decimal places.

Here

$$f(x) = x \log x - 1.2$$

$$f(1) = -1.20000 (-ve)$$

$$f(2) = -0.59794 (-ve)$$

$$f(3) = 0.23136 (+ve)$$

Since $f(2)$ & $f(3)$ have opposite sign. A root lies between 2 & 3

Let initial approximation $x_0 = 2.5$

Now, $f(x) = x \log_{10} x - 1.2$

$$f'(x) = \frac{d}{dx} (x \log_{10} x)$$

$$= x \frac{d}{dx} (\log_{10} e \log_e x) + \log_{10} x$$

$$= \log_{10} x + 0.43429$$

We have, N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

the successive approximations are:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{\log_{10} 2.5 - 1.2}{\log_{10} 2.5 + 0.43429}$$

$$= 2.74651$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.74651 - \frac{2.74651 \log_{10} 2.74651 - 1.2}{\log_{10} 2.74651 + 0.43429}$$

$$= 2.74065$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.74065 - \frac{2.74065 \log_{10} 2.74065 - 1.2}{\log_{10} 2.74065 + 0.43429}$$

$$= 2.74065$$

4) Find a real root of square root of 18, correct to 3 decimal places.

$$\text{let } x = \sqrt{18}$$

$$x^2 = 18$$

$$\therefore f(x) = x^2 - 18$$

$$f(4) = -2.00000$$

$$f(5) = 7(+ve)$$

Since $f(4)$ & $f(5)$ have opp. sign. A root lies betn 4 & 5

Initial approximation $x_0 = 4.5$

$$\text{Here } f(x) = x^2 - 18$$

$$f'(x) = 2x$$

We know,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.2500$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4.2426$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.2426$$

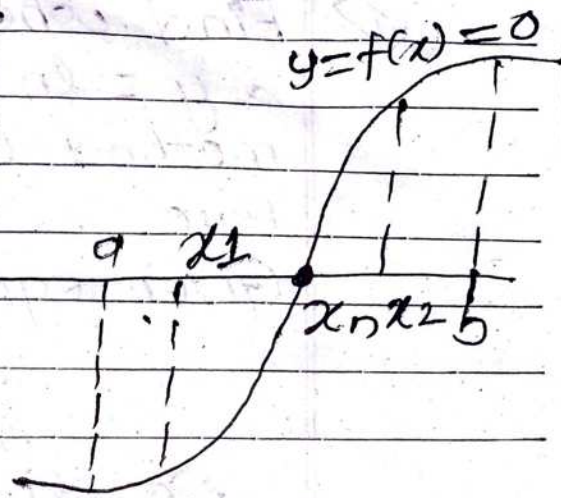
$$\text{So root} = 4.2426$$

Method - II

Bisection method (Bracketing method)

Let $y=f(x)$ be a continuous function between a and b with $f(a)$ & $f(b)$ have opposite sign then a root lies between a & b

Then first approximation
$$x_1 = \frac{a+b}{2}$$



If $f(x_1)=0$ then x_1 is the exact root otherwise the root lies between a & x_1 or x_1 & b according to the sign of $f(x_1)$

In fig $f(a)$ is negative and $f(b)$ is +ve then $x_1 = \frac{a+b}{2}$. Again $f(x_1) = -ve$ so the root lies between x_1 & b . The second approximation is $x_2 = \frac{x_1+b}{2}$. $f(x_2) = +ve$. The root lies between x_1 & x_2 . The third approximation $x_3 = \frac{x_1+x_2}{2}$. This procedure is continued to get root of desired accuracy.

In general $\boxed{x_n = \frac{a+b}{2}}$ is Bisection formula

(X)

Setting: $x = \frac{a+b}{2} : \ln x - x + 3$

2019 Fall

1) Find where the graph of $y = x - 3$ & $y = \ln x$ intersect using bisection method. Get correct to 4 decimal places.

Here

Given equations are

$$y = x - 3$$

$$y = \ln x$$

$$\text{So } \ln x = x - 3$$

$$\text{or } \ln x - x + 3 = 0$$

$$\therefore f(x) = \ln x - x + 3$$

$$\therefore f(4) = 0.38629 (+ve)$$

$$f(5) = -0.39056 (-ve)$$

Since $f(4)$ and $f(5)$ have opp. sign.

A root lies between 4 & 5

$$\text{Let } a = 4, b = 5$$

We know, the bisection formula

$$x_n = \frac{a+b}{2}$$

Tabulating for successive approximations, we get,

Iteration	a (+ve)	b (-ve)	$x = \frac{a+b}{2}$	$f(x) = \ln x - x + 3$
1.	4	5	4.50000	+ve
2.	4.5	5	4.75000	-ve
3)	4.5	4.75	4.625	-ve

4)	4.5	4.625	4.5625	-ve
5)	4.5	4.5625	4.53125	-ve
6)	4.5	4.53125	4.51563	-ve
7)	4.5	4.51563	4.50782	-ve
8)	4.5	4.50782	4.50391	+ve
9)	4.50391	4.50782	4.50587	-ve
10)	4.50391	4.50587	4.50489	+ve
11)	4.50489	4.50587	4.50538	-ve
12)	4.50489	4.50538	4.50514	+ve
13)	4.50514	4.50538	4.50526	-ve
14)	4.50514	4.50526	4.50520	+ve
15)	4.50520	4.50526	4.50523	END

Hence $x_{14} = x_{15}$

The required root = 4.5052 Ans

2) Find a root of equation $xe^x = 3$ by using bracketing method correct to 3 decimal places ($x_1 = 1, x_2 = 1.5$)

Soln:

$$f(x) = xe^x - 3$$

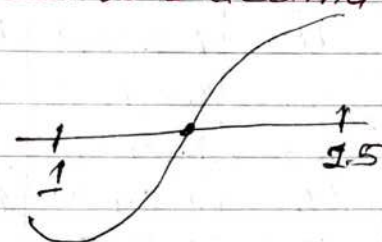
$$f(1) = -0.2817 (-ve)$$

$$f(1.5) = 3.7225 (+ve)$$

Since $f(1)$ & $f(1.5)$ have opposite sign, a root lies between 1 & 1.5

$$\text{Let } a = x_1 = 1$$

$$b = x_2 = 1.5$$



We know

The bisection formula:

$$x_n = \frac{a+b}{2}$$

Tabulating for successive approximations, we get.

Iteration	a (ve)	b (ve)	$x = \frac{a+b}{2}$	$f(x) = x^2 - 3$
1	1	1.5	1.2500	+ve
2	1	1.2500	1.1250	+ve
3	1	1.125	1.0625	+ve
4	1	1.0625	1.0313	-ve
5	1.0313	1.0625	1.0469	-ve
6	1.0469	1.0625	1.0547	+ve
7	1.0469	1.0547	1.0508	+ve
8	1.0469	1.0508	1.0489	-ve
9	1.0489	1.0508	1.0499	-ve
10	1.0499	1.0508	1.0504	+ve
11	1.0499	1.0504	1.0502	+ve
12	1.0499	1.0502	1.0501	+ve
13	1.0499	1.0501	1.0500	+ve
14	1.0499	1.0500	1.0499	-ve
15	1.0499	1.0500	1.0499	END

$$\text{root} = 1.0499$$

3) Find negative root

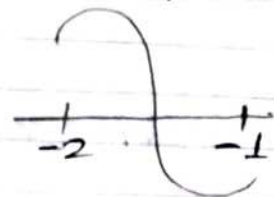
$x^2 - 4x - 10$ correct to 2 decimal

$$f(x) = x^2 - 4x - 10$$

$$f(0) = -10 (-ve)$$

$$f(-1) = -5 (-ve)$$

$$f(-2) = 2 (+ve)$$



Since $f(-1)$ and $f(-2)$ have opposite sign so root lies between -2 and -1
So $a = -2$

$$b = -1$$

Then By bisection formula

$$x_n = \frac{a+b}{2}$$

Now Tabulating the successive approximation

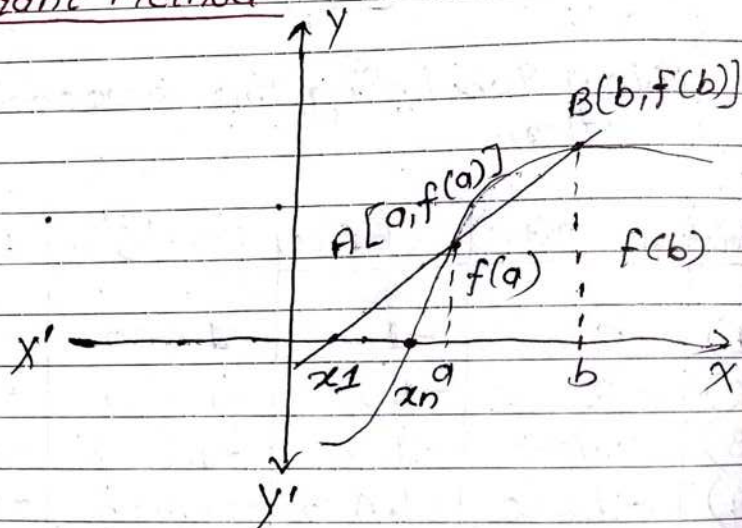
Iteration	a (ve)	b (ve)	$x = \frac{a+b}{2}$	$f(x) = x^2 - 4x - 10$
1	-2	-1	-1.500	-ve
2	-2	-1.5	-1.750	+ve
3	-1.750	-1.5	-1.625	-ve
4	-1.750	-1.625	-1.688	-ve
5	-1.750	-1.688	-1.719	-ve
6	-1.750	-1.719	-1.735	-ve
7	-1.750	-1.735	-1.743	+ve
8	-1.743	-1.735	-1.739	-ve
9	-1.743	-1.739	-1.741	-ve

10)	1.743	-1.741	-1.742	+ve
11	-1.742	-1.741	-1.742	-ve

Ans = -1.742

Solution of System of Linear Equations

Method III Secant Method:



This method is an important improvement over the method of false position. It does not require the condition $f(a)$ & $f(b)$ must have the

opposite sign.

Let a & b are two initial limits of the interval. The equation of secant (chord) joining the points $A[a, f(a)]$, $B[b, f(b)]$ is given by.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$\text{or } y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a)$$

But this line cuts the x -axis at $(x_1, 0)$

$$\therefore 0 - f(a) = \frac{f(b) - f(a)}{b - a} (x_1 - a)$$

$$\text{or, } x_1 - a = \frac{-f(a)(b - a)}{f(b) - f(a)}$$

$$\text{or, } x_1 = \frac{-f(a)(b - a)}{f(b) - f(a)} + a$$

$$\text{or, } x_1 = \frac{-bf(a) + af(a) + af(b) - af(a)}{f(b) - f(a)}$$

$$\therefore x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

which is an first approximation. Continuing this procedure we get desire solution

$$\text{In general } x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

setting $x =$
No sign

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- 1) Using secant method, Find a real root of $x e^x = \cos x$ correct to 4 decimal places.

Soln:

Here,

$$f(x) = x e^x - \cos x$$

$$f(0) = -1 \text{ (-ve)}$$

$$f(1) = 2.17798 \text{ (+ve)}$$

$$\text{let } a = 0, b = 1$$

$$f(a) = -1, f(b) = 2.17798 \text{ (+ve)}$$

We know, secant formula

$$x_n = \frac{a f(b) - f(a) b}{f(b) - f(a)}$$

Tabulating for successive approximation.

	A	B	C	D	$a f(b) - b f(a)$	
Iteration	a	f(a)	b	f(b)	$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$	$x e^x - \cos x$
1.	0	-1	1	2.17798	0.31467	-0.51987
2.	0.31467	-0.51987	0	-1	0.65538	0.46938
3.	0.65538	0.46938	0.31467	-0.51987	0.49372	-0.07167
4.	0.49372	-0.07167	0.65538	0.46938	0.51513	-0.00796
5.	0.51513	-0.00796	0.49372	-0.07167	0.51780	0.00014
6.	0.51780	0.00014	0.51513	-0.00796	0.51775	-0.00001
7.	0.51775	-0.00001	0.51780	0.00014	0.51775	0.00000
8.						(END)
9.						

Required root = 0.5177

setting $x = \frac{AD - CB}{D - B} = f(x)$

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2. Determine root of $e^x = x^3 + \cos 25x$. Using secant method correct to 4 decimal places

Soln:

$$f(x) = e^x - x^3 - \cos 25x$$

$$f(1) = 0.72708$$

$$f(2) = -1.57591$$

Let

$$a = 1, \Rightarrow f(a) = 0.72708$$

$$b = 2 \Rightarrow f(b) = -1.57591$$

We know secant formula,

$$x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Tabulating for successive approximation

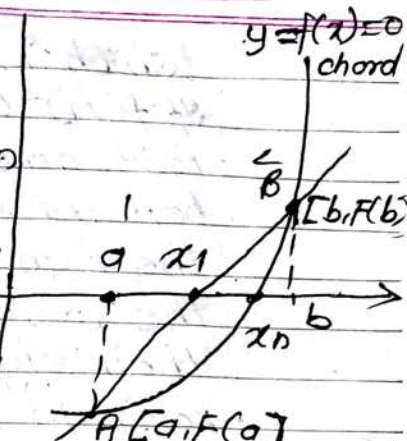
	A	B	C	D		
It ⁿ	a	f(a)	b	f(b)	$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$	$e^x - x^3 - \cos 25x$
1	1	0.72708	2	-1.57591	1.31571	1.35598
2	1.31571	1.35598	1	0.72708	0.63500	2.61704
3	0.63500	2.61704	1.31571	1.35598	2.04766	-1.43698
4	2.04766	-1.43698	0.63500	2.61704	1.54693	0.43338
5	1.54693	0.43338	2.04766	-1.43698	1.66295	1.41919
6	1.66295	1.41919	1.54693	0.43338	1.49593	0.16085
7	1.49593	0.16085	1.66295	1.41919	1.47458	0.49142
8	1.47458	0.49142	1.49593	0.16085	1.50632	0.09311
9	1.50632	0.09311	1.47458	0.49142	1.51374	0.08550
10	1.51374	0.08550	1.50632	0.09311	1.59711	1.47622
11	1.59711	1.47622	1.51374	0.08550	1.50861	0.08711

12	1.50861	0.08711	1.59711	1.47622	1.50306	0.10723
13	1.50306	0.10723	1.50861	0.08711	1.53264	0.21454
14	1.53264	0.21454	1.50306	0.10723	1.47350	0.51396
15	1.47350	0.51396	1.53264	0.21454	1.57501	1.02899
16	1.57501	1.02899	1.47350	0.51396	1.37220	2.32855
17	1.37220	2.32855	1.57501	1.02899	1.73559	0.38590
18	1.73559	-0.38590	1.37220	2.32855	1.68399	0.91834
19	1.68399	0.91834				

Method-IV

False Position Method

This method is oldest method for computing the real roots of an equation $f(x)=0$. For this method we find two numbers a & b such that $f(a)$ & $f(b)$ are of different signs. Hence a root lies between a & b . The curve $y=f(x)=0$ must cross the x -axis betⁿ a & b .



The equation of chord joining the points $A[a, f(a)]$ & $B[b, f(b)]$ is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$\text{or } y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a)$$

But this line cuts x -axis at $(x_1, 0)$

$$\therefore 0 - f(a) = \frac{f(b) - f(a)}{b - a} (x_1 - a)$$

$$\text{or, } (x_1 - a) = \frac{-f(a)(b - a)}{f(b) - f(a)}$$

$$\text{or } x_1 = \frac{-f(a)(b - a) + a}{f(b) - f(a)}$$

$$= \frac{-f(a)b + af(a) + af(b) - af(a)}{f(b) - f(a)}$$

$$= \frac{af(b) - bf(a)}{f(b) - f(a)}$$

which is first approximation
 If $f(x_1) = 0$ then x_1 is the exact root. otherwise the root lies betⁿ a & x_1 or x_1 & b according to the sign of $f(x_1)$. Replacing the value by x_1 & $f(x_1)$ as their sign, we get other approx. This procedure is continued to get root of desired accuracy.

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

1) Find a real roots $\log x - \cos x = 0$ correct to 4 decimal places.

$$f(x) = \log x - \cos x$$

$$f(0) = \text{Not defined}$$

$$f(1) = -0.54030 \text{ (-ve)}$$

$$f(2) = 0.71718 \text{ (+ve)}$$

Since $f(1)$ & $f(2)$ have opp sign. A root lies betⁿ 1 & 2

$$\text{let } a = 1, f(a) = -0.54030$$

$$b = 2, f(b) = 0.71718$$

We know,

False position formula

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Tabulating for successive approximation we get

	-ve		+ve			
It ⁿ	a	f(a)	b	f(b)	$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$	f(x)
1)	1	-0.54030	2	0.71718	1.42967	0.01458 (+ve)
2)	1	-0.54030	1.42967	0.01458	1.41838	-0.00003 (-ve)
3)	1.41838	-0.00003	1.42967	0.01458	1.41840	-0.000004 (-ve)
4)	1.41840	-0.000004	1.42967	0.01458	1.41840	END

Reqⁿ root = 1.4184 (Ans)

1) Find a real root of $e^{\cos x} - \sin x - 1 = 0$ correct to 4 decimal places using false position method.

soln:

$$f(x) = e^{\cos x} - \sin x - 1$$

$$f(0) = 1.71828 \text{ (+ve)}$$

$$f(1) = -0.12495 \text{ (-ve)}$$

Now,

$$a = 0, f(a) = 1.71828$$

$$b = 1, f(b) = -0.12495$$

Since $f(1)$ and $f(2)$ have opp. sign
So root lies betⁿ 1 and 2

Now,

From False position:

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Tabulating successive approximation

Iteration	a	^{+ve} f(a)	b	^{-ve} f(b)	$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x_n)$
1	0	1.71828	1	-0.12495	0.93221	0.01201 (+ve)
2	0.93221	0.01201	1	-0.12495	0.93815	-0.00017 (-ve)
3	0.93221	0.01201	0.93815	-0.00017	0.93807	0.000005 (+ve)
4	0.93807	0.000005	0.93815	-0.00017	0.93807	END

Ans = 0.93807

Fixed Point Iteration Method

Let $f(x) = 0$ be the given equation where roots are to be determined. In iteration method we write the given eqⁿ in the form of $x = \phi(x)$.

Let $x = x_0$ be the initial approximation of the required root, then the first approximation is given by $x_1 = \phi(x_0)$

Second, third & other approximations are

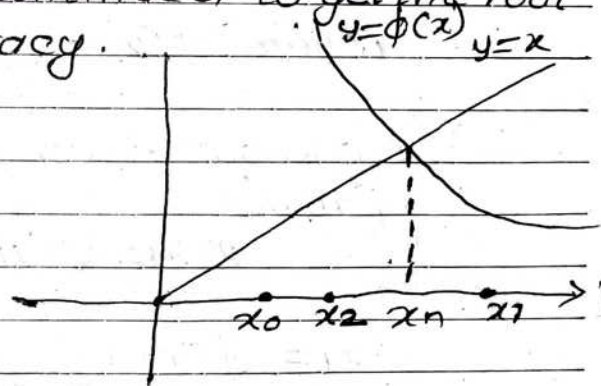
$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2) \text{ \& so on}$$

In General

$$x_{n+1} = \phi(x_n)$$

It is called fixed point iteration formula. This process is continued to get the root of desired accuracy.



1) By using fixed point iteration
Find a positive root of $x^3 + x^2 - 1 = 0$
correct to 4 decimal places
Soln:

$$f(x) = x^3 + x^2 - 1$$

$$f(0) = -1 \text{ (-ve)}$$

$$f(1) = 1 \text{ (+ve)}$$

Since $f(0)$ & $f(1)$ have opp sign
A root lies betⁿ 0 & 1.

Let initial approximation $x_0 = 0.5$

Now, Rearranging given eqⁿ
we get

$$x^3 + x^2 - 1 = 0$$

$$\text{or } x^2(x+1) = 1$$

$$\text{or } x = \frac{1}{x^2 + 1} = \phi(x)$$

$$\text{where } \phi(x) = \frac{1}{x^2 + 1}$$

We know fixed point iteration
formula

$$x_{n+1} = \phi(x_n)$$

$$x_1 = \phi(x_0) = \frac{1}{0.5^2 + 1} = 0.8650$$

$$x_2 = \phi(x_1) = \frac{1}{0.8650^2 + 1} = 0.74196$$

$$x_3 = \phi(x_2) = \frac{1}{0.74196^2 + 1} = 0.75767$$

$$x_4 = \phi(x_3) = \frac{1}{0.75767^2 + 1} = 0.75428$$

$$x_5 = \phi(x_4) = \frac{1}{0.75428^2 + 1} = 0.75501$$

$$x_6 = \phi(x_5) = \frac{1}{0.75501^2 + 1} = 0.75485$$

$$x_7 = \phi(x_6) = \frac{1}{0.75485^2 + 1} = 0.75485$$

Hence $x_6 = x_7$

So req^d root = 0.7548

$$\text{Error} = |0.75488 - 0.75485| = 0.00003$$

2) Using fixed point iteration method. Find
square root of 5. correct to 4 decimal
places.

Here $x = \sqrt{5}$

$$\text{or } x^2 = 5$$

$$\text{or } x^2 - 5 = 0$$

$$\therefore f(x) = x^2 - 5$$

$$f(0) = -5$$

$$f(2) = -1 \text{ (-ve)}$$

$$f(3) = 4 \text{ (+ve)}$$

Since $f(2)$ & $f(3)$ have opp sign
root lies betⁿ 2 & 3

Let initial approximation

$$x_0 = 2.5$$

Now Rearranging the given
eqⁿ we get

Now,

$$x^2 - 5 = 0$$

$$\text{or } x^2 = 5$$

$$\therefore x = \frac{5}{x} = \phi(x)$$

$$\text{where } \phi(x) = \frac{5}{x}$$

By formula $x_{n+1} = \phi(x_n)$

The successive approximat^{ns}
are

$$x_1 = \phi(x_0) = \frac{5}{2.5} = 2$$

$$x_2 = \phi(x_1) = \frac{5}{2} = 2.5$$

$$x_3 = \phi(x_2) = \frac{5}{2.5} = 2$$

$$x_4 = \phi(x_3) = \frac{5}{2} = 2.5$$

This process does not converge
to the solution. This is called
oscillatory divergence.

Again, rearranging the given eqⁿ we get

$$x^2 = 5$$

$$\text{or } x = \frac{5}{x}$$

$$\text{or } x + x = \frac{5}{x} + x$$

$$\text{or } 2x = \frac{5 + x^2}{x}$$

$$\therefore x = \frac{5 + x^2}{2x} = \phi(x)$$

Now, the successive approximations

$$x_1 = \phi(x_0) = \frac{5 + 2.5^2}{2(2.5)} = 2.25000$$

$$x_2 = \phi(x_1) = \frac{5 + 2.25000^2}{2(2.25000)} = 2.23611$$

$$x_3 = \phi(x_2) = \frac{5 + 2.23611^2}{2(2.23611)} = 2.23607$$

$$x_4 = \phi(x_3) = \frac{5 + 2.23607^2}{2(2.23607)} = 2.23607$$

3) $2x - \log_{10} x = 7$ 4 decimal

$$f(x) = 2x - \log_{10} x - 7$$

$$f(1) = -5 \text{ (-ve)}$$

$$f(3) = -1.47712 \text{ (-ve)}$$

$$f(4) = 0.39739 \text{ (+ve)}$$

Here $f(3)$ and $f(4)$ have opp sign. So ~~3.00~~ root lies betⁿ 3 and 4

Now, $x_0 = 3.2$

Rearranging the given eqⁿ is

$$2x - \log_{10} x = 7$$

$$\therefore x = \frac{7 + \log_{10} x}{2}$$

Then the successive approx

$$x_{n+1} = \phi(x_n)$$

$$x_1 = \phi(x_0) = \frac{7 + \log(x_0)}{2} = 3.75257$$

$$x_2 = \phi(x_1) = \frac{7 + \log(x_1)}{2} = 3.78716$$

$$x_3 = \phi(x_2) = \frac{7 + \log(x_2)}{2} = 3.78916$$

$$x_4 = \phi(x_3) = \frac{7 + \log(x_3)}{2} = 3.78927$$

$$x_5 = \phi(x_4) = \frac{7 + \log(x_4)}{2} = 3.78928$$

Now,

Required root = 3.7892

Chapter-Two

Solution of System of linear Equations.

Method I > Gauss Elimination Method.

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular matrix form from which the unknowns are found by back substitution.

Let us consider a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The augmented matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix}$$

With the help of a_1 make a_2 & a_3 to zero we get

$$\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & b_3' & c_3' & : & d_3' \end{bmatrix}$$

With the help of b_2' make b_3' to zero,

$$\begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3''' & : & d_3''' \end{bmatrix}$$

Which is an upper triangular matrix
Using back substitution method

$$\text{From } R_3, c_3''z = d_3'' \Rightarrow z$$

$$\text{From } R_2, b_2'y + c_2'z = d_2' \Rightarrow y$$

$$\text{From } R_1, a_1x + b_1y + c_1z = d_1 \Rightarrow x$$

Hence we get value of x, y & z .

1) Using Gauss Elimination method,

$$2x + y + z = 4$$

$$4x + 2y + 3z = 4$$

$$x - y + z = 0$$

Here augmented matrix form is

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 4 & 2 & 3 & 4 \\ 1 & -1 & 1 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow 2R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & 0 & 1 & -4 \\ 0 & -3 & 1 & -4 \end{array} \right]$$

$R_2 \leftrightarrow R_3$, we get

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & -3 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{array} \right]$$

Then it is an upper triangular matrix
Using back substitution method

$$\text{From } R_3, 1z = -4 \Rightarrow z = -4$$

$$\text{From } R_2, -3y + z = -4$$

$$\text{or } -3y - 4 = -4$$

$$\therefore y = 0$$

$$\text{From } R_1, 2x + y + z = 4$$

$$2x + 0 + (-4) = 4$$

$$\text{or } 2x = 8$$

$$\therefore x = 4$$

$$\text{Hence } x = 4$$

$$y = 0$$

$$z = -4$$

Solve the system of linear equations
by using Gauss Elimination method.

Soln:

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$$

$$12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$$

$$3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$$

$$-6x_1 + 4x_2 + x_3 - 18x_4 = -34$$

Here augmented matrix form

$$\left[\begin{array}{cccc|c} 3 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array} \right]$$

$$R_1 \rightarrow R_1 \div 2$$

$$R_2 \rightarrow R_2 \div 2, \text{ we get.}$$

$$\left[\begin{array}{cccc|c} 3 & -1 & 1 & 2 & 8 \\ 6 & -4 & 3 & 5 & 13 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 + 2R_1$$

$$\left[\begin{array}{cccc|c} 3 & -1 & 1 & 2 & 8 \\ 0 & -2 & 1 & 1 & -3 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & -18 \end{array} \right]$$

$$R_2 \rightarrow R_2 \div 2$$

$$R_4 \rightarrow R_4 + R_2$$

$$\left[\begin{array}{cccc|c} 3 & -1 & 1 & 2 & 8 \\ 0 & -2 & 1 & 1 & -3 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 3 & -1 & 1 & 2 & 8 \\ 0 & -2 & 1 & 1 & -3 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array} \right]$$

which is an upper triangular matrix
Using back substitution we get

$$\text{From } R_4: -3x_4 = -3 \Rightarrow x_4 = 1$$

$$\text{From } R_3: 2x_3 - 5x_4 = -9 \Rightarrow x_3 = -2$$

$$\text{From } R_2: -2x_2 + x_3 + x_4 = -3 \Rightarrow x_2 = 1$$

$$\text{From } R_1: 3x_1 - x_2 + x_3 + 2x_4 = 8 \Rightarrow x_1 = 3$$

$$\text{Hence } x_1 = 3$$

$$x_2 = 1$$

$$x_3 = -2$$

$$x_4 = 1$$

answer

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Gauss Elimination Method with partial pivoting:

In the first step of Gauss Elimination method, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging 1st equation with the equation having largest coefficient of x .

In second step, the ~~un~~ numerically largest, coefficient of y is chosen from the remaining equations and brought as the second pivot by interchanging the second equation with the equation having largest coefficient of y . This process is continued till we arrive at eqn with single variable. This modified procedure is called partial pivoting.

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1)
Solve the following by using Gauss Elimination with partial pivoting:

$$2x + 2y + z = 6$$

$$4x + 2y + 3z = 4$$

$$x - y + z = 0$$

Soln:

Here the augmented matrix form is

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & -1 & 1 & 0 \end{array} \right]$$

$R_1 \leftrightarrow R_2$, we get

$$\left[\begin{array}{ccc|c} \textcircled{4} & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & -1 & 1 & 0 \end{array} \right]$$

$R_2 \rightarrow 2R_2 - R_1$

$R_3 \rightarrow 4R_3 - R_1$

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & 2 & -1 & 8 \\ 0 & -6 & 1 & -4 \end{array} \right]$$

$R_2 \leftrightarrow R_3$ we get

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & -6 & 1 & -4 \\ 0 & 2 & -1 & 8 \end{array} \right]$$

$R_3 \rightarrow 3R_3 + R_2$

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & -6 & 1 & -4 \\ 0 & 0 & -2 & 20 \end{array} \right]$$

which is an upper triangular matrix.

Using back substitution, we get

From R_3 , $-2Z = 20 \Rightarrow Z = -10$

From R_2 , $-6Y + Z = -4$

or, $-6Y - 10 = -4$

or, $-6Y = 6$

$\therefore Y = -1$

From R_1 , $4X + 2Y + 3Z = 4$

or, $4X - 2 - 30 = 4$

or, $4X = 36$

$\therefore X = 9$

Hence

$X = 9$

$Y = -1$

$Z = -10$

Hns: 1, 0, -1, 2

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2) Solve the equations by using Gauss Elimination method with partial pivoting

$$x + y + z + w = 2$$

$$x + y + 3z - 2w = -6$$

$$2x + 3y - z + 2w = 7$$

$$x + 2y + z - w = -2$$

The augmented matrix form

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{array} \right]$$

Now, $R_1 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & -1 & -2 \end{array} \right]$$

$R_2 \rightarrow 2R_2 - R_1$

$R_3 \rightarrow 2R_3 - R_1$

$R_4 \rightarrow 2R_4 - R_1$

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 0 & -1 & 7 & -6 & -19 \\ 0 & -1 & 3 & 0 & -3 \\ 0 & 1 & 3 & -4 & -11 \end{array} \right]$$

$R_4 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 3 & -1 & 2 & : & 7 \\ 0 & 1 & 3 & -4 & : & -11 \\ 0 & -1 & 3 & 0 & : & -3 \\ 0 & -1 & 7 & -6 & : & -19 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$R_4 \rightarrow R_3 + R_2$$

$$\begin{bmatrix} 2 & 3 & -1 & 2 & : & 7 \\ 0 & 1 & 3 & -4 & : & -11 \\ 0 & 0 & 6 & -4 & : & -14 \\ 0 & 0 & 10 & -10 & : & -30 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$\begin{bmatrix} 2 & 3 & -1 & 2 & : & 7 \\ 0 & 1 & 3 & -4 & : & -11 \\ 0 & 0 & 10 & -10 & : & -30 \\ 0 & 0 & 6 & -4 & : & -14 \end{bmatrix}$$

$$R_4 \rightarrow 5R_4 - 3R_3$$

$$\begin{bmatrix} 2 & 3 & -1 & 2 & : & 7 \\ 0 & 1 & 3 & -4 & : & -11 \\ 0 & 0 & 10 & -10 & : & -30 \\ 0 & 0 & 0 & 10 & : & 20 \end{bmatrix}$$

$$\text{From } R_4 \quad 10x = 20$$

$$\text{From } R_3 \quad 10x - 10z = -30$$

$$10x = -10$$

$$x = -1$$

$$\text{From } R_2 \quad 1y + 3z - 4w = -11$$

$$\text{or } y - 3 - 4 \times 2 = -11$$

$$y = 0$$

$$\text{From } R_1 \quad 2x + 3y - 1z + 2w = 7$$

$$2x = 7 - 3 \times 0 + 1 - 2$$

$$x = 1$$

$$\text{Hence } x = 1$$

$$y = 0$$

$$z = -1$$

$$w = 2$$

Method II

Gauss Jordan Method

This method is a modified form of Gauss - Elimination method. In this method the coefficient matrix is reduced to unit matrix and get the unknowns directly without using back substitution.

Let us consider a system,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The augmented matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 : d_1 \\ a_2 & b_2 & c_2 : d_2 \\ a_3 & b_3 & c_3 : d_3 \end{bmatrix}$$

$R_1 \rightarrow \frac{R_1}{a_1}$, we get

$$\begin{bmatrix} 1 & \frac{b_1'}{a_1} & \frac{c_1'}{a_1} : d_1' \\ a_2 & b_2 & c_2 : d_2 \\ a_3 & b_3 & c_3 : d_3 \end{bmatrix}$$

With help of 1 making a_2 & a_3 to zero, we get

$$\begin{bmatrix} 1 & b_1' & c_1' : d_1' \\ 0 & b_2' & c_2' : d_2' \\ 0 & b_3' & c_3' : d_3' \end{bmatrix}$$

$R_2 \rightarrow R_2 \div b_2'$ we get

$$\begin{bmatrix} 1 & b_1' & c_1' : d_1' \\ 0 & 1 & c_2'' : d_2'' \\ 0 & b_3' & c_3' : d_3' \end{bmatrix}$$

With help of 1 make b_1' & b_3' to zero, we get

$$\begin{bmatrix} 1 & 0 & c_1'' : d_1'' \\ 0 & 1 & c_2'' : d_2'' \\ 0 & 0 & c_3'' : d_3' \end{bmatrix}$$

$R_3 \rightarrow R_3 \div c_3''$ we get

$$\begin{bmatrix} 1 & 0 & c_1'' : d_1'' \\ 0 & 1 & c_2'' : d_2'' \\ 0 & 0 & 1 : d_3'' \end{bmatrix}$$

Making c_1'' & c_2'' to zero with help of 1 we get

$$\begin{bmatrix} 1 & 0 & 0 : d_1''' \\ 0 & 1 & 0 : d_2''' \\ 0 & 0 & 1 : d_3''' \end{bmatrix}$$

Hence $x = d_1'''$

$y = d_2'''$

$z = d_3'''$

Solution of System of Linear Equations

Solve:

$$2x_1 + 4x_2 + 6x_3 = -8$$

$$x_1 + 3x_2 + x_3 = 10$$

$$2x_1 - 4x_2 - 2x_3 = -12$$

Here augmented matrix form is

$$\begin{bmatrix} 2 & 4 & -6 : -8 \\ 1 & 3 & 1 : 10 \\ 2 & -4 & -2 : -12 \end{bmatrix}$$

$R_1 \rightarrow R_1 \div 2$, we get

$$\begin{bmatrix} \textcircled{1} & 2 & -3 & : & -4 \\ 1 & 3 & 1 & : & 10 \\ 2 & -4 & -2 & : & -12 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -3 & : & -4 \\ 0 & \textcircled{1} & 4 & : & 14 \\ 0 & -8 & 4 & : & -4 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 2R_2$

$R_3 \rightarrow R_3 + 8R_2$

$$\begin{bmatrix} 1 & 0 & -11 & : & -32 \\ 0 & 1 & 4 & : & 14 \\ 0 & 0 & 36 & : & 108 \end{bmatrix}$$

$R_3 \rightarrow R_3 \div 36$

$$\begin{bmatrix} 1 & 0 & -11 & : & -32 \\ 0 & 1 & 4 & : & 14 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$R_1 \rightarrow R_1 + 11R_3$

$R_2 \rightarrow R_2 - 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Hence $x_1 = 1$ $x_3 = 3$
 $x_2 = 2$

2) $2x_1 + 2x_2 - x_3 + x_4 = 4$

$4x_1 + 3x_2 - x_3 + 2x_4 = 6$

$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12$

$x_1 + 3x_2 - 2x_3 + 2x_4 = 6$

The augmented matrix form

$$\begin{bmatrix} 2 & 2 & -1 & 1 & : & 4 \\ 4 & 3 & -1 & 2 & : & 6 \\ 8 & 5 & -3 & 4 & : & 12 \\ 1 & 3 & -2 & 2 & : & 6 \end{bmatrix}$$

$R_1 \rightarrow R_1 \div 2$

$$\begin{bmatrix} \textcircled{1} & 1 & -\frac{1}{2} & \frac{1}{2} & : & 2 \\ 4 & 3 & -1 & 2 & : & 6 \\ 8 & 5 & -3 & 4 & : & 12 \\ 3 & 3 & -2 & 2 & : & 6 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4R_1$, $R_3 \rightarrow R_3 - 8R_1$, $R_4 \rightarrow R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & : & 2 \\ 0 & -1 & 1 & 0 & : & -2 \\ 0 & -3 & 1 & 0 & : & -4 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & : & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 \div -1$

$$\begin{bmatrix} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & : & 2 \\ 0 & \textcircled{1} & -1 & 0 & : & 2 \\ 0 & -3 & 1 & 0 & : & -4 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & : & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$

$R_3 \rightarrow R_3 + 3R_2$

$$0 + \frac{1}{2}x_2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & : & 0 \\ 0 & 1 & -1 & 0 & : & 2 \\ 0 & 0 & -2 & 0 & : & 2 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & : & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 \div -2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & : & 0 \\ 0 & 1 & -1 & 0 & : & 2 \\ 0 & 0 & 1 & 0 & : & -1 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & : & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - \frac{1}{2}R_3$$

$$R_2 \rightarrow R_2 + R_3$$

$$R_4 \rightarrow R_4 + \frac{1}{2}R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & : & \frac{1}{2} \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & 1 & 0 & : & -1 \\ 0 & 0 & 0 & \frac{1}{2} & : & -\frac{1}{2} \end{bmatrix}$$

$$R_4 \rightarrow R_4 \times 2$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & : & \frac{1}{2} \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & 1 & 0 & : & -1 \\ 0 & 0 & 0 & 1 & : & -1 \end{bmatrix}$$

$$4 - \frac{1}{2}x_1 - 1$$

$$0 - \frac{1}{2}x_1 + 0 + \frac{1}{2}x_2$$

$$0 - \frac{1}{2}x_1 + \frac{1}{2}x_2$$

$$R_1 \rightarrow R_1 - \frac{1}{2}R_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & 1 & 0 & : & -1 \\ 0 & 0 & 0 & 1 & : & -1 \end{bmatrix}$$

$$x_1 = 1$$

$$x_2 = 1$$

$$x_3 = -1$$

$$x_4 = -1$$

Inverse of a matrix:

1) By using Gauss Elimination Method

We know that X will be the inverse of A if $AX = I$ where I is the unit matrix of the same order as A

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a given matrix

$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ be inverse of A such that

$$AX = I$$

The augmented matrix form is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & : & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & : & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & : & 0 & 0 & 1 \end{bmatrix}$$

Using Gauss Elimination procedure, it reduces to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & : & a & 0 & 0 \\ 0 & a_{22}' & a_{23}' & : & b & c & 0 \\ 0 & 0 & a_{33}'' & : & d & e & f \end{bmatrix}$$

This can be divide into 3 systems:

$$\text{System I} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$$

Using back substitution we get $\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}$

System II $\rightarrow$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ e \end{bmatrix}$$

Using back substitution we get $\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}$

System III $\rightarrow$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$$

1) Find inverse of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\text{let } X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

be inverse of A such that

$$AX = I$$

The augmented matrix form

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow 2R_2 - 3R_1$$

$$R_3 \rightarrow 2R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 7 & 17 & -1 & 0 & 2 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 7R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array} \right]$$

which is upper triangular matrix form

It can write in 3 systems

System

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 20 \end{bmatrix}$$

From back substitution

$$\text{From } R_3 \quad -4x_{31} = 20 \Rightarrow x_{31} = -5$$

$$\text{From } R_2 \quad x_{21} + 3x_{31} = -3 \Rightarrow x_{21} = 12$$

$$\text{From } R_1 \quad 2x_{11} + x_{21} + x_{31} = 1 \Rightarrow x_{11} = -3$$

$$\therefore \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ -5 \end{bmatrix}$$

System II

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -14 \end{bmatrix}$$

or, From back substitution

$$\text{From } R_3 \quad -4x_{32} = -14$$

$$\therefore x_{32} = \frac{7}{2}$$

$$\text{From } R_2 \quad x_{22} + 3x_{32} = 2$$

$$\text{or, } x_{22} = \frac{-17}{2}$$

$$\text{From } R_1 \quad 2x_{12} + x_{22} + x_{32} = 0$$

$$x_{12} = \frac{15}{2}$$

From system III

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

From R_3

$$-4x_{33} = 2$$

$$\therefore x_{33} = -\frac{1}{2}$$

From R_2

$$x_{23} + 3x_{33} = 0$$

$$\text{or, } x_{23} = -3x_{33}$$

$$= \frac{3}{2}$$

From R_1

$$2x_{13} + x_{23} + x_{33} = 0$$

$$\therefore x_{13} = \frac{1-3}{2} = \frac{-2}{2}$$

$$\therefore x_{13} = -1$$

Hence

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & -\frac{1}{2} \end{bmatrix}$$

2) Using Gauss Elimination method.
Find inverse of

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$\text{Now, } AX = I$$

The augmented matrix form

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 + 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & \textcircled{2} & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & -1 & 1 & 1 \end{array} \right]$$

which is upper triangular matrix form.

Now,

System I:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

From R_3 ,

$$-4x_{31} = 1$$

$$\therefore x_{31} = -\frac{1}{4}$$

From R_2

$$2x_{21} - 6x_{31} = -1$$

$$\text{or } 2x_{21} = -1 + 6 \times \left(-\frac{1}{4}\right)$$

$$\therefore x_{21} = -\frac{5}{4}$$

From R_1

$$x_{11} + x_{21} + 3x_{31} = 1$$

$$\therefore x_{11} = 1 + \frac{5}{4} + 3 \times \left(-\frac{1}{4}\right)$$

$$= 3$$

System II)

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

From R_3

$$-4x_{32} = 1$$

$$\therefore x_{32} = -\frac{1}{4}$$

From R_2

$$2x_{22} - 6x_{32} = 1$$

$$\therefore x_{22} = -\frac{1}{4}$$

From R_1

$$x_{12} + x_{22} + 3x_{32} = 0$$

$$\therefore x_{12} = -\frac{5}{4}$$

System III

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From R_3

$$-4x_{33} = 1$$

$$\therefore x_{33} = -\frac{1}{4}$$

From R_2

$$2x_{32} - 6x_{33} = 0$$

$$\therefore x_{32} = \frac{6x_{33}}{2}$$

$$= -\frac{3}{4}$$

From R_1

$$x_{13} + x_{32} + 3x_{33} = 0$$

$$\therefore x_{13} = \frac{3}{2}$$

Hence

$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Solution of System of linear Equations.

Inverse of a matrix

b) By Gauss-Jordan Method

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let X be the inverse of A such that $AX = I$

The augmented matrix form is:

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array} \right]$$

By using Gauss-Jordan procedure it is reduced to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{array} \right]$$

then $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

17 Find inverse of:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

The augmented matrix form is

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array} \right]$$

$$R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & -8 & -3 & 1 & 0 \end{array} \right]$$

$$R_1 \rightarrow R_1 + R_2$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -8 & 0 & 1 & -3 \end{array} \right]$$

$$R_3 \rightarrow \frac{R_3}{-8}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/8 & 3/8 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 2R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/4 & -1/4 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/8 & 3/8 \end{array} \right]$$

$$A = \begin{bmatrix} 0 & 1/4 & -1/4 \\ -1 & 0 & 1 \\ 0 & -1/8 & 3/8 \end{bmatrix}$$

$$-4 - 3 \times 0$$

Method III > Factorisation Method:

This method is based on the fact that ^{every} square matrix A can be expressed as the product of lower triangular matrix and an upper triangular matrix. Let us consider a system of linear eqⁿ

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

In matrix form It can be written as

$$AX = B$$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\text{Let } LU = A$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now eqⁿ ① becomes

$$LUX = B \rightarrow \textcircled{\text{iii}}$$

$$\text{Let } UX = Z \rightarrow \textcircled{\text{iv}}$$

$$\text{Then } LZ = B \rightarrow \textcircled{\text{v}}$$

Solving eqⁿ (v) and then (iv) we get the value of X.

On the basis of diagonal elements there are two methods.

The factorisation with L having unit diagonal is called

a) Factorisation Method ^{is called} _{Doolittle LU}

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

b) The factorisation with U having unit diagonals is called cholesky LU factorisation method

$$LU = A$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & 0 & u_{11} & u_{12} & u_{13} \\ l_{21} & 1 & 0 & 0 & u_{22} & u_{23} \\ l_{31} & l_{32} & 1 & 0 & 0 & u_{33} \end{matrix}$$

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Solve the system by using Doolittle LU factorisation method.

$$3x_1 + 2x_2 + x_3 = 10$$

$$2x_1 + 3x_2 + 2x_3 = 14$$

$$x_1 + 2x_2 + 3x_3 = 14$$

$$u_{11} \quad 0 \quad 0$$

$$l_{21} \quad l_{22} \quad 0$$

$$l_{31} \quad l_{32} \quad l_{33}$$

$$u_{14} \quad u_{12} \quad u_{13}$$

$$0 \quad 1 \quad 1$$

Soln:

Here the given system is written in

$$AX = B$$

where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Now

$$LU = A$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$u_{11}=3, u_{12}=2, u_{13}=1$$

Now,

$$l_{21}u_{11}=2$$

$$l_{21}=\frac{2}{3}$$

$$l_{21}u_{12}+u_{22}=3$$

$$\text{or } \frac{2}{3} \times 2 + u_{22} = 3$$

$$\text{or, } u_{22} = \frac{5}{3}$$

$$l_{21}u_{13}+u_{23}=2$$

$$\frac{2}{3} \times 1 + u_{23} = 2$$

$$\therefore u_{23} = \frac{4}{3}$$

$$l_{31}u_{11}=1$$

$$\therefore l_{31} = \frac{1}{3}$$

$$l_{31}u_{12}+l_{32}u_{22}=2$$

$$\frac{1}{3} \times 2 + \frac{5}{3} u_{32} = 2$$

$$\text{or, } \frac{5}{3} u_{32} = \frac{4}{3}$$

$$\therefore u_{32} = \frac{4}{5}$$

$$l_{31}u_{13}+l_{32}u_{23}+u_{33}=3$$

$$\text{or, } \frac{1}{3} \times 1 + \frac{4}{3} \times \frac{4}{5} + u_{33} = 3$$

$$\text{or } \frac{1}{3} + \frac{4}{3} \times \frac{4}{5} + u_{33} = 3$$

$$\therefore u_{33} = \frac{8}{5}$$

Now,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{bmatrix}$$

From eqⁿ (i)

$$AX=B$$

$$LUX=B \rightarrow \text{(ii)}$$

$$\text{let } UX=Z \rightarrow \text{(iii)}$$

$$\text{From (ii)} \quad LZ=B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Using forward substitution,

$$\text{From } R_1, z_1 = 10$$

From R_2

$$\frac{2}{3} z_1 + z_2 = 14 \Rightarrow z_2 = \frac{22}{3}$$

From R_3

$$\frac{1}{3} z_1 + \frac{4}{5} z_2 + z_3 = 14 \Rightarrow z_3 = \frac{24}{5}$$

$$\therefore \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{22}{3} \\ \frac{24}{5} \end{bmatrix}$$

From eq (iii)

$$UX = Z$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Using back substitution

From R_3 $\frac{8}{5}x_3 = 14$

$$\therefore x_3 = 3$$

From R_2 : $\frac{5}{3}x_2 + \frac{4}{3}x_3 = \frac{22}{3}$
 $\Rightarrow x_2 = 2$

From R_1 : $3x_1 + 2x_2 + x_3 = 10$

$$\therefore x_1 = 1$$

Hence

$$\left. \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array} \right\}$$

2) Using LU factorisation method, solve the system

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

Here the system can be written in matrix form as

$$AX = B \longrightarrow (i)$$

where $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

let $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & u_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Equating the corresponding elements

$$u_{11} = 2, u_{12} = 3, u_{13} = 1$$

$$l_{21}u_{11} = 1 \Rightarrow l_{21} = \frac{1}{2}$$

$$l_{21}u_{12} + u_{22} = 2$$

$$\text{or, } u_{22} = 2 - \frac{1}{2} \times 3$$

$$= 2 - \frac{3}{2}$$

$$= \frac{1}{2}$$

$$l_{21}u_{13} + u_{23} = 3$$

$$\text{or, } u_{23} = 3 - \frac{1}{2} \times 1$$

$$= \frac{5}{2}$$

$$l_{31}u_{11} + l_{32}u_{22} + u_{33} = 2$$

$$\text{or, } u_{33} = 2 - 1 \times$$

$$l_{31}u_{11} = 3$$

$$\therefore u_{31} = \frac{3}{2}$$

$$l_{31}u_{12} + l_{32}u_{22} = 1$$

$$\frac{3}{2} \times 3 + \frac{1}{2} l_{32} = 1$$

$$\therefore l_{32} = -7$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$$

$$\therefore u_{33} = 18$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

From eqⁿ ①

$$AX = B$$

$$\text{or } LUX = B$$

$$\text{Let } UX = Z$$

$$UX = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

From R_1 , $z_1 = 9$

$$R_2, \frac{1}{2}z_1 + z_2 = 6 \Rightarrow z_2 = \frac{3}{2}$$

$$R_3 \div \frac{3}{2}z_1 - 7z_2 + z_3 = 8 \Rightarrow z_3 = 5$$

$$\therefore \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

From eqⁿ (iii)

$$0x = 2$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

Using back substitution

$$\text{From } R_3 \quad 18z = 5 \Rightarrow z = \frac{5}{18}$$

$$\text{From } R_2 \quad \frac{1}{2}y + \frac{5}{2}z = \frac{3}{2}$$

$$\therefore y = \frac{29}{18}$$

$$\text{From } 2x + 3y + z = 5$$

$$\therefore x = 5 - \frac{3}{2} \times \frac{29}{18} - \frac{5}{18}$$

$$= -\frac{35}{18}$$

Using cholesky (Crout's method)

$$2x_1 + 2x_2 + x_3 = 6 \quad (5, 1-6)$$

$$4x_1 + 2x_2 + 3x_3 = 4$$

$$x_1 + x_2 + x_3 = 0$$

$$AX = B$$

The augmented matrix as:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now, let $LU = A$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} l_{11} & l_{11} \times u_{12} & l_{11} u_{13} \\ l_{21} & l_{21} \times u_{12} + l_{22} & l_{21} u_{13} + l_{22} \times u_{23} \\ l_{31} & l_{31} \times u_{12} + l_{32} & l_{31} u_{13} + l_{32} u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Equating the value

$$l_{11} = 2$$

$$l_{11} \times u_{12} = 2$$

$$l_{11} u_{13} = 1$$

$$\therefore u_{12} = 1$$

$$\therefore u_{13} = \frac{1}{2}$$

$$l_{21} = 4$$

$$l_{21} u_{12} + l_{22} = 2$$

$$l_{31} = 1$$

$$4 \times 1 + l_{22} = 2$$

$$\therefore l_{22} = -2$$

$$l_{21} u_{13} + l_{22} \times u_{23} = 3$$

$$4 \times \frac{1}{2} + (-2) \times u_{23} = 3$$

$$u_{23} = -\frac{1}{2}$$

$$l_{31} u_{12} + l_{32} = 1$$

$$1 \times 1 + l_{32} = 1$$

$$\therefore l_{32} = 0$$

$$l_{31} u_{13} + l_{32} u_{23} + l_{33} = 1$$

$$\text{or, } 1 \cdot 1 + 0 \times \left(\frac{-1}{2}\right) + l_{33} = 1$$

$$\text{or, } l_{33} = 1 - \frac{1}{2} = \frac{1}{2}$$

Now,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Here $AX = B$

$$LUX = B$$

$$\text{let } UX = Z$$

$$\text{So } LZ = B$$

$$\text{Now, } \begin{bmatrix} 2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 2z_1 \\ 4z_1 - 2z_2 \\ z_1 + \frac{1}{2}z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

Using forward substitution

$$\text{From } R_1 \quad 2z_1 = 6 \Rightarrow z_1 = 3$$

$$\text{From } R_2 \quad 4z_1 - 2z_2 = 4$$

$$\text{or } 4 \times 3 - 2z_2 = 4$$

$$\text{or, } z_2 = 4$$

$$\text{From } R_3 \quad z_1 + \frac{1}{2}z_3 = 0$$

$$\text{or, } 3 + \frac{1}{2}z_3 = 0$$

$$\text{or, } \frac{1}{2}z_3 = -3$$

$$\therefore z_3 = -6$$

Again $UX = Z$

$$\text{or } \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x_1 + x_2 + \frac{1}{2}x_3 \\ x_2 - \frac{1}{2}x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

From back substitution,

$$\text{From } R_3 \quad x_3 = -6$$

$$\text{From } R_2 \quad x_2 - \frac{1}{2}x_3 = 4$$

$$\text{or, } x_2 = 4 + 3 = 7$$

$$\text{From } R_1 \quad x_1 + 7 - 3 = 3 \Rightarrow x_1 = -1$$

$$\text{Hence } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -6 \end{bmatrix}$$

Method IV

Iterative Method:

a) Gauss-Seidel Method

b) Gauss-Jacobi's Method:

a) Gauss Seidel Method:

↳ This method is a modified form of Gauss-Jacobi's method.

Let us consider a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

If the system is diagonally dominant, it can be written as,

$$\text{From eq (i)} \quad x = \frac{1}{a_1} [d_1 - b_1y - c_1z]$$

$$\text{From eq (ii)} \quad y = \frac{1}{b_2} [d_2 - a_2x - c_2z]$$

$$\text{From eq (iii)} \quad z = \frac{1}{c_3} [d_3 - a_3x - b_3y]$$

Let initial approximation be $x = x_0$, $y = y_0$, $z = z_0$

For 1st approximation

put $y = y_0$ & $z = z_0$ in eq (i) we get,

$$x_1 = \frac{1}{a_1} [d_1 - b_1y_0 - c_1z_0]$$

Put $x = x_1$ & $z = z_0$ in eq (ii) we get

$$y_1 = \frac{1}{b_2} [d_2 - a_2x_1 - c_2z_0]$$

Put $x = x_1$ & $y = y_1$ in eq (iii) we get

$$z_1 = \frac{1}{c_3} [d_3 - a_3x_1 - b_3y_1]$$

As soon as a new approximation for an unknown is found, it is immediately used in next step.

This process is repeated till values of x, y, z are obtained to desired accuracy.

1) Apply Gauss-Seidel Method, solve the systems:

$$20x + y + 2z = 17$$

$$3x + 20y - z = 18$$

$$2x - 3y + 20z = 25$$

Here the given system is diagonally dominant. It can be written as

$$x = \frac{1}{20} [17 - y + 2z]$$

$$y = \frac{1}{20} [-18 - 3x + 2z]$$

setting
 $A = \frac{17-B+2C}{20} : B = \frac{-18-2A+C}{20}$
 $C = \frac{25-2A+3B}{20}$

$$Z = \frac{1}{20} [25 - 2A + 3B]$$

Let initial approximation $x_0 = y_0 = z_0 = 0$
 Tabulating for successive approximations, we get,

It ⁿ	$x = \frac{1}{20} (17 - y + 2z)$	$y = \frac{1}{20} (-18 - 3x + z)$	$z = \frac{1}{20} (25 - 2x + 3y)$
Initial	0	0	0
1	0.85	-1.03	1.01
2)	1.00	-1.00	1.00
3)	1.00	-1.00	1.00

From table,

$$x = 1$$

$$y = -1$$

$$z = 1$$

2) Using Gauss seidal iteration method solve the system.

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned}$$

Here the given system is diagonally dominant. It can be written as,

$$x_1 = \frac{3 + 2x_2 + x_3 + x_4}{10}$$

$$x_2 = \frac{15 + 2x_1 + x_3 + x_4}{10}$$

$$x_3 = \frac{27 + x_1 + x_2 + 2x_4}{10}$$

$$x_4 = \frac{-9 + x_1 + x_2 + 2x_3}{10}$$

Let initial approximation $x_0 = x_2 = x_3 = x_4 = 0$
 Tabulating successive approximation

It ⁿ	x_1	x_2	x_3	x_4
Initial	0	0	0	0
1)	0.30	1.56	2.89	-0.14
2)	0.89	1.95	2.96	-0.02
3)	0.98	1.99	2.99	0.00
4)	1.00	2.00	3.00	0.00
5)	1.00	2.00	3.00	0.00

From table

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

$$x_4 = 0$$

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3) $10x_1 + 6x_2 - 5x_3 = 27$

$$3x_1 + 8x_2 + 10x_3 = 27$$

$$4x_1 + 10x_2 + 3x_3 = 27$$

Rearranging the given eqⁿ to make diagonally dominant we get

$$10x_1 + 6x_2 - 5x_3 = 27$$

$$4x_1 + 10x_2 + 3x_3 = 27$$

$$3x_1 + 8x_2 + 10x_3 = 27$$

Then

$$x_1 = \frac{27 - 6x_2 + 5x_3}{10}$$

$$x_2 = \frac{27 - 4x_1 - 3x_3}{10}$$

$$x_3 = \frac{27 - 3x_1 - 8x_2}{10}$$

Tabulating successive approximation

I^n	x_1	x_2	x_3
Initial	0	0	0
1)	2.70	1.62	0
2)	2.03	1.71	0.59
3)	2.03		0.72

setting:

1, -1, 1

✓ 2017 Spring

$$\frac{17-3+25}{20}:$$

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1) Solve the following set of equations by Gauss-Jacobi method.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Here the system is diagonally dominant
It can be written as

$$x = \frac{1}{20} (17 - y + 2z)$$

$$y = \frac{1}{20} (-18 - 3x + z)$$

$$z = \frac{1}{20} (25 - 2x + 3y)$$

Let initial approximation $x_0 = y_0 = z_0 = 0$
Tabulating for successive approximation.

It^n	x	y	z
Initial	0	0	0
1	0.85	-0.90	1.25
2)	1.02	-0.97	1.03
3)	1.00	-1.00	1.00
4)	1.00	-1.00	1.00

From table

$$x = 1$$

$$y = -1$$

$$z = 1$$

Ans

$$2) \quad 2x_1 + x_2 + x_3 = 5$$

$$3x_1 + 5x_2 + 2x_3 = 15$$

$$2x_1 + x_2 + 4x_3 = 8$$

Here the system is diagonally dominant

It can be written as

$$x_1 = \frac{5 - x_2 - x_3}{2}$$

$$x_2 = \frac{15 - 3x_1 - 2x_3}{5}$$

$$x_3 = \frac{8 - x_2 - 2x_1}{4}$$

Let initial approximation

$$x_0 = y_0 = z_0 = 0$$

tabulating successive approximation

It ⁿ	x_1	x_2	x_3
Initial	0	0	0
1	2.50	3.00	2.00
2)	0.00	0.70	0.00
3)	2.15	3.00	1.83
4)	0.09	0.98	0.18
5)	1.92	2.87	1.71
6)	0.21	1.16	0.32
7)	1.76	2.75	1.61

8)	0.32	1.30	0.43
9	1.64	2.64	1.52
10	0.42	1.41	0.52
11	1.54	2.54	1.44
12	0.51	1.5	0.60
13	1.45	2.45	1.37
14	0.59	1.58	0.66
15	1.38	2.38	1.31
16	0.66	1.65	0.72
17	1.32	2.32	1.26
18	0.71	1.70	0.76
19	1.27	2.27	1.22
20	0.76	1.75	0.80
21	1.23	2.22	1.18
22	0.80	1.79	0.83
23	1.19	2.19	1.15
24	0.83	1.83	0.86
25	1.16	2.16	1.13
26	0.86	1.85	0.88
27	1.14	2.13	1.11
28	0.88	1.87	0.90
29	1.12	2.11	1.09
30	0.9	1.89	0.91
31	1.10	2.10	1.08
32	0.91	1.91	0.93
33	1.08	2.08	1.07
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Eigen value & Eigen vector

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Problems by power Method

Let A be any square matrix of order $n \times n$ with elements a_{ij} we can find a column matrix X and a constant λ such that $AX = \lambda X$ where λ is called the eigen value & X is called the corresponding eigen vector.

Let A be the given matrix & $X^{(0)}$ be the initial approximation of eigen vector. We evaluate $AX^{(0)}$ which is written as $\lambda^{(1)} X^{(1)}$. This gives the first approximation. Similarly we evaluate $AX^{(1)} = \lambda^{(2)} X^{(2)}$ which gives the second approximation. we repeat this procedure to get the repetition of X as well as λ .

Note:

1) For initial approximation, take

$$X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2) Evaluate $AX^{(0)}$ and take numerically largest value as common & divide the whole element of the vector with largest value

1) Find the largest eigen value and corresponding eigen vector of matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Soln:

Here Given matrix is $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Let $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial approximation for eigen vector. Then successive approximations are:

$$AX^{(0)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2 \\ 0 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0.8 \\ 0 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.8 \\ 0 \end{bmatrix}$$

$$= 2.8 \begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 2.86 \\ 0 \end{bmatrix}$$

$$= 2.93 \begin{bmatrix} 1 \\ 0.98 \\ 0 \end{bmatrix}$$

$$= \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.96 \\ 2.98 \\ 0 \end{bmatrix}$$

$$= 2.98 \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(5)} x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 2.98 \\ 0 \end{bmatrix}$$

$$= 2.99 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(6)} x^{(6)}$$

$$Ax^{(6)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(7)} x^{(7)}$$

$$Ax^{(7)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(8)} x^{(8)}$$

Hence, largest eigen value $\lambda = 3$
corresponding eigen
vector $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$2) A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be initial approxi-
-mation for eigen vector. Then
successive approximation

$$Ax^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$$

$$= 7 \begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3.58 \\ 1.86 \\ 0 \end{bmatrix} = 3.58 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$= \lambda^{(4)} X^{(4)}$$

$$AX^4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

largest
Hence eigen value = 4
corresponding eigen vector = $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

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3) Determine the largest eigen value and the corresponding eigenvector of the matrix using power method.

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

Let $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be initial approximation for eigen vector. Then successive approximation

$$AX^{(0)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ -20 \end{bmatrix} = 15 \begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 16.25 \\ -7.50 \\ -15 \end{bmatrix} = 16.25 \begin{bmatrix} 1 \\ -2.17 \\ -0.92 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2.17 \\ -0.92 \end{bmatrix} = \begin{bmatrix} 26.44 \\ -30.52 \\ -23.16 \end{bmatrix} = 30.52 \begin{bmatrix} 0.87 \\ -1 \\ -0.76 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.87 \\ -1 \\ -0.76 \end{bmatrix} = \begin{bmatrix} 19.33 \\ -16.14 \\ -16.84 \end{bmatrix} = 19.33 \begin{bmatrix} 1 \\ -0.83 \\ -0.87 \end{bmatrix}$$

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$$AX(4) = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.83 \\ -0.87 \end{bmatrix} = \begin{bmatrix} 20.93 \\ -11.74 \\ -21.58 \end{bmatrix}$$

$$= 21.58 \begin{bmatrix} 20.93 \\ -0.68 \\ -1 \end{bmatrix}$$

$$= \lambda^5 x^{(5)}$$

$$AX(5) = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.97 \\ -0.68 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.27 \\ -11.86 \\ -20.12 \end{bmatrix}$$

$$= 20.27 \begin{bmatrix} 1 \\ -0.59 \\ -0.99 \end{bmatrix}$$

$$= \lambda^{(6)} x^{(6)}$$

$$AX(6) = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.59 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 20.33 \\ -11.14 \\ -20.38 \end{bmatrix}$$

$$= 20.38 \begin{bmatrix} 0.99 \\ -0.55 \\ -1 \end{bmatrix}$$

$$= \lambda^{(7)} x^{(7)}$$

$$AX(7) = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.99 \\ -0.55 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.05 \\ -10.50 \\ -20.00 \end{bmatrix}$$

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$$= 20.05 \begin{bmatrix} 1 \\ -0.51 \\ -1 \end{bmatrix}$$

$$AX(8) = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.51 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.04 \\ -10.12 \\ -20.04 \end{bmatrix} = 20.04 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

$$AX(9) = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.00 \\ -10 \\ -20.00 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

$$AX^{10} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.00 \\ -10.00 \\ -20.00 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

Hence largest eigen value = 20
corresponding eigen vector = $\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$

Interpolation and Approximation

Computing the value for a tabulated function at a point is called interpolation.

Suppose we are given the following values of $y = f(x)$ for a set of values of x

x	x_0	x_1	x_2	---	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$		$f(x_n)$

Then the process of finding the value of corresponding to any value of x is called interpolation.

Interpolation with unequal intervals

Let $f(x_0), f(x_1), f(x_2), f(x_3), \dots$ be the values corresponding to $x = x_0, x_1, x_2, x_3, \dots$ where $x_0, x_1, x_2, \dots$ are not equispaced. Then we can find value of $f(x)$ by following methods:

- 1) Lagrange's interpolation method.
- 2) Divided difference interpolation method.

Lagrange's Interpolation method

Let us consider a tabulated function:

x	x_0	x_1	x_2	x_3
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Then

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) \\ + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) \\ + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

This formula is called Lagrange's interpolation formula.

Similarly

x	x_0	x_1	x_2	x_3	x_4
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times f(x_0) \\ + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times f(x_1) \\ + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times f(x_2) \\ + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times f(x_3) \\ + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times f(x_4)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

1) Using Lagrange interpolation
Find $f(10)$ from following table

x	5	6	9	11
y	12	13	14	16

Soln:

Here $x_0=5$ $f(x_0)=12$
 $x_1=6$ $f(x_1)=13$
 $x_2=9$ $f(x_2)=14$
 $x_3=11$ $f(x_3)=16$
 $f(x)=?$

Using Lagrang's interpolation formula

$$\begin{aligned} f(10) &= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 \\ &+ \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 \\ &+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 \\ &+ \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16 \\ &= 14.67 \end{aligned}$$

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Using Lagrang's formula evaluate $f(9)$

Soln:

$x=9$ $f(x_0)=150$
 $x_0=5$ $f(x_1)=392$
 $x_1=7$ $f(x_2)=1452$
 $x_2=11$ $f(x_3)=2366$
 $x_3=13$ $f(x_4)=5202$
 $x_4=17$

Now,

$$\begin{aligned} f(9) &= \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\ &+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\ &+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\ &+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\ &+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 \\ &= 810 \end{aligned}$$

Hence $f(9) = 810 //$

Find Eqⁿ $f(x)$

3)	x	0	1	3	4
	y	-12	0	6	12

Solⁿ:

$$x_0 = 0 \quad f(x_0) = -12$$

$$x_1 = 1 \quad f(x_1) = 0$$

$$x_2 = 3 \quad f(x_2) = 6$$

$$x_3 = 4 \quad f(x_3) = 12$$

Now,

$$f(x) = \frac{(x-1)(x-3)(x-4) \times (-12)}{(0-1)(0-3)(0-4)} +$$

$$+ \frac{(x-0)(x-3)(x-4) \times 0}{(1-0)(1-3)(1-4)} +$$

$$+ \frac{(x-1)(x-0)(x-4) \times 6}{(3-1)(3-0)(3-4)} +$$

$$+ \frac{(x-0)(x-1)(x-3) \times 12}{(4-0)(4-1)(4-3)}$$

$$= \frac{-12(x^2 - x + 3)(x-4)}{-1 \times -3 \times -4} +$$

$$\frac{6(x^2 - x)(x-4)}{2 \times 3 \times -9} + \frac{12(x^2 - x)(x-3)}{4 \times 3 \times 1}$$

$$= \frac{-12(x^3 - x^2 + 3x - 4x^2 + 4x - 12)}{-12}$$

$$+ \frac{6(x^3 - 4x^2 - x^2 + 4x)}{6 \times -9} + \frac{12(x^3 - 3x^2 - x^2 + 3x)}{12}$$

$$= (x^3 - 5x^2 + 7x - 12) + x^3 - 4x^2 + 3x$$

$$+ \frac{(x^3 - 5x^2 + 4x)}{-9}$$

$$= \frac{-9(x^3 - 9x^2 + 10x - 12) + x^3 - 5x^2 + 4x}{-9}$$

$$= \frac{-18x^3 + 81x^2 - 90x + 108 + x^3 - 5x^2 + 4x}{-9}$$

$$= \frac{-17x^3 + 76x^2 - 86x + 108}{-9}$$

$$= \frac{17x^3 - 76x^2 + 86x - 108}{9} //$$

4) The function $y = f(x)$ is given at the points $(7, 3)$, $(8, 1)$, $(9, 1)$ & $(10, 9)$. Find the value of y for $x = 9.5$ using Lagrange interpolation formula.

Here

Given table is

x	7	8	9	10
f(x)	3	1	1	9

Now,

$$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10) \times 3}{(7-8)(7-9)(7-10)} +$$

$$+ \frac{(9.5-7)(9.5-9)(9.5-10) \times 1}{(8-7)(8-9)(8-10)} +$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10) \times 1}{(9-7)(9-8)(9-10)} +$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-9) \times 9}{(10-7)(10-8)(10-9)}$$

Find Eqⁿ $f(x)$

3)	x	0	1	3	4
	y	-12	0	6	12

Solⁿ:

$$x_0 = 0 \quad f(x_0) = -12$$

$$x_1 = 1 \quad f(x_1) = 0$$

$$x_2 = 3 \quad f(x_2) = 6$$

$$x_3 = 4 \quad f(x_3) = 12$$

Now,

$$f(x) = \frac{(x-1)(x-3)(x-4) \times (-12)}{(0-1)(0-3)(0-4)} +$$

$$+ \frac{(x-0)(x-3)(x-4) \times 0}{(1-0)(1-3)(1-4)} +$$

$$+ \frac{(x-1)(x-0)(x-4) \times 6}{(3-1)(3-0)(3-4)} +$$

$$+ \frac{(x-0)(x-1)(x-3) \times 12}{(4-0)(4-1)(4-3)}$$

$$= \frac{-12(x^2 - x + 3)(x-4)}{-1 \times -3 \times -4} +$$

$$\frac{6(x^2 - x)(x-4)}{2 \times 3 \times -9} + \frac{12(x^2 - x)(x-3)}{4 \times 3 \times 1}$$

$$= \frac{-12(x^3 - x^2 + 3x - 4x^2 + 4x - 12)}{-12}$$

$$+ \frac{6(x^3 - 4x^2 - x^2 + 4x)}{6 \times -9} + \frac{12(x^3 - 3x^2 + 3x)}{12}$$

$$= (x^3 - 5x^2 + 7x - 12) + x^3 - 4x^2 + 3x$$

$$+ (x^3 - 5x^2 + 4x)$$

$$= \frac{-9(2x^3 - 9x^2 + 10x - 12) + x^3 - 5x^2 + 4x}{-9}$$

$$= \frac{-18x^3 + 81x^2 - 90x + 108 + x^3 - 5x^2 + 4x}{-9}$$

$$= \frac{-17x^3 + 76x^2 - 86x + 108}{-9}$$

$$= \frac{17x^3 - 76x^2 + 86x - 108}{9} //$$

4) The function $y = f(x)$ is given at the points (7, 3), (8, 1), (9, 1) & (10, 9). Find the value of y for $x = 9.5$ using Lagrange interpolation formula.

Here

x	7	8	9	10
f(x)	3	1	1	9

Now,

$$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10) \times 3}{(7-8)(7-9)(7-10)} +$$

$$+ \frac{(9.5-7)(9.5-9)(9.5-10) \times 1}{(8-7)(8-9)(8-10)} +$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10) \times 1}{(9-7)(9-8)(9-10)} +$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-9) \times 9}{(10-7)(10-8)(10-9)}$$

$$= \frac{-3}{16} - \frac{5}{16} + \frac{15}{16} + \frac{45}{16}$$

$$= \frac{58}{16}$$

$$= \cancel{3.63} \quad 3.63$$

PV (2019) Fall

4(b)

Determine the largest eigen value and corresponding eigen vector of matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$

So/n:

$$\text{let } x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Ax^{(0)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$

$$= 6 \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix} = \lambda^1 x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.16 \\ 2.36 \\ 8.03 \end{bmatrix} = 8.03 \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.15 \\ 0.24 \\ 5.99 \end{bmatrix} = 5.99 \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.07 \\ -0.08 \\ 6.26 \end{bmatrix} = 6.26 \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.36 \\ 0.28 \\ 6.95 \end{bmatrix} = 6.95 \begin{bmatrix} 0.34 \\ 0.04 \\ 1 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.22 \\ 0.52 \\ 7.16 \end{bmatrix} = 7.16 \begin{bmatrix} 0.31 \\ 0.07 \\ 1 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

$$Ax^{(6)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.31 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 0.52 \\ 7.07 \end{bmatrix} = 7.07 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$$

$$AX^{(7)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.09 \\ 0.48 \\ 7.01 \end{bmatrix}$$

$$= 7.01 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$$

$$AX^{(8)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.09 \\ 0.48 \\ 7.01 \end{bmatrix}$$

$$= 7.01 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$$

Largest eigen value ≈ 7.01
corresponding eigen vector-

$$= \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} //$$

Method II

Divided Difference Method:-

Divided difference table.

		¹ $\Delta f(x)$
x_0	$y = f(x_0)$	$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1)$
x_1	$f(x_1)$	$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2)$
x_2	$f(x_2)$	$\frac{f(x_3) - f(x_2)}{x_3 - x_2} = f(x_2, x_3)$
x_3	$f(x_3)$	ϕ
		$\Delta^2 f(x)$
$\frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = f(x_0, x_1, x_2)$		$\frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$
$\frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = f(x_1, x_2, x_3)$		
$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$		

Construct a divided difference table of

x	0	2	3	5	6
$f(x)$	1	19	55	241	415

Divided difference table

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	$\frac{19-1}{2-0} = 9$	$\frac{36-9}{3-0} = 9$	$\frac{19-8}{5-0} = 2$
2	19	$\frac{55-19}{3-2} = 36$	$\frac{93-36}{5-2} = 27$	
3	55	$\frac{241-55}{5-3} = 93$	$\frac{174-93}{6-3} = 27$	
5	241	$\frac{415-241}{6-5} = 174$		
6	415			

$\Delta^4 f(x)$
$\frac{2-2}{6-0} = 0$

Interpolation & Approximation

M-

x	5	7	11	13	17
y	150	392	1452	2366	5202

Find Evaluate $f(9)$

Soln:

The divided difference table

x	$f(x)=y$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150	$392-150 = 242$	$\frac{265-121}{7-5} = 121$	$\frac{32-24}{13-5} = 1$	$\frac{1-1}{17-5} = 0$
7	392	$1452-392 = 1060$	$\frac{457-265}{13-7} = 32$		
11	1452	$2366-1452 = 914$	$\frac{709-457}{17-11} = 42$		
13	2366	$5202-2366 = 2836$			
17	5202				

Using Newton's Divided difference.

$$f(9) = 150 + (9-5) \times 121 + (9-5) \times (9-7) \times 24 + (9-5) \times (9-7) \times (9-11) \times 1 = 810$$

3) The following table give percentage of criminals for different age groups. Using suitable interpolation formula, Find percentage of criminals under age 35

Under age	25	30	40	50
% of criminals	52	67.3	84.1	94.4

the divided difference table

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
25	52	$\frac{67.3-52}{30-25} = 3.06$	$\frac{1.68-3.06}{40-25}$	
30	67.3	$\frac{84.1-67.3}{40-30} = 1.68$	-0.092	$-\frac{0.0325+0.092}{50-25}$
40	84.1	$\frac{94.4-84.1}{50-40} = 1.03$	$\frac{1.03-1.68}{50-30} = -0.0325$	0.00238
50	94.4			

$$f(35) = 52 + (35-25) \times 3.06 + (35-25)(35-30) \times -0.092 + (35-25)(35-30)(35-40) \times 0.00238$$

$$= 77.375$$

Find polynomial eqⁿ.

4)

x	0	1	2	5
$f(x)$	2	3	12	147

The divided difference table is:

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2	$\frac{3-2}{1-0} = 1$	$\frac{9-1}{2-0} = 4$	$\frac{9-4}{5-0} = 1$
1	3	$\frac{12-3}{2-1} = 9$	$\frac{45-9}{5-1}$	
2	12	$\frac{147-12}{5-2} = 45$		
5	147			

$$f(x) = 2 + (x-0) \times 1 + (x-0)(x-1) \times 4 + (x-0)(x-1)(x-2) \times 1$$

$$= 2 + x + x(x-1) \times 4 + x(x-1)(x-2)$$

$$= 2 + x + 4x^2 - 4x + x(x^2 - 3x + 2)$$

$$= 2 + x + 4x^2 - 4x + x^3 - 3x^2 + 2x$$

$$= 2 + 3x + x^2 + x^3 - 4x$$

$$= 2 - x + x^2 + x^3 = x^3 + x^2 - x + 2$$

1st

pu 2016 (Spring)

5) x	0	1	2	4	5	6
f(x)	1	14	15	5	6	19

The divided difference table

x	f(x)	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1	$\frac{14-1}{1-0} = 13$	$\frac{1-13}{2-0} = -6$	$\frac{-2+6}{4-0} = 1$	$\frac{1-1}{5-0} = 0$
1	14	$\frac{15-14}{2-1} = 1$	$\frac{-5-1}{4-1} = -2$	$\frac{2+2}{5-1} = 1$	
2	15	$\frac{5-15}{4-2} = -5$	$\frac{1+5}{5-2} = 2$	$\frac{1-1}{6-1} = 0$	
4	5	$\frac{6-5}{5-4} = 1$	$\frac{6-2}{6-4} = 1$		
5	6	$\frac{19-6}{6-5} = 13$			
6	19				

$$f(3) = 1 + (3-0)13 + (3-0)(3-1) \times -6 + (3-0)(3-1)(3-2) \times 1 + (3-0)(3-1)(3-2)(3-4) \times 0$$

$$= 10.$$

Method III Interpolation With Equal Interval Newton Forward Interpolation Method

Formula:

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \dots$$

where $p = \frac{x-x_0}{h}$

h = size of intervals.

This formula is used if x is near to the beginning value of given tabulated function. $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ & others are found in difference table.

Method IV Newton's Backward Interpolation method

Formula: $f(x) = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \dots$

$$\frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \dots$$

where $p = \frac{x-x_0}{h}$

h = class size

$\nabla y_0, \nabla^2 y_0, \nabla^3 y_0$ & others are found in difference table.

This formula is used if x is near to the ending value of tabulated function

Q No. 1 From following table, estimate the number of students who obtained marks 45.

Marks (x)	40	50	60	70	80
No. of std ($f(x)$)	31	73	124	159	190

Using suitable interpolation
Soln.

Here $x = 45$ is near to the beginning value of tabulated functⁿ, Newton's Forward interpolation formula is suitable formula.

The difference table is given below

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42			
50	73	51	9		
60	124	35	-16	-25	
70	159	31	-4	420	37
80	190				

From table:

$$x = 45$$

$$x_0 = 40 \quad \therefore p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

$$h = 10$$

We know Newton's Forward interpolation formula.

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2!} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times (-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times 37$$

$$= 31 + 21 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times (-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times 37$$

$$= 47.8672$$

Hence, number of students who obtained marks $45 = 47.8672$
 $= 48 //$

2) The following table gives the popⁿ of the town during last six census. Estimate using Newton's interpolation formula, the increase in popⁿ during period 1946 & 1948.

Year	1911	1921	1931	1941	1951	1961
pop ⁿ (000)	12	13	20	27	39	52

Here, $x = 1946$ & 1948 are near to the ending value of the tabulated functⁿ. Newton's backward interpolation formula is suitable method

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1911	12	1				
1921	13		6			
1931	20	7		-6		
1941	27	7	0	5	11	
1951	39	12	5	-9	-20	
1961	52	13	1	-4		
		$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	$\nabla^5 y_0$	
x_0	y_0					

For 1946

$$h = 10$$

$$x = 1946$$

$$x_0 = 1961$$

$$p = \frac{x - x_0}{h} = \frac{1946 - 1961}{10} = -1.5$$

We know Newton's interpolation formula

$$f(x) = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0 + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!} \nabla^5 y_0$$

$$+ \frac{p(p+1)(p+2)(p+3)(p+4)}{5!} \nabla^5 y_0$$

$$+ \frac{p(p+1)(p+2)(p+3)(p+4)}{5!} \nabla^5 y_0$$

$$= 52 + (-1.5) \times 13 - 1.5 \frac{(-1.5+1)}{2} \times 1$$

$$- 1.5 \frac{(-1.5+1)(-1.5+2)}{3!} (-4) +$$

$$1.5 \frac{(-1.5+1)(-1.5+2)(-1.5+3)}{4!} \times (-9)$$

$$- 1.5 \frac{(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5!} \times (-20)$$

$$= 32.180$$

11) For 1948

$$x = 1948$$

$$x_0 = 1961$$

$$\therefore p = \frac{x - x_0}{h} = -1.3$$

Then

$f(x) =$

$$f(1948) = 52 + (-1.3 \times 13) - 1.3 \frac{(-1.3+1)}{2} \times 1$$

$$- 1.3 \frac{(-1.3+1)(-1.3+2)}{3!} \times (4)$$

$$- 1.3 \frac{(-1.3+1)(-1.3+2)(-1.3+3)}{4!} \times (-9)$$

$$+ 1.3 \frac{(-1.3+1)(-1.3+2)(-1.3+3)(-1.3+4)}{5!} \times (-20)$$

$$= 34.7301$$

Now,

$$\text{Increase in pop}^n = 34.7301 - 32.180$$

$$= 2.5501$$

3) From following data, estimate the no. of persons earning hourly wages between 60 & 70 rupees.

Wage Rs	20-40	40-60	60-80	80-100	100-120
pop ⁿ (000)	250	120	100	70	50

Here the difference table is as below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250	120			
" 60	370	100	-20		
" 80	470	70	-30	-10	
" 100	540	50	-20	10	20
" 120	590				

Below 70

From table

$x_0 = 40$

$h = 20$

$x = 70$

$$p = \frac{x - x_0}{h} = \frac{70 - 40}{20} = 1.5$$

From Newton's Forwarded interpolation formula

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)\Delta^2 y_0}{2!} + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \frac{(p-3)\Delta^4 y_0}{4!}$$

$$\therefore f(70) = 423.594$$

$f(60) = 370$

Then

No. of person's earning below 60
= 370No. of person's betⁿ 60 & 70

$= 423.594 - 370$

$= 53.594 \text{ thousand}$

$= 53594$

4) The following table gives the displacement of an object at various of time. Find the velocity & acceleration of object at $t=1.6$ using suitable interpolation

x	t(sec)
y	

4

Estimate the value of $\sin \theta$ at $\theta = 25^\circ$
Using Newton's forward interpolation

θ	10	20	30	40	50
$y = \sin \theta$	0.1736	0.3420	0.5000	0.6428	0.7660

the forward difference table is

θ	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	0.1736	0.1684			
20	0.3420		-0.0104		
		0.1580		-0.0098	
30	0.5000		-0.0152		0.0004
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

$$f(25) =$$

$$x = 25^\circ \quad h = 10$$

$$x_0 = 10$$

$$p = \frac{x - x_0}{h} = \frac{25 - 10}{10} = 1.5$$

Then

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 0.4226 //$$

- 4) Fitting of a str. line
 b) Fitting of a parabola
 c) Fitting of an exponential curve

Fitting of a Curve:

a) Fitting of a straight line:

The equation of straight line is

$$y = a + bx \quad \text{--- (i)}$$

Fitting of a curve means finding the value of parameters a & b with help of normal equations

The normal equations are

$$na + b \sum x = \sum y \quad \text{--- (i)}$$

$$a \sum x + b \sum x^2 = \sum xy \quad \text{--- (ii)}$$

Tabulating for $\sum x$, $\sum y$, $\sum x^2$, $\sum xy$ then solving (i) & (ii) we get value of a & b . Substituting value of a & b in eqⁿ (i) we get required eqⁿ of straight line

1) Fit a straight line by using least square method to following data

x	1	2	3	4	5
y	14	27	40	55	68

Estimate value of y when $x = 6$

Solⁿ:

Let eqⁿ of straight line to be fit is

$$y = a + bx \quad \text{--- (i)}$$

Normal eqⁿ are:

$$na + b \sum x = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

Tabulating for $\sum x$, $\sum y$, $\sum x^2$ & $\sum xy$, we get

x	y	x^2	$\sum xy$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\sum x = 15$	$\sum y = 204$	$\sum x^2 = 55$	$\sum xy = 748$

$$n = 5$$

Normal eqⁿ are

$$na + b \sum x = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

Now

$$5a + 15b = 204$$

$$15a + 55b = 748$$

We get $a = 0$

$$b = 13.6$$

From eqⁿ (i) $y = 13.6x$

when $x = 6$, $y = 13.6 \times 6 = 81.6$

Hence reqⁿ eqⁿ of str line is

$$y = 13.6x$$

$$\text{at } x=6, y=81.6$$

PV 2014 (Fall)

The following table gives height x cm & weight y kg of five persons.

x	175	165	160	155	145
y	68	58	55	52	48

Assuming linear relationship betⁿ obtain regression line. Obtain x when $y=40$.

Let linear relationship betⁿ x & y is

eqⁿ of straight line $y=a+bx$

Normal eqⁿ are

$$na + b \sum x = \sum y \quad \text{--- (i)}$$

$$a \sum x + b \sum x^2 = \sum xy \quad \text{--- (ii)}$$

x	y	x^2	xy
175	68	30625	11900
165	58	27225	9570
160	55	25600	8800
155	52	24025	8060
145	48	21025	6960
$\sum x = 800$	$\sum y$	$\sum x^2$	$\sum xy$

$$= 281 = 128500 = 45290$$

Then

$$na + b \sum x = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

Then

$$5a + b \times 800 = 281$$

$$800a + b \times 128500 = 45290$$

We get

$$\therefore a = -49.4$$

$$b = 0.66$$

Then $y = a + bx$

$$40 = a + bx$$

$$\text{or } 40 = -49.4 + 0.66x$$

$$\therefore x = 135.45 //$$

The required eqⁿ is $y = -49.4 + 0.66x$
 $x = 135.45$.

3) If p is pull required to lift a load w by means of pulley. Find a linear law of form $p = mw + c$ using following data

w	50	70	100	120
p	12	15	21	25

where p & w are taken in kg

Here

Given eqⁿ to be fit be

$$p = c + mw$$

Normal eqⁿs are

$$nc + m \sum w = \sum p$$

$$c \sum w + m \sum w^2 = \sum wp$$

w	p	w^2	pw
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
$\sum w = 340$	$\sum p = 73$	$\sum w^2 = 31800$	$\sum pw = 6750$

Then

$$4c + m340 = 73$$

$$340c + 31800m = 6750$$

We get $c = 2.276$

$$m = 0.188$$

$$\text{So } p = 0.188w + 2.276 //$$

4) Find missing value from following data $y = a + bx$

x	3	6	7	9	10
y	168	—	120	72	73

Here

Given data is

x	3	7	9	10
y	168	120	72	73

We have to find value of y when $x = 6$

Given eqⁿ of st. line be

$$y = a + bx$$

Normal eqⁿs are

$$na + b \sum x = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

x	y	x^2	xy
3	168	9	504
7	120	49	840
9	72	81	648
10	73	100	730
$\sum x = 29$	$\sum y = 433$	$\sum x^2 = 239$	$\sum xy = 2722$

Then

$$4a + 29b = 433$$

$$29a + 239b = 2722$$

We get $a = 213.46$
 $b = -14.513$

Now,

$$y = 213.46 - 14.513x$$

At $x = 6$

$$y = 213.46 - 14.513 \times 6$$

$$= 126.382 //$$

b) Fitting of a parabola

The equation of a parabola of degree two is $y = a + bx + cx^2 \rightarrow (i)$

The normal equations are

$$na + b\sum x + c\sum x^2 = \sum y \rightarrow (i)$$

$$a\sum x + b\sum x^2 + c\sum x^3 = \sum xy \rightarrow (ii)$$

$$a\sum x^2 + b\sum x^3 + c\sum x^4 = \sum x^2y \rightarrow (iii)$$

Tabulating for $\sum x, \sum y, \sum x^2, \sum x^3, \sum x^4, \sum xy$ & $\sum x^2y$ and solving normal eq^{ns} we get a, b & c

1) Fit a second degree parabola

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Find difference between actual value & calculated the value of y when $x=2$

Soln^g

Let eqⁿ of parabola to be fit is:

$$y = a + bx + cx^2 \rightarrow (1)$$

Normal equations are:

$$na + b\sum x + c\sum x^2 = \sum y \rightarrow (i)$$

$$a\sum x + b\sum x^2 + c\sum x^3 = \sum xy \rightarrow (ii)$$

$$a\sum x^2 + b\sum x^3 + c\sum x^4 = \sum x^2y \rightarrow (iii)$$

Tabulating for $\sum x, \sum y, \sum xy, \sum x^2, \sum x^3, \sum x^4$ & $\sum x^2y$

x	y	xy	x ²	x ³	x ⁴	x ² y
0	1	0	0	0	0	0
1	1.8	1.8	1	1	1	1.8
2	1.3	2.6	4	8	16	5.2
3	2.5	7.5	9	27	81	22.5
4	6.3	25.2	16	64	256	100.8
$\sum x = 10$	$\sum y = 12.9$	$\sum xy = 37.1$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum x^2y = 130.3$

$$\begin{aligned} 5a + 10b + 30c &= 12.9 \\ 10a + 30b + 100c &= 37.1 \\ 30a + 100b + 354c &= 130.3 \end{aligned}$$

We get

$$a = 1.42$$

$$b = -1.07$$

$$c = 0.55$$

Then $y = a + bx + cx^2$

$$y = 1.42 - 1.07x + 0.55x^2$$

Also when $x = 2$

$$y = 1.48$$

Then difference $= 1.38 - 1.48$
 $= -0.18$

Hence req^d eqⁿ $\therefore y = 1.42 - 1.07x + 0.55x^2$

A difference $= -0.18$

2) Fit the eqⁿ of parabola

x	1	2	3	4
y	6	11	18	27

Let eqⁿ of parabola to be fit is:

$$y = a + bx + cx^2$$

Normal equations are:

$$na + b \sum x + c \sum x^2 = \sum y$$

$$a \sum x + b \sum x^2 + c \sum x^3 = \sum xy$$

$$a \sum x^2 + b \sum x^3 + c \sum x^4 = \sum x^2 y$$

Tabulating for $\sum x, \sum x^2, \sum x^3, \sum x^4, \sum xy, \sum x^2 y$

x	y	x^2	x^3	x^4	xy	$x^2 y$
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
$\sum x = 10$	$\sum y = 62$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 190$	$\sum x^2 y = 644$

Now,

$$4a + 10b + 30c = 62$$

$$10a + 30b + 100c = 190$$

$$30a + 100b + 354c = 644$$

We get

$$a = 3$$

$$b = 2$$

$$c = 1$$

Then $y = a + bx + cx^2$
 $= 3 + 2x + x^2$
 $= x^2 + 2x + 3$

c) Fitting of an exponential curve:

i) $y = ax^b$

Taking log on both sides, we get

$$\log y = \log(ax^b)$$

$$\log y = \log a + b \log x$$

$$\text{or } y = A + bx \text{ --- (1)}$$

where $\log y = Y$

$$\log a = A$$

$$\log x = X$$

$$b = b$$

Eqⁿ (1) is eqⁿ of str line

Normal equations are

$$nA + b \sum X = \sum Y$$

$$A \sum X + b \sum X^2 = \sum XY$$

Tabulating for $\sum x, \sum y, \sum x^2, \sum xy$
then solving normal eqⁿ's we get
 a & b .

ii) $y = ae^{bx}$

Taking log on both sides

$$\log y = \log(ae^{bx})$$

$$\log y = \log a + \log e^{bx}$$

$$\log y = \log a + bx \log e$$

$$y = A + Bx \text{ --- (1)}$$

Normal eqⁿ is

$$nA + B \sum X = \sum Y$$

$$A \sum X + B \sum X^2 = \sum XY$$

iii) $xy^a = b$

Taking log on both sides

$$\log(xy^a) = \log b$$

$$\text{or } \log x + a \log y = \log b$$

$$\text{or, } a \log y = \log b - \log x$$

$$\log y = \frac{\log b - \log x}{a}$$

1) Fit a curve of form $y = ae^{bx}$ to the following data

x	0	2	4
y	5.012	10	31.62

Here the curve to be fit

$$y = ae^{bx}$$

Taking log on both sides

$$\log y = \log ae^{bx}$$

$$\text{or } \log y = \log a + \log e^{bx}$$

$$\text{or } \log y = \log a + bx \log e$$

$$\text{or } \log y = \log a + b \log e \cdot x$$

$$\therefore y = A + Bx$$

The eqⁿ (1) is eqⁿ of straight lines

The normal eqⁿs are

$$nA + B \sum X = \sum Y \text{ --- (i)}$$

$$A \sum X + B \sum X^2 = \sum XY \text{ --- (ii)}$$

x	y	$y = \log y$	xy	x^2
0	5.02	0.7	0.7	0
2	10	1	2	4
4	31.62	1.5	6	16
$\Sigma x = 6$		$\Sigma y = 3.2$	$\Sigma xy = 8$	$\Sigma x^2 = 20$

From Normal equations:

$$3A + 6B = 3.2$$

$$6A + 20B = 8$$

We get $A = 0.6667$, $B = 0.2$

Now,

$$\log a = A \Rightarrow a = \text{Antilog } A = 4.6419$$

$$b \log e = B \Rightarrow b = \frac{B}{\log e} = 0.4605$$

Then

$$y = ae^{bx}$$

$$= 4.6419 e^{0.4605x}$$

2) Fit a power curve $y = ax^b$ to the following data

x	61	26	7	2.6
y	350	400	500	600

Here $y = ax^b$

Taking log on both sides

$$\log y = \log a + b \log x$$

$$\text{or, } y = A + bX \rightarrow (1)$$

Where

$$Y = \log y$$

$$A = \log a$$

$$b = b$$

$$X = \log x$$

Then eqⁿ (i) is straight line

$$nA + b \Sigma X = \Sigma Y$$

$$A \Sigma X + b \Sigma X^2 = \Sigma XY$$

Finding ΣX , ΣX^2 , ΣY , ΣXY

x	y	X	Y	XY	X ²
61	350	1.7853	2.5441	4.5420	3.1873
26	400	1.4150	2.6021	3.6820	2.0022
7	500	0.8451	2.6990	2.2805	0.7142
2.6	600	0.4150	2.7782	1.1530	0.1722
$\Sigma X =$		$\Sigma Y =$	$\Sigma XY =$	$\Sigma X^2 =$	
4.4604		10.6234	11.6579	6.0759	

Normal eqⁿ

$$4A + 4.4604B = 10.6234$$

$$4.4604A + 6.0759B = 11.6579$$

Solving eqⁿs we get

$$A = 2.8463$$

$$B = -0.1708$$

So,

$$\log a = A = 2.8463$$

$$\therefore a = 701.94$$

$$\text{So, } y = ax^b = 701.94x^{-0.1708}$$

3) The pressure and volume of gas are related by the eqⁿ $pV^\gamma = C$ where γ & C being constant. Fit this eqⁿ to the following data

x kg/cm ²	P	0.5	1.0	1.5	2	2.5	3.0
y litres	V	1.62	1.00	0.75	0.62	0.52	0.46

Here

$$pV^\gamma = C$$

Taking log on both sides

$$\log(pV^\gamma) = \log C$$

$$\text{or, } \log p + \gamma \log V = \log C$$

$$\text{or, } \gamma \log V = \log C - \log p$$

$$\text{or, } \log V = \frac{1}{\gamma} \log C - \frac{1}{\gamma} \log p$$

$$Y = A + BX \rightarrow \text{①}$$

where

$$Y = \log V$$

$$A = \frac{1}{\gamma} \log C$$

$$B = -\frac{1}{\gamma}$$

$$X = \log p$$

Eqⁿ ① is eqⁿ of straight line.

The normal equations are

$$nA + B \sum X = \sum Y \rightarrow \text{①}$$

$$A \sum X + B \sum X^2 = \sum XY \rightarrow \text{②}$$

Tabulating for $\sum X, \sum Y, \sum XY, \sum X^2$, we get

x(P)	y(V)	X=log p	Y=log V	XY	X ²
0.5	1.62	-0.3010	0.2095	-0.0631	0.0906
1	1	0	0	0	0
1.5	0.75	0.1761	-0.1249	-0.0220	0.0310
2	0.62	0.3010	-0.2076	-0.0625	0.0906
2.5	0.52	0.3979	-0.2840	-0.1130	0.1583
3	0.46	0.4771	-0.3372	-0.1609	0.2276
$\sum X =$ 10.5		$\sum X =$ 1.0511	$\sum Y =$ -0.7442	$\sum XY =$ -0.4215	$\sum X^2 =$ 0.5981

From normal eqⁿ

$$6A + 1.0511B = -0.7442$$

$$1.0511A + 0.5981B = -0.4215$$

We get $A = -0.0008$

$$B = -0.7033$$

Now,

$$\frac{-1}{Y} = B \Rightarrow r = \frac{-1}{B} = \frac{-1}{0.7033} = 1.4219$$

Then

$$\frac{1}{Y} \log C = A \Rightarrow \log C = A \times Y$$

$$\log C = (-0.0008 \times 1.4219)$$

$$\therefore C = 0.9974$$

So,

$$PV^r = C$$

$$PV^{1.4219} = 0.9974 //$$

4) Fit curve $y = e^{a+bx}$

x	-5	-4	-3	-1	0
y	12.96	6.94	4.63	2.11	0.09

Here $y = e^{a+bx}$

$$\log y = (a+bx) \log e$$

$$\text{or } \log y = a \log e + b \log e x$$

$$Y = A + Bx \quad \text{--- (i)}$$

$$Y = \log y$$

$$A = a \log e$$

$$B = b \log e$$

$$x = x$$

Eqⁿ (i) is eqⁿ of straight line

Normal equations are:

$$nA + B \sum x = \sum Y$$

$$A \sum x + B \sum x^2 = \sum xY$$

x	y	Y = log y	x ²	xy
-5	12.96	1.1126	25	-5.5630
-4	6.94	0.8414	16	-3.3656
-3	4.63	0.6656	9	-1.9968
-1	2.11	0.3243	1	-0.3243
0	0.09	-1.0458	0	0
$\sum x = -13$		$\sum Y = 1.8981$	$\sum x^2 = 51$	$\sum xy = -11.2497$

$$5A + 13B = 1.8981 \quad \text{--- (i)}$$

$$-13A + 51B = -11.2497 \quad \text{--- (ii)}$$

We get $A = -0.5749$

$$B = -0.3671$$

Then,

$$a = \frac{A}{\log e} = \frac{-0.5749}{\log e} = -1.3238$$

$$b = \frac{B}{\log e} = \frac{-0.3671}{\log e} = -0.8453$$

Hence $-1.3238 + 0.8453x$

$$y = e$$

Chapter-Four

Numerical Differentiation and Integration



Numerical Differentiation

↳ Numerical differentiation is the process of calculating derivatives of a given tabulated function at any point. If (x_i, y_i) are the given set of values then the process of computing values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... etc is called numerical differentiation.

If the values of derivatives are required at a point near to the beginning value we use Newton's forward difference formula for derivatives & if near to the ending values we use backward formula.

a) Newton's forward difference formula for derivatives.

We know Newton's forward difference interpolation formula

$$y = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \dots$$

$$\text{or } y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0 + \frac{p^4 - 6p^3 + 11p^2 - 6p}{4!} \Delta^4 y_0 + \dots$$

Diff both side wr to p

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 + \dots$$

$$\text{where } p = \frac{x - x_0}{h}$$

$$\therefore \frac{dp}{dx} = \frac{1}{h}$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

$$= \frac{1}{h} \left[\Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 + \dots \right]$$

Similarly

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[\frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0 + \frac{12p^2-36p+22}{4!} \Delta^4 y_0 + \dots \right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3} \left[\frac{6}{3!} \Delta^3 y_0 + \frac{24p-36}{4!} \Delta^4 y_0 + \dots \right]$$

1) From the data compute $\frac{dy}{dx}$ at $x=1$

$\frac{d^2y}{dx^2}$ at $x=1$

x	1	2	3	4	5	6
y	1	8	27	64	125	216

Here $x=1$ is the beginning value of given function, we use Newton forward difference formula for derivatives. The table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1	7	12		
2	8	19	18	6	0
3	27	37	24	6	
4	64	61	30	6	0
5	125	81			
6	216				

From table

$$x_0 = 1$$

$$x = 1 \quad \therefore p = \frac{x - x_0}{h} = 0$$

$$h = 1$$

We know Forward difference interpolation formula.

$$y = \Delta y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$my = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0 + \frac{p^4 - 6p^3 + 11p^2 - 6p}{4!} \Delta^4 y_0$$

Now, diff both sides wr to x we get,

$$\frac{dy}{dx} = \frac{1}{h} \left[\Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 \right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[\frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0 + \frac{12p^2-36p+22}{4!} \Delta^4 y_0 \right]$$

At $x=1, p=0$

$$\frac{dy}{dx} = \frac{1}{1} \left[7 - \frac{1 \times 12}{2!} + \frac{2 \times 6}{3!} - \frac{6 \times 0}{4!} \right]$$

$$= [7 - 6 + 2]$$

$$= 3$$

$$\frac{d^2y}{dx^2} = \frac{2}{2!} \times 12 + \frac{6}{3!} \times 6 + \frac{22}{4!} \times 0$$

$$= 12 - \frac{36}{6}$$

$$= 12 - 6$$

$$= 6$$

Hence $\frac{dy}{dx} = 3$, $\frac{d^2y}{dx^2} = 6$.

Q) The popⁿ of a certain town is given below.

Year(x)	1931	1941	1951	1961	1971
pop ⁿ (y)	40.6	60.8	79.9	103.6	132.7

Here 1961 is near to ending value so we use Newton backward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.6				
1941	60.8	20.20			
1951	79.9	19.1	-1.1		
1961	103.6	23.70	4.6		
1971	132.7	29.1	5.4	0.8	
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$

From table

$$x = 1961$$

$$x_0 = 1931$$

$$h = 10$$

$$p = \frac{x - x_0}{h} = \frac{1961 - 1931}{10} = \frac{30}{10} = 3$$

Then

$$y = y_0 + p \Delta y_0 + \frac{p(p+1)}{2!} \Delta^2 y_0 + \frac{p(p+1)(p+2)}{3!} \Delta^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \Delta^4 y_0$$

$$y = y_0 + p \Delta y_0 + \frac{p^2 + p}{2!} \Delta^2 y_0 + \frac{p^3 + 3p^2 + 2p}{3!} \Delta^3 y_0 + \frac{p^4 + 6p^3 + 11p^2 + 6p}{4!} \Delta^4 y_0$$

$$y = y_0 + p \Delta y_0 + \frac{p^2 + p}{2!} \Delta^2 y_0 + \frac{p^3 + 3p^2 + 2p}{3!} \Delta^3 y_0 + \frac{p^4 + 6p^3 + 11p^2 + 6p}{4!} \Delta^4 y_0$$

$$\frac{dy}{dp} = p + \frac{2p+1}{2!} \Delta^2 y_0 + \frac{3p^2 + 3p + 2}{3!} \Delta^3 y_0 + \frac{4p^3 + 18p^2 + 22p + 6}{4!} \Delta^4 y_0$$

Then

$$= \frac{1}{h} \left[p y_0 + \frac{2p+1}{2!} \nabla y_0 + \frac{3p^2+6p+2}{3!} \nabla^2 y_0 + \frac{4p^3+18p^2+22p+6}{4!} \nabla^3 y_0 \right]$$

$$= \frac{1}{10} \left[29.1 + \frac{-2+1}{2!} \times 5.4 + \frac{3-6+2}{3!} \times 0.8 - \frac{4+18+22+6}{4!} \times (-4.9) \right]$$

$$= 2.667 //$$

3) PU 10 times

A rod is rotating a plane. The following table gives the angle θ radian through which the rod has turned for various values of the time t seconds.

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

calculate angular velocity and angular acceleration at $t = 0.6$ s.

$\Rightarrow$ Here $t = 0.6$ is near to the ending value of given table. Newton's backward difference formula is used

The difference table

$x=t$	$y=0$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.	0	0.12			
0.2	0.12	0.37	0.25		
0.4	0.49	0.63	0.26	0.01	
0.6	1.12	0.9	0.27	0.01	0
0.8	2.02	1.18	0.28	0.01	0
1.0	3.2	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$

From table

$$x_0 = 1$$

$$x = 0.6$$

$$h = 0.2$$

$$\therefore p = \frac{x - x_0}{h} = -2$$

Then we know

$$y = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0$$

$$\text{or } y = y_0 + p \nabla y_0 + \frac{p^2+p}{2!} \nabla^2 y_0 + \frac{p^3+3p^2+2p}{3!} \nabla^3 y_0 + \frac{p^4+6p^3+11p^2+6p}{4!} \nabla^4 y_0$$

$$\frac{dy}{dx} = \frac{1}{h} \left[\nabla y_0 + \frac{2p+1}{2!} \nabla^2 y_0 + \frac{3p^2+6p+2}{3!} \nabla^3 y_0 + \frac{4p^3+18p^2+22p+6}{4!} \nabla^4 y_0 \right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[\frac{2}{2!} \nabla^2 y_0 + \frac{6p+6}{3!} \nabla^3 y_0 + \frac{12p^2+36p+22}{4!} \nabla^4 y_0 \right]$$

Then

$$v = \frac{dy}{dx} = \frac{1}{0.2} \left[1.18 + \frac{-4+1}{2!} \times 0.28 + \frac{12-12+2}{3!} \times 0.01 + \frac{4 \times 8 + 18 \times 4 - 22 \times 2 + 6 \times 0}{4!} \right]$$

$$= 3.8167 //$$

$$a = \frac{d^2y}{dx^2} = \frac{1}{0.2^2} \left[1 \times 0.28 + \frac{6 \times (-2) + 6}{3!} \times 0.01 + 0 \right]$$

$$= 6.75 //$$

$$v = 3.8167 \text{ rad/s}$$

$$a = 6.75 \text{ rad/sec}^2$$

4) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ & $\frac{d^3y}{dx^3}$ at $x=1.6$

x	1	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.75	10.03

Soln:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	7.989					
1.1	8.403	0.4140	-0.0360			
1.2	8.781	0.3780	-0.0300	0.0060		
1.3	9.129	0.3480	-0.0260	0.004	-0.002	
1.4	9.451	0.3220	-0.0230	0.003	-0.001	
1.5	9.75	0.2990	-0.0180	0.005	0.002	
1.6	10.03	0.2810				

From table

$$x = 1.6$$

$$x_0 = 1.6$$

$$h = 0.1$$

$$\therefore p = \frac{x - x_0}{h} = \frac{1.6 - 1.6}{0.1} = 0$$

$$y = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0$$

$$\frac{dy}{dp} = y_0 + p \nabla y_0 + \frac{p^2+p}{2!} \nabla^2 y_0 + \frac{p^3+3p^2+2p}{3!} \nabla^3 y_0 + \frac{p^4+6p^3+11p^2+6p}{4!} \nabla^4 y_0$$

Now

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{h} \left[\nabla y_0 + \frac{p+1}{2!} \nabla^2 y_0 + \frac{3p^2+6p+2}{3!} \nabla^3 y_0 + \frac{4p^3+18p^2+22p+6}{4!} \nabla^4 y_0 \right] \\ &= \frac{1}{0.1} \left[0.281 + \frac{1}{2!} (-0.0180) + \frac{2}{3!} (0.005) + \frac{6}{4!} \times 0.02 \right] \\ &= 2.7867 \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{h^2} \left[\frac{2}{2!} \nabla^2 y_0 + \frac{6p+6}{3!} \nabla^3 y_0 + \frac{12p^2+36p+22}{4!} \nabla^4 y_0 \right] \\ &= \frac{1}{0.1^2} \left[\frac{2}{2!} (-0.0180) + \frac{6}{3!} \times 0.005 + \frac{22}{4!} \times 0.02 \right] \end{aligned}$$

$$= \frac{1}{0.01} [-0.018 + 0.005 + 0.0018]$$

$$= \frac{1}{0.01} \times -0.0112 = -1.12$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{1}{h^3} \left[\frac{6}{3!} \nabla^3 y_0 + \frac{24p+36}{4!} \nabla^4 y_0 \right] \\ &= \frac{1}{0.1^3} [0.005 + 0] \\ &= 5 \end{aligned}$$

Numerical Differentiation

Maxima & minima of tabulated function

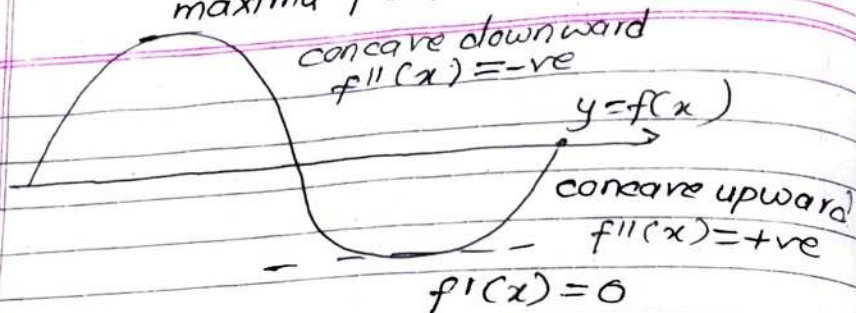
Note

i) For maxima & minima: $\frac{dy}{dx} = 0$

ii) For maxima $\frac{d^2y}{dx^2} = -ve$ / minima

For minima $\frac{d^2y}{dx^2} = +ve$ min

maxima $f'(x) = 0$



We know Newton's forward interpolation
Forward Formula:

$$y = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0$$

$$\text{or } y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0 + \dots$$

Diff both sides wrt p we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0$$

$$\frac{d^2 y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0 + \dots$$

For maxima & minima

$$\frac{dy}{dp} = 0$$

$$\text{or } \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0 = 0$$

Taking the value of Δy_0 , $\Delta^2 y_0$ & $\Delta^3 y_0$ from difference table only up to third difference of solving we get value of p

If $\frac{d^2 y}{dp^2} = +ve$ y has minima

If $\frac{d^2 y}{dp^2} = -ve$ y has maxima

From the table below, for what value of x , y is maximum? Also this value of y

x	3	4	5	6	7	8
y	0.205	0.24	0.259	0.262	0.25	0.224

Here different table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
3	0.205	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
4	0.240	0.035	-0.016	0
5	0.259	0.019	-0.016	0.001
6	0.262	0.003	-0.015	0.001
7	0.25	-0.012	-0.014	
8	0.224	-0.026		

We know Newton's forward interpolation formula

$$y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0$$

Diff both sides wr to p we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0$$

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

For maxima & minima

$$\frac{dy}{dp} = 0$$

$$\text{or } \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 = 0$$

$$\text{or } 0.035 + \frac{2p-1}{2} (-0.008) + \frac{3p^2-6p+2}{6} \times 0 = 0$$

$$\text{or } 0.035 + (2p-1)(-0.008) = 0$$

$$\text{or } (2p-1) = \frac{-0.035}{-0.008}$$

$$\therefore p = 2.688$$

When $p = 2.688$

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

$$= -0.016 \text{ (-ve)}$$

This shows that y has maxima at $p = 2.688$
Now,

$$p = \frac{x - x_0}{h}$$

$$\text{or } 2.688 = \frac{x - 3}{1}$$

$$\text{or, } x = 5.688$$

Then

$$y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0$$

$$= 0.205 + 2.688 \times 0.035 + \frac{2.688^2 - 2.688 \times 2.688}{2!}$$

$$(-0.016) + 0$$

$$= 0.2628$$

Hence maximum value of $y = 0.263$ at
 $x = 5.688$

27 From following table, find x correct to 4 decimal places for which y is maximum & find this value of y

x	1.2	1.3	1.4	1.5	1.6
y	0.932	0.9636	0.985	0.9975	0.9996

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2	0.9320	0.0316	-0.0097	-0.0002
1.3	0.9636			
1.4	0.9855	0.0219	-0.0099	0
1.5	0.9975	0.0120	-0.0099	
1.6	0.9996	0.0021		

Using Newton's forward interpolation formula

$$y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0$$

Diff wrto p

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0$$

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

For maxima & minima

$$\frac{dy}{dp} = 0$$

$$\text{or, } \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 = 0$$

$$\text{or, } 0.0316 + \frac{2p-1}{2} \times (-0.0097) + \frac{3p^2-6p+2}{6}$$

$$\times -0.0002 = 0$$

$$\text{or, } 0.0316 + 0.0049(2p-1) - \frac{3p^2-6p+2}{6}$$

$$0.0002 = 0$$

$$\text{or, } 0.0316 - 0.0049 \times 2p + 0.0049 - \frac{0.0006p^2 + 0.0012p + 0.0004}{6} = 0$$

$$\text{or, } 0.1896 - 0.0588p + 0.0049 - 0.0006p^2 + 0.0012p + 0.0004 = 0$$

$$\text{or, } 0.0006p^2 + 0.0576 - 0.1949 = 0$$

$$\therefore p = 3.2721$$

$$p = -99.2721$$

→

$$\text{Now, } 0.0316 - 0.0049 \times 2p + 0.0049 = 0$$

$$\text{or } 0.0365 - 0.0098p = 0$$

$$\therefore p = 3.7245$$

Now,

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

$$= 2 \times (-0.0097) + \frac{2 \times 3.7245 - 6}{6}$$

$$\times (-0.0002)$$

$$= -0.0194 - \text{ve}$$

So there is maxima

$$p = \frac{x-x_0}{h}$$

$$\text{or, } 3.7245 = \frac{x-1.2}{0.1} \therefore x = 1.5725$$

$$y = y_0 + p \Delta y_0 + \frac{p^2}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0$$

$$\Delta^3 y_0$$

$$= 0.9320 + 3.7245 \times 0.0316 + \frac{3.7245^2 - 3.7245 \times (-0.0097)}{2!}$$

$$= 1.0005 //$$

Hence maximum value of $y = 1.0005$
at $x = 1.5725$

Trapezoidal and Simpson's
 $\frac{1}{3}$ and $\frac{3}{8}$

1) Trapezoidal rule

$$\int_a^b f(x) dx = \frac{b-a}{2} [y_0 + y_b + 2(y_1 + y_2 + \dots + y_{n-1})]$$

2) Simpson's 1/3 rule

$$\int_a^b f(x) dx = \frac{b-a}{3} [y_0 + y_b + 2y_2 + 4y_1 + 4y_3 + 2y_4 + 4y_5 + \dots]$$

Simpson's $\frac{3}{8}$ rule

$$\int_a^b f(x) dx = \frac{3h}{8} [y_0 + y_b + 3(y_1 + y_2 + y_4 + y_5) + 3(y_3)]$$

Numerical Integration

Method IV > Romberg's Integration method

In Romberg's method we use trapezoidal rule to find

i) I_1 with $n=2$

ii) I_2 with $n=4$

iii) I_3 with $n=8$

then after

$$I_4 = I_2 + \frac{I_2 - I_1}{3}$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3}$$

$$I_6 = I_4 + \frac{I_4 - I_3}{3}$$

This process is continued to get two equal successive values of I

↓ Evaluate

$\int_0^1 \frac{1}{1+x^2} dx$ by using Romberg's formula.

Here comparing $\int_0^1 \frac{1}{1+x^2} dx$ with $\int_a^b f(x) dx$

we get

$$f(x) = \frac{1}{1+x^2}$$

$$a = 0$$

$$b = 1$$

Using Trapezoidal formula, we have to find I_1, I_2 & I_3

i) When $n=2$

$$h = \frac{b-a}{n} = \frac{1}{2}$$

x	0	0.5	1
$f(x)$	1	0.8	0.5
	y_0	y_1	y_2

$$I_1 = \frac{h}{2} [y_0 + y_2 + 2y_1]$$

$$= \frac{1}{4} [1 + 0.5 + 2 \times 0.8]$$

$$= 0.7750$$

ii) When $n=4$

$$h = \frac{b-a}{n} = \frac{1}{4}$$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.9411	0.8	0.64	0.5

$$I_2 = \frac{h}{2} [y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{8} [1 + 0.5 + 2(0.9411 + 0.8 + 0.64)]$$

$$= 0.7828$$

iii) When $n=8$

$$h = \frac{b-a}{n} =$$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	1	0.9846	0.9411	0.8767	0.8	0.7191	0.64	0.5664	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$I_3 = \frac{h}{2} [y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= 0.7847$$

Now,

Using Romberg's formula

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.7828 + \frac{0.7828 - 0.7750}{3}$$

$$= 0.7854$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.7847 + \frac{0.7847 - 0.7828}{3}$$

$$= 0.7855$$

Here $I_4 = I_5$ upto decimal places

$$\int_0^1 \frac{1}{1+x^2} dx = 0.785$$

Apply Romberg's method to evaluate

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}} dx$$

Soln:

Here $a=0$

$b=\pi/2$

Using Trapezoidal formula we find

I_1, I_2, I_3

i) When $n=2$

$$h = \frac{b-a}{n} = \frac{\pi}{4}$$

Tabulating:

x	0	0.7854	1.5708
$f(x)$	1	0.5412	0

Then

$$I_1 = \frac{h}{2} [y_0 + y_2 + 2y_1]$$

$$= \frac{\pi}{8} [1 + 0 + 2 \times 0.5412]$$

$$= 0.8178$$

ii) When $n=4$

$$h = \frac{b-a}{n} = \frac{\pi}{8}$$

Tabulating

x	0	0.3927	0.7854	1.1781	1.5708
$f(x)$	1	0.7857	0.5412	0.2759	0.0
	y_0	y_1	y_2	y_3	y_4

$$I_2 = \frac{h}{2} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= 0.8258$$

When $n=8$

$$h = \frac{b-a}{n} = \frac{\pi}{16}$$

x	0	0.1963	0.3927	0.5890	0.7854	0.9817	1.1781	1.3744	1.5708
$f(x)$	1	0.8972	0.7857	0.6667	0.5412	0.4105	0.2759	0.1386	0

$$I_3 = \frac{h}{2} [y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= 0.8278$$

$$\text{Then } I_4 = \frac{I_2 + I_3 - I_1}{3}$$

$$= \frac{0.8258 + 0.8278 - 0.8178}{3}$$

$$= 0.8285$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3}$$

$$= 0.8278 + \frac{0.8278 - 0.8258}{3}$$

$$= 0.8285$$

Here $I_4 = I_5$

$$\text{So } \int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}} = 0.8285$$

Method V > Gauss - Legendre point method

Gauss Legendre integration is based on the concept that accuracy of numerical integration can be improved by choosing the sampling points than on the basis of equal intervals.

Formula: $\int_{-1}^1 f(x) dx = \sum_{i=1}^n w_i f(x_i)$

i) Two point formula

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^2 w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2)$$

$$\text{where } w_1 = 1 \quad x_1 = -\frac{1}{\sqrt{3}}$$

$$w_2 = 1 \quad x_2 = \frac{1}{\sqrt{3}}$$

ii) Three point formula

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^3 w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

where

$$w_1 = \frac{5}{9} \quad x_1 = -\sqrt{\frac{3}{5}}$$

$$w_2 = \frac{8}{9} \quad x_2 = 0$$

$$w_3 = \frac{5}{9} \quad x_3 = \sqrt{\frac{3}{5}}$$

1) Evaluate $\int_{-1}^1 e^x dx$ by using Gauss

Legendre 2-point, 3-point formula.

Soln:

Here $a = -1$

$b = 1$

$$f(x) = e^x$$

We know Gauss Legendre

2 point formula

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + w_2 f(x_2)$$

$$= 1e^{-\frac{1}{\sqrt{3}}} + 1e^{\frac{1}{\sqrt{3}}}$$

$$= 2.3427$$

ii) 3-point formula

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$= \frac{5}{9} e^{-\frac{1}{\sqrt{3}}} + \frac{8}{9} e^0 + \frac{5}{9} e^{\frac{1}{\sqrt{3}}}$$

$$= 2.3503$$

Hence

$$\int_{-1}^1 e^x dx = 2.3427 \text{ by 2-point formula}$$

$$= 2.3503 \text{ by 3-point formula}$$

Changing limits of integration

$$\text{Let } \int_a^b f(x) dx = c \int_{-1}^1 g(z) dz$$

Assume this transformation between x & new variable z satisfies the relation

$$x = Az + B$$

$$\text{When } x = a, z = -1$$

$$\text{When } x = b, z = 1$$

$$\text{Then } a = -A + B$$

$$b = A + B$$

solving we get

$$A = \frac{b-a}{2}, B = \frac{a+b}{2}$$

From eqⁿ (i)

$$x = \frac{b-a}{2} z + \frac{a+b}{2}$$

$$\text{or } \frac{dx}{dz} = \frac{b-a}{2}$$

$$\text{or } dx = \frac{b-a}{2} dz \Rightarrow c = \frac{b-a}{2}$$

Now,

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 g(z) dz$$

$$= \frac{b-a}{2} \sum_{i=1}^n w_i g(z_i)$$

2) Evaluate the integral $\int_0^1 \frac{1}{1+x} dx$

Using Gauss Legendre 3-point formula.

Here $I = \int_0^1 \frac{1}{1+x} dx$

$a=0, b=1, f(x) = \frac{1}{1+x}$

changing limits of integration we get

$$x = \frac{b-a}{2}z + \frac{a+b}{2}$$

$$x = \frac{z}{2} + \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{1+x}$$

$$f(z) = \frac{1}{1 + \left(\frac{z}{2} + \frac{1}{2}\right)}$$

$$= \frac{2}{z+3}$$

We know, Gauss Legendre 3-point formula,

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 g(z) dz$$

$$= \frac{b-a}{2} \sum_{i=1}^3 (w_i g(z_i))$$

$$= \frac{b-a}{2} [w_1 g(z_1) + w_2 g(z_2) + w_3 g(z_3)]$$

$$= \frac{b-a}{2} \left[w_1 \left(\frac{2}{z_1+3} \right) + w_2 \left(\frac{2}{z_2+3} \right) + w_3 \left(\frac{2}{z_3+3} \right) \right]$$

$$= \frac{1-0}{2} \left[\frac{5}{9} \left(-\sqrt{\frac{3}{5}} + 3 \right) + \frac{8}{9} \left(\frac{2}{0+3} \right) + \frac{5}{9} \left(\frac{2}{\sqrt{\frac{3}{5}} + 3} \right) \right]$$

$$= \frac{1}{2} \times 0.6931$$

3) ~~2000~~ Using Gauss Legendre point formula evaluate

$$I = \int_3^5 \frac{1}{x^6+5} dx$$

Here $a=3, b=5, f(x) = \frac{1}{x^6+5}$

changing limits of integration

$$x = \frac{(b-a)}{2}z + \frac{a+b}{2}$$

$$= \left(\frac{5-3}{2} \right) z + \frac{8}{2}$$

$$\therefore x = z + 4$$

$$\therefore f(x) = \frac{1}{x^6+5}$$

$$g(z) = \frac{1}{(z+4)^6+5}$$

We know Gauss Legendre 2-point formula,

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 g(z) dz$$

$$= \frac{5-3}{2} \sum_{i=1}^2 (w_i g(z_i))$$

$$= [w_1 g(z_1) + w_2 g(z_2)]$$

$$= w_1 \left[\frac{1}{(z+4)^6+5} \right] + w_2 \left[\frac{1}{(z+4)^6+5} \right]$$

$$= \frac{5}{9} \left[\frac{1}{(-\sqrt{\frac{3}{5}}+4)^6+5} \right] + \frac{8}{9} \left[\frac{1}{(0+4)^6+5} \right]$$

$$= 0.0007$$

Then For 3 point formula

$$\int_a^b f(x) dx = \frac{b-a}{2} \left[w_1 g(z_1) + w_2 g(z_2) + w_3 g(z_3) \right]$$

$$= \frac{5-3}{2} \left[\frac{5}{9} \left[\frac{1}{(z_1+4)^6+5} \right] + \frac{8}{9} \left[\frac{1}{(z_2+4)^6+5} \right] + \frac{5}{9} \left[\frac{1}{(z_3+4)^6+5} \right] \right]$$

$$= \frac{5}{9} \left[\frac{1}{(-\sqrt{\frac{3}{5}}+4)^6+5} \right] + \frac{8}{9} \times \frac{1}{(0+4)^6+5} + \frac{5}{9} \left[\frac{1}{(\sqrt{\frac{3}{5}}+4)^6+5} \right]$$

$$= 0.0061$$

CHAPTER-5 Solution of ordinary Differential Eq^{ns}

Solution of ordinary differential equations means finding the value of y in the equation $\frac{dy}{dx} = f(x, y)$ with initial conditⁿ

$y(x_0) = y_0$ & step size h . In this method we apply step by step procedure where

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$x_3 = x_0 + 3h \text{ \& so on}$$

Methods

i) Taylor's series method

ii) Picard's method

iii) Euler's method

iv) Runge-kutta method.

v) Heun's method

vi) Shooting method

I) Taylor's series method

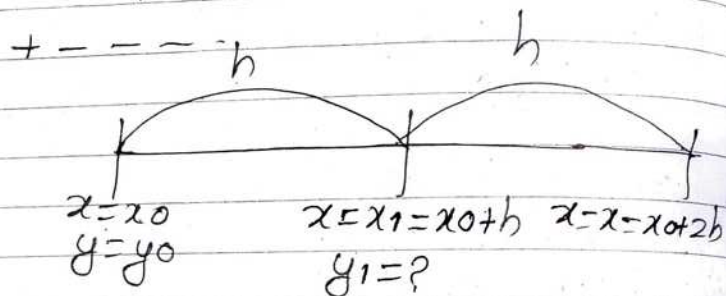
Let $\frac{dy}{dx} = y' = f(x, y)$ be a differential eqⁿ

with initial condition $y(x_0) = y_0 \Rightarrow x = x_0$, $y = y_0$ & step size h . Then

Taylor's expansion at point x_0 is given by $f(x) = f(x_0) + \frac{x-x_0}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \dots$

This can be written as

$$y = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \dots$$



But $x = x_1 = x_0 + h \Rightarrow x - x_0 = h$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV}$$

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{IV}$$

We have to evaluate up to 4th derivative

17) Using Taylor's series evaluate $y(0.3)$ $\frac{dy}{dx} =$

$x+y$ with initial condition.

$y(0) = 1$, Take $h = 0.1$

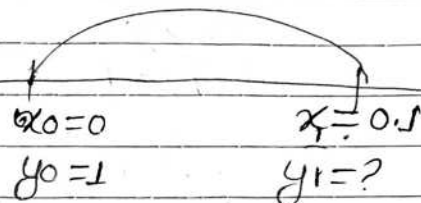
Soln:

$$y' = x + y$$

$$y(0) = 1 \Rightarrow x_0 = 0$$

$$y_0 = 1$$

$$h = 0.1$$



$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = ?$$

Now,

$$y' = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} \rightarrow \text{①}$$

where

$$y_1' = x + y \Rightarrow y_0' = x_0 + y_0 = 0 + 1 = 1$$

$$y'' = 1 + y' \Rightarrow y_0'' = 1 + y_0' = 1 + 1 = 2$$

$$y''' = 0 + y'' \Rightarrow y_0''' = y_0'' = 2$$

$$y^{IV} = y''' \Rightarrow y_0^{IV} = y_0''' = 2$$

From eqn ①

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV}$$

$$= 1 + 0.1 \times 1 + \frac{0.1^2 \times 2}{2} + \frac{0.1^3 \times 2}{3!} + \frac{0.1^4 \times 2}{4!}$$

$$= 1.1103$$

Hence, $y(0.1) = 1.1103$,

// Apply the Taylor's series method to find the value of $y(1.1)$ & $y(1.2)$

$\frac{dy}{dx} = xy^{\frac{1}{3}}, y(1) = 1, h = 0.1$

Here

$y' = xy^{\frac{1}{3}}$

$y(1) = 1$

$x_0 = 1$

$y_0 = 1$

$h = 0.1$

$y_1 = ?$

Now,

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$

where

$y' = xy^{\frac{1}{3}} \Rightarrow y_0' = x_0 y_0^{\frac{1}{3}} = 1 \cdot 1^{\frac{1}{3}} = 1$

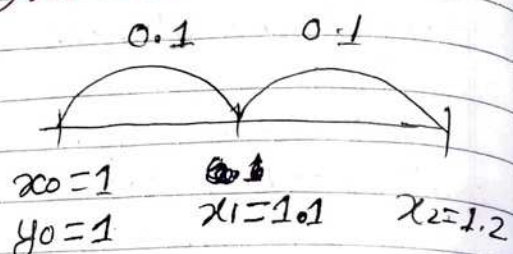
$y'' = y^{\frac{1}{3}}$

$y'' = y^{\frac{1}{3}} + \frac{x}{3y^{\frac{2}{3}}} y' \Rightarrow y_0'' = y_0^{\frac{1}{3}} + x_0 y_0'$

where

$y' = xy^{\frac{1}{3}} \Rightarrow y_0' = x_0 y_0^{\frac{1}{3}} = 1 \cdot 1^{\frac{1}{3}} = 1$

$y'' = y^{\frac{1}{3}} + \frac{x}{3y^{\frac{2}{3}}} y' \Rightarrow y_0'' = y_0^{\frac{1}{3}} + \frac{x_0 y_0'}{3 y_0^{\frac{2}{3}}}$
 $= 1 + \frac{1}{3} = \frac{4}{3}$



$y'' = y^{\frac{1}{3}} + \frac{x}{3y^{\frac{2}{3}}} y' = y^{\frac{1}{3}} + \frac{x^2 y^{-\frac{1}{3}}}{3}$

$y_0'' = 1 + \frac{1 \cdot 1}{3} = \frac{4}{3}$

$y''' = \frac{1}{3} y^{-\frac{2}{3}} + \frac{1}{3} (y^{-\frac{1}{3}} \cdot 2x + x^2 (-\frac{1}{3}) y^{-\frac{4}{3}}) y'$
 $= \frac{1}{3y^{\frac{2}{3}}} + \frac{2x}{3y^{\frac{1}{3}}} - \frac{x^2 y^{-\frac{1}{3}}}{3} \cdot xy^{\frac{1}{3}}$
 $= \frac{1}{3} y^{-\frac{2}{3}} + \frac{2}{3} xy^{-\frac{1}{3}} - \frac{x^3 y^{-1}}{3}$

$y_0''' = \frac{1}{3} y_0^{-\frac{2}{3}} + \frac{2}{3} x_0 y_0^{-\frac{1}{3}} - \frac{x_0^3 y_0^{-1}}{3}$
 $= \frac{1}{3} + \frac{2}{3} - \frac{1}{3}$
 $= \frac{2}{3}$

$y^{iv} = -\frac{2}{9} y^{-\frac{5}{3}} + \frac{2}{3} [y^{-\frac{1}{3}} - \frac{1}{3} xy^{-\frac{4}{3}}] - \frac{1}{3} [3x^2 y^{-1} - x^3 y^{-2}]$

$y_0^{iv} = -\frac{2}{9} y_0^{-\frac{5}{3}} + \frac{2}{3} [y_0^{-\frac{1}{3}} - \frac{1}{3} x_0 y_0^{-\frac{4}{3}}] - \frac{1}{3} [3x_0^2 y_0^{-1} - x_0^3 y_0^{-2}]$
 $= -\frac{2}{9} + 2 [1 - \frac{1}{3} - \frac{1}{3} (3 - 1 - 1)]$
 $= -\frac{2}{9} + 2 [1 - \frac{1}{3} - 1 + \frac{2}{3}] = \frac{4}{9}$

Then

$$y_1 = 1 + 0.1 \times 1 + \frac{0.1^2 \times 4}{2!} + \frac{0.1^3 \times 2}{3!} +$$

$$\frac{0.1^4 \times 4}{4!} + \frac{0.1^5 \times 2}{5!}$$

$$= 1.1068$$

Also

$$y_0' = 1.1 \times 1.1068^{1/2} = 1.1378$$

$$y_0'' = 1.1^{1/3} +$$

Picard's method of successive approximations
 Suppose the given differential equation is

$$\frac{dy}{dx} = f(x, y) \rightarrow (1)$$

Let us solve the given eqⁿ with the given initial condition $y = y_0$ when $x = x_0$. The given differential eqⁿ can be written as

$$dy = f(x, y) dx \rightarrow (2)$$

Integrating eqⁿ (2) from x_0 to x wrt to x , we get,

$$\int_{y_0}^y dy = \int_{x_0}^x f(x, y) dx$$

$$\text{or } [y]_{y_0}^y = \int_{x_0}^x f(x, y) dx$$

$$\text{or } y - y_0 = \int_{x_0}^x f(x, y) dx$$

$$\text{or, } y = y_0 + \int_{x_0}^x f(x, y) dx \rightarrow (3)$$

As a first approximation, we replace y by y_0 in $f(x, y)$ we get

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

Similarly replacing y by y_1 in $f(x, y)$ we get second approximation

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx \text{ \& so on}$$

$$y_3 = y_0 + \int_{x_0}^x f(x, y_2) dx$$

$$y_{n+1} = y_0 + \int_{x_0}^x f(x, y_n) dx$$

We continue the process we get two successive approximation are equal.

Q. No 1 Using picard's method, find y when $x=0.2$ show that $y(0)=1, \frac{dy}{dx}=x-y$
soln:

Here, Given that

$$\frac{dy}{dx} = f(x, y) = x - y$$

$$y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$$

$$x = 0.2, y = ?$$

We know picard's formula,

$$y_{n+1} = y_0 + \int_{x_0}^x f(x, y_n) dx$$

The successive approximations are

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$= 1 + \int_0^{0.2} (x - y_0) dx$$

$$= 1$$

$$= 0.82$$

$$2) y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx$$

$$= y_0 + \int_{x_0}^x (x - y_1) dx$$

$$= 1 + \int_0^{0.2} (x - 0.82) dx$$

$$= 0.8560$$

$$3) y_3 = y_0 + \int_{x_0}^x f(x, y_2) dx$$

$$= y_0 + \int_{x_0}^x (x - y_2) dx$$

$$= 1 + \int_0^{0.2} (x - 0.8560) dx$$

$$= 0.8488$$

$$4) y_4 = y_0 + \int_{x_0}^x f(x, y_3) dx$$

$$= 1 + \int_0^{0.2} (x - y_3) dx$$

$$= 0.8502$$

$$5) y_5 = 0.8500$$

$$y(0.2) = 0.85 \text{ Ans}$$

Q) Use Picard's method to approximate the value of y when $x=0.1, x=0.2, x=0.4$

~~Soln~~

Given that $y=1$, when $x=0$ &
 $\frac{dy}{dx} = 1+x+xy$ correct to three decimal

places (upto 3rd approximation)
 Soln:

Here Given that $\frac{dy}{dx} = f(x,y) = 1+xy$

$$y(0)=1 \Rightarrow x_0=0 \\ y_0=1$$

When $x=0.1, y=?$

When $x=0.2, y=?$

When $x=0.4, y=?$

i) For $x=0.1$

We know Picard's formula

$$y_{n+1} = y_0 + \int_{x_0}^x f(x, y_n) dx$$

The successive approximations are

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$= y_0 + \int_0^{0.1} (1+x y_0) dx \\ = 1 + \int_0^{0.1} (1+x \cdot 1) dx$$

$$= 1.1050$$

$$y_2 = 1 + \int_0^{0.1} (1+x \cdot 1.1050) dx = 1.1055$$

$$y_3 = 1 + \int_0^{0.1} (1+x \cdot 1.1055) dx = 1.1055$$

$$\text{So } y(0.1) = 1.1055$$

ii) for $x=0.2$

$$y_1 = y_0 + \int_{x_0}^x (1+x y_0) dx \\ = 1 + \int_0^{0.2} (1+x \cdot 1) dx \\ = 1.2200$$

$$y_2 = 1 + \int_0^{0.2} (1+x \cdot 1.22) dx \\ = 1.2244$$

$$y_3 = 1.2245 \\ \text{So } y(0.2) = 1.2245$$

ii) For $x=0.4$

$$y_1 = y_0 + \int_{x_0, y_0}^x f(x, y_0) dx$$

$$= 1 + \int_0^{0.4} (1+x \times 1) dx$$

$$= 1 + \int_0^{0.4} (1+x \times 1) dx$$

$$= 1.48$$

$$y_2 = 1 + \int_0^{0.4} (1+x \times 1.48) dx$$

$$= 1.5189$$

$$y_3 = 1 + \int_0^{0.4} (1+x \times 1.5189) dx$$

$$= 1.5215$$

Solution of ordinary Differential Equation

⇓

Method III > Euler's method

Consider the first order differential eqⁿ

$$\frac{dy}{dx} = f(x, y) \rightarrow (1)$$

Let us solve this differential equation under the condition $y(x_0) = y_0$. The solⁿ of (1) gives y as a function of x which may be written as $y = f(x) \rightarrow (2)$

The graph of (2) is a curve in xy plane & since a smooth curve is practically straight for a short distance from any point on it. We have from

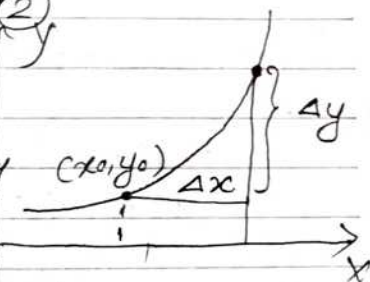


figure.

$$\tan \theta = \frac{\Delta y}{\Delta x}$$

$$\text{or, } \Delta y = \Delta x \tan \theta$$

$$\text{or, } \Delta y = \Delta x \left(\frac{dy}{dx} \right)_0$$

$$\therefore \text{ slope at } (x_0, y_0) = \left(\frac{dy}{dx} \right)_0$$

Now, $y_1 = y_0 + \Delta y$
 or $y_1 = y_0 + \left(\frac{dy}{dx}\right)_0 \Delta x$
 or $y_1 = y_0 + f(x_0, y_0) h$
 $= y_0 + h y_0'$

where $h = \Delta x$

Similarly,

$$y_2 = y_1 + h \left(\frac{dy}{dx}\right)_1$$

$$= y_1 + h y_1'$$

$$y_3 = y_2 + h y_2' \text{ \& so on}$$

In general

$$y_{n+1} = y_n + h y_n'$$

this is called Euler's formula.

1) Solve the equation $\frac{dy}{dx} = 1-y$ with the

initial condition $y(0) = 0$ using Euler's method & tabulate the solution at

$x = 0.1, 0.2, 0.3$

soln:

Here $\frac{dy}{dx} = f(x, y) = 1-y$

$$y(0) = 0 \Rightarrow x_0 = 0$$

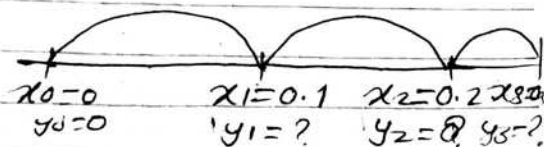
$$y_0 = 0$$

let $h = 0.1$

$$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1, y_1 = ?$$

$$x_2 = x_0 + 2h = 0 + 0.1 \times 2 = 0.2, y_2 = ?$$

$$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3, y_3 = ?$$



We know the Euler's formula,

$$y_{n+1} = y_n + h y_n'$$

Now, $y_1 = y_0 + h y_0'$
 $= y_0 + h f(x_0, y_0)$
 $= y_0 + h [1 - y_0]$
 $= 0 + 0.1 [1 - 0]$
 $= 0.1$

$$y_2 = y_1 + h y_1'$$

$$= 0.1 + h [1 - y_1]$$

$$= 0.1 + 0.1 [1 - 0.1]$$

$$= 0.19$$

$$y_3 = y_2 + h y_2'$$

$$= 0.19 + h [1 - 0.19]$$

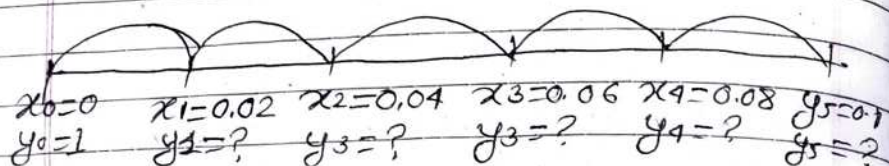
$$= 0.19 + 0.1 [1 - 0.19]$$

$$= 0.271$$

2) Solve by Euler's method:

$$\frac{dy}{dx} = \frac{y-x}{y+x}, y(0)=0.1, h=0.02, \text{ for}$$

$$x=0.1$$



$$\text{We know } y_{n+1} = y_n + h y_n'$$

$$y(0)=0.1 \Rightarrow x_0=0, y_0=0.1$$

$$\begin{aligned} \textcircled{i} \quad y_1 &= y_0 + h y_0' \\ &= y_0 + h f(x_0, y_0) \\ &= y_0 + h \left[\frac{y_0 - x_0}{y_0 + x_0} \right] \end{aligned}$$

$$= 1 + 0.02 \left[\frac{0.1 - 0}{0.1 + 0} \right]$$

$$= 1.02$$

$$\begin{aligned} \text{ii) } y_2 &= y_1 + h y_1' \\ &= 1.02 + 0.02 \left[\frac{y_1 - x_1}{y_1 + x_1} \right] \end{aligned}$$

$$= 1.02 + 0.02 \left[\frac{1.02 - 0.02}{1.02 + 0.02} \right]$$

$$= 1.0392$$

$$\begin{aligned} \text{iii) } y_3 &= y_2 + h y_2' \\ &= 1.0392 + 0.02 \left[\frac{1.0392 - 0.04}{1.0392 + 0.04} \right] \\ &= 1.0577 \end{aligned}$$

$$\begin{aligned} \text{iv) } y_4 &= y_3 + h y_3' \\ &= 1.0577 + 0.02 \left[\frac{1.0577 - 0.06}{1.0577 + 0.06} \right] \\ &= 1.0756 \end{aligned}$$

$$\begin{aligned} \text{v) } y_5 &= y_4 + h y_4' \\ &= 1.0756 + 0.02 \left[\frac{1.0756 - 0.08}{1.0756 + 0.08} \right] \\ &= 1.0785 \quad 1.0928 \end{aligned}$$

$$\begin{aligned} \text{vi) } y_6 &= y_5 + h y_5' \\ &= 1.0928 + 0.02 \left[\frac{1.0928 - 0.1}{1.0928 + 0.1} \right] \\ &= 1.1094 \end{aligned}$$

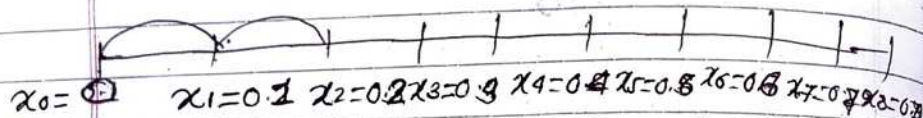
3) $\frac{dy}{dx} = x+y$, $y(0)=1$, $h=0.1$ for $x=1$

soln:

$$\frac{dy}{dx} = x+y$$

$$y(0)=1 \Rightarrow y_0=1, x_0=0$$

$$h=0.1$$



$$x_0=0 \quad x_1=0.1 \quad x_2=0.2 \quad x_3=0.3 \quad x_4=0.4 \quad x_5=0.5 \quad x_6=0.6 \quad x_7=0.7 \quad x_8=0.8$$

$$y_0=1 \quad y_1=? \quad y_2=? \quad y_3=? \quad y_4=? \quad y_5=? \quad y_6=? \quad y_7=? \quad y_8=?$$

$$x_9=0.9$$

$$y_9=?$$

$$x_{10}=1$$

$$y_{10}=?$$

We know

$$y_{n+1} = y_n + h y_n'$$

Then

$$y_1 = y_0 + h y_0'$$

$$= 1 + 0.1 [x_0 + y_0]$$

$$= 1 + 0.1 [0 + 1]$$

$$= 1.1$$

$$y_2 = y_1 + h y_1'$$

$$= 1.1 + 0.1 [0.1 + 1.1]$$

$$= 1.22$$

$$\begin{aligned} y_3 &= y_2 + h y_2' \\ &= 1.22 + 0.1 [0.2 + 1.22] \\ &= 1.362 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + h y_3' \\ &= 1.362 + 0.1 [0.3 + 1.362] \\ &= 1.5282 \end{aligned}$$

$$\begin{aligned} y_5 &= y_4 + h y_4' \\ &= 1.5282 + 0.1 [0.4 + 1.5282] \\ &= 1.7210 \end{aligned}$$

$$\begin{aligned} y_6 &= y_5 + h y_5' \\ &= 1.7210 + 0.1 [0.5 + 1.7210] \\ &= 1.9431 \end{aligned}$$

$$\begin{aligned} y_7 &= y_6 + h y_6' \\ &= 1.9431 + 0.1 [0.6 + 1.9431] \\ &= 2.1974 \end{aligned}$$

$$\begin{aligned} y_8 &= y_7 + h y_7' \\ &= 2.1974 + 0.1 [0.7 + 2.1974] \\ &= 2.4871 \end{aligned}$$

$$\begin{aligned} y_9 &= y_8 + h y_8' \\ &= 2.4871 + 0.1 [0.8 + 2.4871] \\ &= 2.8158 \end{aligned}$$

$$y_0 = y_9 + h y_9'$$

$$= 2.8158 + 0.1 [0.9 + 2.8158]$$

$$= 3.1874$$

$$y_{10} = y_9 + h y_9'$$

$$= 3.1874 + 0.1 [1 + 3.1874]$$

$$= 3.6061$$

Method IV Runge-Kutta Method

This method was devised by Runge, about the year 1934 and extended by Kutta a few years later. Therefore we call this method as Runge-Kutta method.

I) Fourth order Runge-Kutta method

This method is commonly used and is often referred to as Runge-Kutta method only or classical Fourth order R-K method.

Let $\frac{dy}{dx} = f(x, y)$ be a given differential

Equation to be solved under the initial condition $y(x_0) = y_0$. If h be the length of interval between equidistant values. Then

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$$

$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{2K_2}{2})$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

Similarly

$$y_2 = y_1 + \frac{1}{6} (K_2 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = hf(x_1, y_1)$$

$$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$$

$$K_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2})$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

& so on.

1) Apply Runge-Kutta 4th order method to find an approximate value of y when $x=0.2$. Given that $\frac{dy}{dx}=x+y$, $y(0)=1$, $h=0.2$

Soln:

$$\frac{dy}{dx} = f(x, y) = x + y$$

$$y(0)=1 \Rightarrow x_0=0, y_0=1$$

$$h=0.2$$

$$x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$y_1 = ?$$

We know R-K 4th order formula

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4) \quad \text{--- (1)}$$

where

$$K_1 = hf(x_0, y_0)$$

$$= 0.2 f(0, 1)$$

$$= 0.2 [0 + 1]$$

$$= 0.2$$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$$

$$= 0.2 f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2})$$

$$= 0.2 f(0.1, 1.1)$$

$$= 0.2 [0.1 + 1.1]$$

$$= 0.24$$

$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$$

$$= 0.2 f(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2})$$

$$= 0.2 f(0.1, 1.12)$$

$$= 0.2 [0.1 + 1.12]$$

$$= 0.244$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= 0.2 f(0 + 0.2, 1 + 0.244)$$

$$= 0.2 f(0.2, 1.244)$$

$$= 0.2 [0.2 + 1.244]$$

$$= 0.2888$$

Now,

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6} (0.2 + 2 \times 0.24 + 2 \times 0.244 + 0.2888)$$

$$y(0.2) = 1.2428$$

2) Solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0)=1$ at $x=0.4$

when $x=0.2$ Given $h=0.2$

Soln:

$$\frac{dy}{dx} = f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$$

$$\text{Here } y(0)=1 \Rightarrow x_0=0$$

$$y_0=1$$

$$h=0.2$$

$$\begin{array}{lcl} x_0=0 & x_1=0.2 & x_2=0.4 \\ y_0=1 & y_1=? & y_2=? \end{array}$$

$$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.2 = 0.4$$

We know

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4) \rightarrow \textcircled{1}$$

where

$$K_1 = hf(x_0, y_0)$$

$$= 0.2 f(0, 1)$$

$$= 0.2 \times \left(\frac{1^2 - 0^2}{1^2 + 0^2} \right)$$

$$= 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= 0.2 f(0.1, 1.1)$$

$$= 0.2 \left(\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2} \right)$$

$$= 0.1967$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1967}{2}\right)$$

$$= 0.2 f(0.1, 1.0984)$$

$$= 0.2 \left(\frac{1.0984^2 - 0.1^2}{1.0984^2 + 0.1^2} \right)$$

$$= 0.1967$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= 0.2 f(0 + 0.2, 1 + 0.1967)$$

$$= 0.2 f(0.2, 1.1967)$$

$$= 0.2 \left(\frac{1.1967^2 - 0.2^2}{1.1967^2 + 0.2^2} \right)$$

$$= 0.1891$$

Then

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6} (0.2 + 2 \times 0.1967 + 2 \times 0.1967 + 0.1891)$$

$$= 1.1960$$

Again,

$$y_2 = y_1 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$K_1 = hf(x_1, y_1)$$

$$= 0.2 f(0.2, 1.1960)$$

$$= 0.2 \left(\frac{1.1960^2 - 0.2^2}{1.1960^2 + 0.2^2} \right)$$

$$= 0.1891$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$= 0.2 f\left(0.2 + 0.1, 1.1960 + \frac{0.1891}{2}\right)$$

$$= 0.2 f(0.3, 1.2906)$$

$$= 0.2 \left(\frac{1.2906^2 - 0.3^2}{1.2906^2 + 0.3^2} \right)$$

$$= 0.1795$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= 0.2 f\left(0.2 + 0.1, 1.1960 + \frac{0.1795}{2}\right)$$

$$= 0.2 f(0.3, 1.2858)$$

$$= 0.2 \left(\frac{1.2858^2 - 0.3^2}{1.2858^2 + 0.3^2} \right)$$

$$= 0.1793$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

$$= 0.2 f(0.2 + 0.2, 1.1960 + 0.1793)$$

$$= 0.2 f(0.4, 1.3753)$$

$$= 0.1688$$

Then

$$y_2 = y_1 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.1960 + \frac{1}{6} (0.1891 + 2 \times 0.1795 + 2 \times 0.1793 + 0.1688)$$

$$= 1.3753$$

$$\therefore y_2(0.4) = 1.3753$$

II Solution of simultaneous differential eqⁿ

$$\text{Let } \frac{dy}{dx} = f(x, y, z)$$

$$\frac{dz}{dx} = \phi(x, y, z)$$

be simultaneous differential eqⁿs with initial conditions $y(x_0) = y_0$ & $z(x_0) = z_0$ & interval size h Then

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

where

$$K_1 = hf(x_0, y_0, z_0)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$$

Similarly

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$l_2 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_3 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_4 = h\phi(x_0 + h, y_0 + K_3, z_0 + l_3)$$

Replacing x_0, y_0, z_0 by x, y, z we get y_2 & z_2 & so on,

3) Solve $\frac{dy}{dx} = yz + x$, $\frac{dz}{dx} = xz + y$ with

initial conditions, $y(0) = 1$ & $z(0) = -1$
 $h = 0.1$ for $x = 0.1$

Soln:

Here Given that,

$$\frac{dy}{dx} = f(x, y, z) = yz + x$$

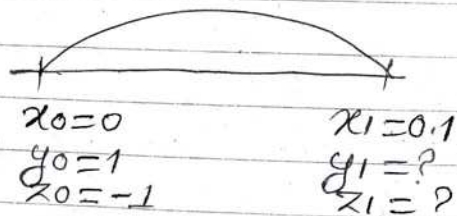
$$\frac{dz}{dx} = g(x, y, z) = xz + y$$

Initial conditions $y(0) = 1 \Rightarrow x = 0$
 $y_0 = 1$

$z(0) = -1, x_0 = 0$
 $z_0 = -1$

$h = 0.1$

$$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$$



We know R-K formula,

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

where

$$\begin{aligned} K_1 &= hf(x_0, y_0, z_0) \\ &= 0.1 f(0, 1, -1) \\ &= 0.1 [1 \times -1 + 0] \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} l_1 &= hg(0, 1, -1) \\ &= 0.1 [0 \times -1 + 1] \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} K_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right) \\ &= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{-0.1}{2}, -1 + \frac{0.1}{2}\right) \\ &= 0.1 f(0.05, 0.95, -0.95) \\ &= 0.1 (-0.95 \times 0.95 + 0.05) \\ &= -0.0853 \end{aligned}$$

$$\begin{aligned} l_2 &= hg(0.05, 0.95, -0.95) \\ &= 0.1 (0.05 \times -0.95 + 0.95) \\ &= 0.0903 \end{aligned}$$

$$\begin{aligned} K_3 &= hf\left(x_0 + h, y_0 + K_2, z_0 + l_2\right) \\ &= 0.1 f(0.1, 0.9549, -0.9549) \\ &= 0.1 (-0.9549 \times 0.9549 + 0.1) \\ &= -0.0864 \end{aligned}$$

$$\begin{aligned} K_4 &= hf\left(x_0 + h, y_0 + K_3, z_0 + l_3\right) \\ &= 0.1 f(0.1, 0.9136, -0.9090) \\ &= 0.1 (-0.9090 \times 0.9136 + 0.1) \\ &= -0.0823 \end{aligned}$$

$$\begin{aligned} l_3 &= hg(0.1, 0.9574, -0.9549) \\ &= 0.1 (0.1 \times -0.9549 + 0.9574) \\ &= 0.0910 \end{aligned}$$

$$\begin{aligned} K_4 &= hf\left(x_0 + h, y_0 + K_3, z_0 + l_3\right) \\ &= 0.1 f(0.1, 0.9136, -0.9090) \\ &= 0.1 (-0.9090 \times 0.9136 + 0.1) \\ &= -0.0823 \end{aligned}$$

$$= -0.0730$$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}(-0.1 + 2 \times -0.0853 - 2 \times 0.0864 - 0.0750)$$

$$= 0.9139$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= -1 + \frac{1}{6}(0.1 + 2 \times 0.0903 + 2 \times 0.0910 + 0.0823)$$

$$= -0.9092$$

$$\text{Hence } y(0.1) = 0.9139$$

$$z(0.1) = -0.9092$$

Q.3. Use R-K method, solve the system
 $\frac{dy}{dx} = x+z, \frac{dz}{dx} = x-y, y(0)=0, z(0)=1$

h=0.1, for $x=0.2$

Soln:

$$\frac{dy}{dx} = f(x, y, z) = x+z$$

$$\frac{dz}{dx} = \phi(x, y, z) = x-y$$

$$y(0)=0 \Rightarrow x_0=0, y_0=0$$

$$z(0)=1 \Rightarrow x_0=0, z_0=1$$

$$h=0.1$$

$$\begin{array}{ccc} x_0=0 & x_1=0.1 & x_2=0.2 \\ y_0=0 & y_1=? & y_2=? \\ z_0=1 & z_1=? & z_2=? \end{array}$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_2 = ?$$

$$z_2 = ?$$

R-K method

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \rightarrow (1)$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \rightarrow (2)$$

where

$K_1 = hf(x_0, y_0, z_0)$	$l_1 = h\phi(x_0, y_0, z_0)$
$= 0.1 f(0, 0, 1)$	$= 0.1 \phi(0, 0, 1)$
$= 0.1 [0+1]$	$= 0.1 (0-0)$
$= 0.1$	$= 0$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2})$	$l_2 = h\phi(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2})$
$= 0.1 f(0.05, 0.05, 1)$	$= 0.1 \phi(0.05, 0.05, 1)$
$= 0.1 (0.05+1)$	$= 0$
$= 0.1050$	

$K_3 = hf(x_0 + h, y_0 + K_2, z_0 + l_2)$	$l_3 = h\phi(x_0 + h, y_0 + K_2, z_0 + l_2)$
$= 0.1 f(0.05 + 0.0525, 1)$	$= 0.1 (0.05 - 0.0525)$
$= 0.1 (0.05+1)$	$= -0.0003$
$= 0.1050$	

$$\begin{aligned}
 K_4 &= h \phi(x_0 + h, y_0 + K_3, z_0 + l_3) \quad l_4 = h \phi(0.1, 0.1050, 0.9997) \\
 &= h f(0.1, 0.1050, 0.9997) = 0.1(0.1 - 0.1050) \\
 &= 0.1(0.1 + 0.9997) = -0.0005 \\
 &= 0.1100
 \end{aligned}$$

Then

$$\begin{aligned}
 y_1 &= y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\
 &= 0 + \frac{1}{6}(0.1 + 2 \times 0.1050 + 2 \times 0.1050 + 0.11) \\
 &= 0.1050
 \end{aligned}$$

$$\begin{aligned}
 z_1 &= z_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + l_4) \\
 &= 1 + \frac{1}{6}(0 + 2 \times 0 + 2 \times (-0.0008) + 0.0005) \\
 &= 0.9998
 \end{aligned}$$

Again

$$\begin{aligned}
 y_2 &= y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\
 &= 0.1050 + \frac{1}{6}(0.1 + 2 \times 0.1050 + 2 \times 0.1050 + 0.11) \\
 &= 0.2100 \\
 z_2 &= z_1 + \frac{1}{6}(K_1 + 2l_2 + 2l_3 + l_4) \\
 &= 0.9998 + \frac{1}{6}(0 + 2 \times 0 - 0.0003 \times 2 + 0.0005) \\
 &= 0.9996
 \end{aligned}$$

$$\begin{aligned}
 K_1 &= h f(x_1, y_1, z_1) \\
 &= 0.1 f(0.1, 0.1050, 0.9998) \\
 &= 0.1(0.1 + 0.9998) \\
 &= 0.11
 \end{aligned}$$

$$\begin{aligned}
 l_1 &= h \phi(0.1, 0.1050, 0.9998) \\
 &= 0.1 \phi(0.1 - 0.1050) \\
 &= -0.0005
 \end{aligned}$$

$$\begin{aligned}
 K_2 &= h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}, z_1 + \frac{l_1}{2}\right) \\
 &= 0.1 f(0.15, 0.16, 0.9996) \\
 &= 0.1(0.15 + 0.9996) \\
 &= 0.1150
 \end{aligned}$$

$$\begin{aligned}
 l_2 &= h \phi(0.15, 0.16, 0.9996) \\
 &= 0.1(0.15 - 0.16) \\
 &= -0.001
 \end{aligned}$$

$$\begin{aligned}
 K_3 &= h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}, z_1 + \frac{l_2}{2}\right) \\
 &= 0.1 f(0.15, 0.1625, 0.9993) \\
 &= 0.1(0.15 + 0.9993) \\
 &= 0.1149
 \end{aligned}$$

$$\begin{aligned}
 l_3 &= h \phi\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}, z_1 + \frac{l_2}{2}\right) \\
 &= 0.1 \phi(0.15, 0.1625, 0.9993) \\
 &= 0.1(0.15 - 0.1625) \\
 &= -0.0013
 \end{aligned}$$

$$\begin{aligned}
 k_4 &= h f(x_1+h, y_1+k_3, z_1+l_3) \\
 &= 0.1 f(0.2, 0.2199, 0.9985) \\
 &= 0.1(0.2+0.9985) \\
 &= 0.1199
 \end{aligned}$$

$$\begin{aligned}
 l_4 &= h \phi(0.2, 0.2199, 0.9985) \\
 &= 0.1(0.2-0.2199) \\
 &= -0.0020
 \end{aligned}$$

So,

$$y_2 = y_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.1050 + \frac{1}{6} [0.11 + 2 \times 0.1150 + 2 \times 0.1149 + 0.1199]$$

$$= 0.2200$$

$$z_2 = z_1 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 0.9998 + \frac{1}{6} (-0.0005 - 2 \times 0.001 - 2 \times 0.0013 - 0.002)$$

$$= 0.9986 //$$

Model III >

Solution of Second order differential eqⁿ by Runge Kutta method.

Let us consider the second order differential equations

$$\frac{d^2 y}{dx^2} = \phi(x, y, \frac{dy}{dx}) \quad \text{--- (1)}$$

If we put $\frac{dy}{dx} = z$ --- (2) Then

$$\frac{d^2 y}{dx^2} = \frac{dz}{dx} \quad \text{--- (3)}$$

From eqⁿ (1) $\frac{dz}{dx} = \phi(x, y, z)$

Now, we have two simultaneous differential eqⁿ

$$\frac{dy}{dx} = f(x, y, z)$$

$$\& \frac{dz}{dx} = \phi(x, y, z)$$

with initial conditions $y(x_0) = y_0$ & $z(x_0) = z_0$
solving these eqⁿs we get,

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

where

$$k_1 = hf(x_0, y_0, z_0)$$

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_2 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_3 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Replacing x_0, y_0, z_0 by x_1, y_1, z_1
we get x_2, z_2 & so on.

5) Using R-K method of 4th order,
solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ for $x=0.2$

with initial condⁿ $y(0)=1, y'(0)=0$
take $h=0.2$

soln:

Here, Given that

$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$$

Let $\frac{dy}{dx} = z$ then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$f \frac{dy}{dx} = z$$

Now, we have following simultaneous eq^{ns}

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = xz^2 - y^2$$

with $y(0)=1 \Rightarrow x_0=0, y_0=1$

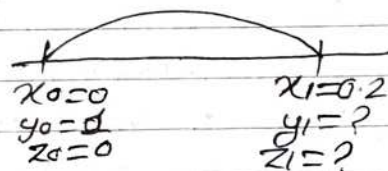
$$y'(0)=0 \Rightarrow z(0)=0 \Rightarrow x_0=0, z_0=0$$

$$h=0.2$$

$$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$y_1 = ?$$

$$z_1 = ?$$



We know R-K formula,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3) \rightarrow ①$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 4l_2 + l_3) \rightarrow ②$$

where

$$k_1 = hf(x_0, y_0, z_0) \quad l_1 = h\phi(x_0, y_0, z_0)$$

$$= hf(0, 1, 0) \quad = 0.2\phi(0, 1, 0)$$

$$= 0.2(0.0^2 - 1^2) \quad = 0.2(0.0^2 - 1^2)$$

$$= +0.2 \times 0 \quad = -0.2$$

$$= 0$$

$$K_2 = h \times f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2})$$

$$= 0.2 f(0.1, 1, -0.1)$$

$$= 0.2 (-0.1)$$

$$= -0.02$$

$$L_2 = h \phi(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2})$$

$$= 0.2 \phi(0.1, 1, -0.1)$$

$$= 0.2 (0.1 \times (-0.1)^2 - 1)$$

$$= -0.1998$$

$$K_3 = h f(x_0 + h, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2})$$

$$= 0.2 f(0.1, -0.09, -0.0999)$$

$$= 0.2 (-0.0999)$$

$$= -0.02$$

$$L_3 = h \phi(0.1, -0.09, -0.0999)$$

$$= 0.2 (0.1 (-0.0999)^2 - 0.99^2)$$

$$= 0.0002 - 0.1958$$

$$K_4 = h f(x_0 + h, y_0 + K_3, z_0 + L_3)$$

$$= 0.2 f(0.2, 0.9800, -0.1958)$$

$$= -0.0392$$

$$L_4 = h \phi(0.2, 0.98, -0.1958)$$

$$= 0.2 (0.2 (0.1958)^2 - 0.98^2)$$

$$= -0.1905$$

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6} (0 + 2 \times 0.2 - 2 \times 0.2 + 0.0392)$$

$$= 0.9801$$

$$Z_1 = z_0 + \frac{1}{6} (L_1 + 2L_2 + 2L_3 + L_4)$$

$$= 0 + \frac{1}{6} (-0.2 - 2 \times 0.1998 - 2 \times 0.1958 - 0.1905)$$

$$= -0.1970$$

b) Solve

$$y'' = xy' - y, y(0) = 3, y'(0) = 0 \text{ to } y(0.1)$$

Here

$$\frac{d^2y}{dx^2} = x \frac{dy}{dx} - y$$

$$\text{Let } \frac{dy}{dx} = z$$

$$\frac{dz}{dx} = z - y$$

From eqn ①

$$\frac{dz}{dx} = xz - y$$

Now we have two simultaneous eqns

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\begin{array}{c} \text{---} h=0.1 \text{---} \\ | \quad \quad \quad | \\ x_0=0 \quad x_1=0.1 \\ y_0=3 \\ z_0=0 \end{array}$$

$$\frac{dz}{dx} = \phi(x, y, z) = xz - y$$

with initial condⁿ

$$y(0)=3 \Rightarrow x_0=0$$

$$y_0=3$$

$$y'(0)=0 \Rightarrow x_0=0$$

$$z_0=0$$

$$\text{Taking } h=0.1$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = ?$$

$$z_1 = ?$$

We know -R-K formula

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$z_1 = z_0 + \frac{1}{6} (\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$$

Now,

$$K_1 = hf(x_0, y_0, z_0)$$

$$= 0.1 f(0, 3, 0)$$

$$= 0.1 \times 0$$

$$= 0$$

$$\ell_1 = h \phi(x_0, y_0, z_0)$$

$$= 0.1 \phi(0, 3, 0)$$

$$= 0.1 (0, 0 - 3)$$

$$= -0.3$$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{\ell_1}{2})$$

$$= 0.1 f(0.05, 3, -0.15)$$

$$= 0.1 (-0.15) = -0.015$$

$$\begin{aligned} \ell_2 &= h \phi(0.05, 3, -0.15) \\ &= 0.1 [0.05 \times (-0.15 - 3)] \\ &= -0.3008 \end{aligned}$$

$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{\ell_2}{2})$$

$$= 0.1 f(0.05, 2.9250, -0.1504)$$

$$= 0.1 (-0.1504)$$

$$= -0.015$$

$$\ell_3 = h \phi(0.05, 2.9250, -0.1504)$$

$$= 0.1 (-0.05 \times 0.1504 - 2.9250)$$

$$= -0.2928 = -0.3$$

$$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + \ell_3)$$

$$= 0.1 f(0.1, 2.9850, -0.2928)$$

$$= 0.1 (-0.2928)$$

$$= -0.293$$

$$= -0.03$$

$$\ell_4 = -0.3015$$

$$\text{Then } y_1 = y_0 + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 3 + \frac{1}{6} [0 + 2 \times (-0.015) - 2 \times 0.015 - 0.03]$$

$$= 2.9850$$

$$z_1 = z_0 + \frac{1}{6} [\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4]$$

$$= 0 + \frac{1}{6} [-0.3 - 2 \times 0.3008 - 2 \times 0.3 - 0.3015]$$

$$= -0.3005$$

$$7) \cdot 20 \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - 4y = 5$$

$$y(0) = 0, y'(0) = 0, h = 0.025$$

for $0 \leq x \leq 0.5$

Soln:

$$20 \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - 4y = 5$$

$$\text{Let } \frac{dy}{dx} = z$$

$$\text{So, } \frac{d^2 y}{dx^2} = \left[5 + 4y - 2 \frac{dy}{dx} \right] \frac{1}{20}$$

$$\frac{dz}{dx} = \left[5 + 4y - 2z \right] \frac{1}{20}$$

Now,

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = \frac{1}{20} [5 + 4y - 2z]$$

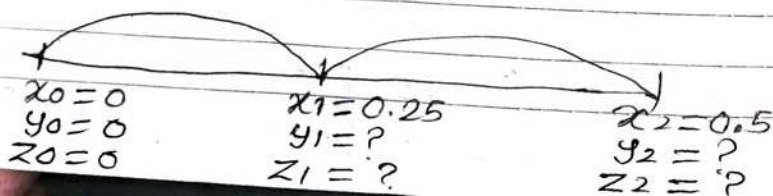
$$\text{Here } y(0) = 0 \Rightarrow x_0 = 0$$

$$y_0 = 0$$

$$y'(0) = 0 \Rightarrow z_0 = 0$$

$$z_0 = 0$$

Then,



Now, From R-K formula

$$y_1 = y_0 + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4]$$

$$z_1 = z_0 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

Then

$$\begin{aligned} K_1 &= hf(x_0, y_0, z_0) & l_1 &= h\phi(x_0, y_0, z_0) \\ &= 0.25f(0, 0, 0) & &= 0.25\phi(0, 0, 0) \\ &= 0.25 \times 0 & &= 0 \\ &= 0 & & \end{aligned}$$

$$\begin{aligned} K_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right) & l_2 &= h\phi\left(0.125, 0, 0\right) \\ &= 0.25f(0.125, 0, 0) & &= 0.25\left[\frac{5}{20}\right] \\ &= 0 & &= 0.0625 \end{aligned}$$

$$\begin{aligned} K_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= 0.25(0.125, 0, 0.0313) \\ &= 0.25(0.0313) \\ &= 0.0078 \end{aligned}$$

$$\begin{aligned} l_3 &= h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= 0.25\phi(0.125, 0, 0.0313) \\ &= 0.25\left[\frac{1}{20} (5 + 0 - 2 \times 0.0313)\right] \\ &= 0.0617 \end{aligned}$$

$$\begin{aligned}
 K_4 &= hf(x_0+h, y_0+K_3, z_0+l_3) \\
 &= 0.25f(0.25, 0.0078, 0.0617) \\
 &= 0.25 \times 0.0617 \\
 &= 0.0154
 \end{aligned}$$

$$\begin{aligned}
 l_4 &= hf(0.25, 0.0078, 0.0617) \\
 &= 0.25 \left[\frac{1}{20} (5 + 4 \times 0.0078 - 2 \times 0.0617) \right] \\
 &= 0.0613
 \end{aligned}$$

Then

$$\begin{aligned}
 y_1 &= y_0 + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4] \\
 &= 0 + \frac{1}{6} [0 + 2 \times 0 + 2 \times 0.0078 + 0.0154] \\
 &= 0.0052
 \end{aligned}$$

$$\begin{aligned}
 z_1 &= z_0 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4] \\
 &= 0 + \frac{1}{6} [0 + 2 \times 0.0625 + 2 \times 0.0617 + 0.0613] \\
 &= 0.0516
 \end{aligned}$$

$$\begin{aligned}
 \text{Again } K_1 &= hf(x_1, y_1, z_1) \\
 &= 0.25f(0.25, 0.0052, 0.0516) \\
 &= 0.25 \times 0.0516 \\
 &= 0.0129
 \end{aligned}$$

$$\begin{aligned}
 l_1 &= hf(x_1, y_1, z_1) \\
 &= 0.25f(0.25, 0.0052, 0.0516) \\
 &= 0.25 \left[\frac{1}{20} (5 + 4 \times 0.0052 - 2 \times 0.0516) \right] \\
 &= 0.0615
 \end{aligned}$$

$$\begin{aligned}
 K_2 &= hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}, z_1 + \frac{l_1}{2}) \\
 &= 0.25f(0.375, 0.0117, 0.0824) \\
 &= 0.25 \times 0.0824 \\
 &= 0.0206
 \end{aligned}$$

$$\begin{aligned}
 l_2 &= hf(0.375, 0.0117, 0.0824) \\
 &= 0.25 \left[\frac{1}{20} (5 + 4 \times 0.0117 - 2 \times 0.0824) \right] \\
 &= 0.0610
 \end{aligned}$$

$$\begin{aligned}
 K_3 &= hf(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}, z_1 + \frac{l_2}{2}) \\
 &= 0.25f(0.375, 0.0155, 0.0821) \\
 &= 0.25 \times 0.0821 \\
 &= 0.0205
 \end{aligned}$$

$$\begin{aligned}
 l_3 &= hf(0.375, 0.0155, 0.0821) \\
 &= 0.25 \left[\frac{1}{20} (5 + 4 \times 0.0155 - 2 \times 0.0821) \right] \\
 &= 0.0612
 \end{aligned}$$

$$\begin{aligned}
 K_4 &= hf(x_1+h, y_1+K_3, z_1+l_3) \\
 &= 0.25f(0.5, 0.0257, 0.1128) \\
 &= 0.25 \times 0.1128 = 0.0282
 \end{aligned}$$

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$$k_4 = h \phi(0.5, 0.0257, 0.1128)$$

$$= 0.25 \left[\frac{d}{dx} (5 + 4 \times 0.0257 - 2 \times 0.1128) \right]$$

$$= 0.0610$$

Then,

$$y_2 = y_1 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 0.0052 + \frac{1}{6} [0.0129 + 2 \times 0.0206 + 2 \times 0.0205 + 0.0282]$$

$$= 0.0258$$

$$z_3 = z_1 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

$$= 0.0516 + \frac{1}{6} (0.0615 + 2 \times 0.0610 + 2 \times 0.0612 + 0.0610)$$

$$= 0.1128$$

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Method V) Heun's Method :-

(R-K 2nd order method)

or modified Euler's Method.)

Formula:

$$y' = \frac{dy}{dx} = f(x, y) \text{ with initial condition}$$

$y(x_0) = y_0$ & step size h . Then

$$y_1 = y_0 + \frac{1}{2} (k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

Replacing x_0, y_0 by x, y we get
 y_2 & so on.

17) Using Heun's method, find an appropriate value of y for $x = 0.3$. Given $\frac{dy}{dx} = x + y$,
 $y(0) = 1, h = 0.1$

Soln:

$$\text{Given } \frac{dy}{dx} = f(x, y) = x + y$$

$$y(0) = 1 \Rightarrow x_0 = 0$$

$$y_0 = 1$$

$$h = 0.1$$

$$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.1 = 0.2$$

$$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3$$

We know Heun's formula

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

For y_1 where $k_1 = hf(x_0, y_0)$

$$= 0.1 f(0, 1)$$

$$= 0.1 [0 + 1]$$

$$= 0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$= 0.1 f(0.1, 1.1)$$

$$= 0.1 (0.1 + 1.1)$$

$$= 0.12$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}(0.1 + 0.12)$$

$$= 1.11$$

Then For y_2

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = hf(x_1, y_1)$$

$$= 0.1 f(0.1, 1.11)$$

$$= 0.1 (0.1 + 1.11)$$

$$= 0.1210$$

$$k_2 = hf(x_1 + h, y_1 + k_1)$$

$$= 0.1 f(0.2, 1.11 + 0.121)$$

$$= 0.1 f(0.2, 1.232)$$

$$= 0.1 (0.2 + 1.232)$$

$$= 0.1431$$

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

$$= 1.2421$$

For y_3

$$y_3 = y_2 + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = hf(x_2, y_2)$$

$$= 0.1 f(0.2, 1.2421)$$

$$= 0.1 \times (0.2 + 1.2421)$$

$$= 0.1442$$

$$k_2 = hf(x_2 + h, y_2 + k_1)$$

$$= 0.1 f(0.3, 1.3863)$$

$$= 0.1686$$

Then

$$y_3 = y_2 + \frac{1}{2}(k_1 + k_2)$$

$$= 1.2421 + \frac{1}{2}(0.1442 + 0.1686)$$

$$= 1.3985$$

$$\text{Hence } \boxed{y(0.3) = 1.3985}$$

Heun's method for second order differential

Solve for $y(0.2)$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x, y(0)=0, y'(0)=1$$

$$h=0.2$$

Soln's

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$$

$$\text{Put } \frac{dy}{dx} = z$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From eqn (1)

$$\frac{dz}{dx} + 2z - 3y = 6x$$

$$\text{or } \frac{dz}{dx} = 6x + 3y - 2z$$

Now we have simultaneous diff eqn

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = 6x + 3y - 2z$$

with initial conditions

$$y(0)=0 \Rightarrow x_0=0$$

$$y_0=0$$

$$y'(0)=1 \Rightarrow z_0=1, x_0=0$$

$$x_0=0$$

$$y_0=0$$

$$z_0=1$$

$$x_1=0.2$$

$$y_1=?$$

$$z_1=?$$

$$\therefore x_1 = x_0 + h = 0 + 0.2, y_1 = ?$$

$$z_1 = ?$$

We know Heun's formula

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$z_1 = z_0 + \frac{1}{2}(L_1 + L_2)$$

where

$$K_1 = hf(x_0, y_0, z_0) \quad L_1 = h\phi(x_0, y_0, z_0)$$

$$= 0.2 f(0, 0, 1) \quad = 0.2 \phi(0, 0, 1)$$

$$= 0.2(1) \quad = 0.2[6(0) + 3(0) - 2(1)]$$

$$= 0.2 \quad = -0.4$$

$$K_2 = hf(x_0 + h, y_0 + K_1, z_0 + L_1) \quad L_2 = h\phi(x_0 + h, y_0 + K_1, z_0 + L_1)$$

$$= 0.2 f(0.2, 0.2, +0.6) \quad = 0.2[6(0.2) + 3(0.2) - 2(0.6)]$$

$$= 0.2 \times 0.6 \quad = 0.12$$

$$= 0.12$$

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$= 0 + \frac{1}{2}(0.2 + 0.12)$$

$$= 0.16$$

$$z_1 = z_0 + \frac{1}{2}(L_1 + L_2)$$

$$= 0 + \frac{1}{2}(-0.4 + 0.12)$$

$$= -0.14$$

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solve $\frac{dy}{dx} = 1+xz$, $\frac{dz}{dx} = -xy$ for
 $y(0.6), z(0.6)$, Given $y=0, z=1$
 at $x=0$ by using Heun's method
 Take $h=0.3$.

Soln:

Here Given

$$\frac{dy}{dx} = 1+xz$$

$$\frac{dz}{dx} = \phi(x, y, z) = -xy$$

$$y(0)=0 \text{ \& } z(0)=1 \Rightarrow x_0=0$$

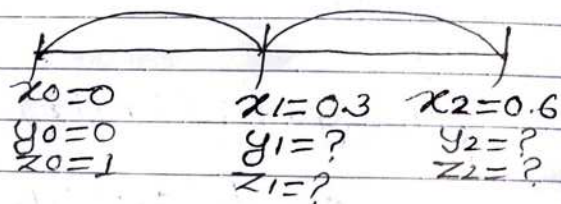
$$\text{Now, } x_1 = x_0 + h = 0 + 0.3 \quad y_0 = 0 \\ = 0.3 \quad z_0 = 1$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.3 = 0.6$$

$$y_2 = ?$$

$$z_2 = ?$$

Now,

For y_1

$$\begin{aligned} K_1 &= hf(x_0, y_0, z_0) & L_1 &= h\phi(x_0, y_0, z_0) \\ &= 0.3(0, 0, 1) & &= 0.3\phi(0, 0, 1) \\ &= 0.3(1 + 0 \times 1) & &= 0.3(-0 \times 0) \\ &= 0.3 & &= 0 \end{aligned}$$

$$\begin{aligned} K_2 &= hf(x_0+h, y_0+K_1, z_0+L_1) & L_2 &= h\phi(0.3, 0.3, 1) \\ &= 0.3f(0.3, 0.3, 1) & &= 0.3(-0.3 \times 0.3) \\ &= 0.3(1 + 0.3 \times 1) & &= -0.027 \\ &= 0.39 \end{aligned}$$

Then

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2}(K_1 + K_2) \\ &= 0 + \frac{1}{2}(0.3 + 0.39) \end{aligned}$$

$$= 0.345$$

$$z_1 = z_0 + \frac{1}{2}(L_1 + L_2)$$

$$= 1 + \frac{1}{2}(0 - 0.027)$$

$$= 0.9865$$

Again For y_2

$$\begin{aligned} K_1 &= hf(x_1, y_1, z_1) \\ &= 0.3f(0.3, 0.345, 0.9865) \\ &= 0.3(1 + 0.3 \times 0.9865) \\ &= 0.3888 \end{aligned}$$

$$\begin{aligned} L_1 &= h\phi(0.3, 0.345, 0.9865) \\ &= 0.3(-0.3 \times 0.345) \\ &= -0.0311 \end{aligned}$$

$$\begin{aligned} K_2 &= hf(x_1+h, y_1+K_1, z_1+L_1) \\ &= 0.3f(0.6, 0.7338, 0.9554) \\ &= 0.3(1 + 0.6 \times 0.9554) \\ &= 0.4720 \end{aligned}$$

$$\begin{aligned} L_2 &= 0.3(-0.6 \times 0.7338) \\ &= -0.1321 \end{aligned}$$

$$y_2 = y_1 + \frac{1}{2} (K_1 + K_2)$$

$$= 0.3450 + \frac{1}{2} (0.3888 + 0.4720)$$

$$= 0.7754$$

$$z_2 = z_1 + \frac{1}{2} (l_1 + l_2)$$

$$= 0.9865 + \frac{1}{2} (-0.0311 - 0.1321)$$

$$= 0.8$$

$$= 0.9049$$

Hence $y(0.6) = 0.7754$
 $z(0.6) = 0.9049 //$

Method VI Shooting Method

Consider the eqⁿ

$$\frac{d^2 y}{dx^2} = \phi(x, y, \frac{dy}{dx}) \text{ or } y'' = \phi(x, y, y')$$

$y(a) = A$ & $y(b) = B$
 Let $\frac{dy}{dx} = z$

$$\frac{d^2 y}{dx^2} = \frac{dz}{dx}$$

$$\therefore \frac{dz}{dx} = \phi(x, y, z)$$

In order to solve this set as an initial value problem, we need two cond^{ns} at $x=a$. We have one condition for $f(a)=A$ and therefore require another condition for z at $x=a$.

Let us assume that $z(a) = M_1$, where M_1 is the guess. M_1 represents the slope y' at $x=a$. Thus $y' = z$, $y(a) = A$
 $z' = \phi(x, y, z)$ $z(a) = M_1$ (1)

Eqⁿ (1) can be solved for y & z using Heun's method using steps of h until $x=b$ be.

If $B_1 = B$ then we have obtained the required solution. If $B_1 \neq B$ then we obtain the solution with another guess. Say $z(a) = M_2$. Let the new estimate of y at $x=b$ be B_2 . If $B_2 \neq B$ then process may be continued till we obtain the correct estimate of y .

However the procedure can be accelerated by using an improved guess for $z(a)$ after estimates of B_1 & B_2 . Let us consider that $z(a) = M_3$ leads to the value $y(b) = B$. If we assume that the values of M & B are linearly related then

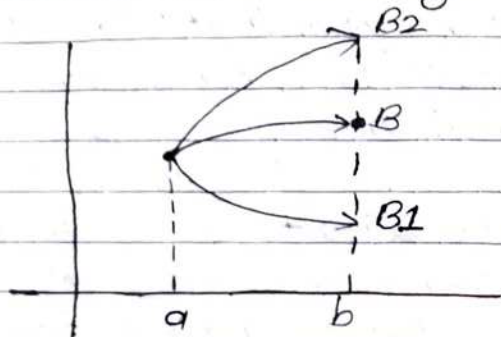
$$\frac{M_3 - M_1}{B - B_1} = \frac{M_2 - M_1}{B_2 - B_1}$$

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$$\text{or, } M_3 = M_2 + \frac{B - B_2}{B_2 - B_1} \times (M_2 - M_1)$$

$$= M_2 - \frac{B_2 - B}{B_2 - B_1} \times (M_2 - M_1)$$

Now with $z(a) = M_3$, we can again obtain solution of y .



1) Using shooting method, solve the eqⁿ $\frac{d^2y}{dx^2} = 6x$, $y(1)=2$, $y(2)=9$ in

interval $(1, 2)$

Soln:

Here $\frac{d^2y}{dx^2} = 6x \rightarrow (1)$

Let $\frac{dy}{dx} = z$ then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From eqⁿ (1) $\frac{dz}{dx} = 6x$

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$$\therefore \frac{dz}{dx} = \phi(x, y, z) = 6x$$

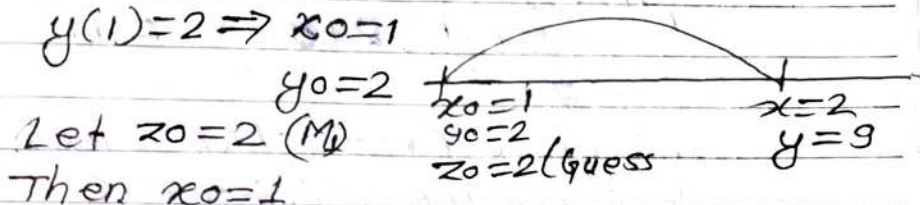
$$\frac{dy}{dx} = f(x, y, z) = z$$

Initial condⁿ

$$y(a) = A \text{ or } y(1) = 2 \Rightarrow a = 1, A = 2$$

$$y(b) = B \text{ or } y(2) = 9 \Rightarrow b = 2, B = 9$$

$$y(1) = 2 \Rightarrow x_0 = 1$$



Let $z_0 = 2$ (M)

Then $x_0 = 1$

$$y_0 = 2$$

$$z_0 = 2$$

let, $h = 1$ $x_1 = x_0 + h = 1 + 1 = 2$, $y_1 = ?$, $z = ?$

Using Heun's method we get

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

Where $K_1 = hf(x_0, y_0, z_0)$
 $= 1f(1, 2, 2)$
 $= 2$

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$= h\phi(1, 2, 2)$$

$$= 6 \times 1 \times 1$$

$$= 6$$

$$k_2 = hf(x_0+h, y_0+k_1, z_0+k_2) \\ = 1f(2, 8, 8) \\ = 8$$

$$l_2 = 1\phi(2, 4, 8) \\ = 12$$

Then

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2) \quad z_1 = z_0 + \frac{1}{2}(l_1+l_2) \\ = 2 + \frac{1}{2}(2+8) \quad = 11 \\ = 7$$

This gives $B_1 = 7$ which is less than 9.
Now assume $z(1) = 4 = M_2$

Then $x_0 = 1$

$$y_0 = 2$$

$$z_0 = 4$$

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2) = 2 + \frac{1}{2}(6+6)$$

$$z_1 = z_0 + \frac{1}{2}(l_1+l_2)$$

where

$$k_1 = hf(x_0, y_0, z_0) \quad l_1 = h\phi(1, 2, 4) \\ = 1f(1, 2, 4) \quad = 1(6 \times 1) \\ = 4 \quad = 6$$

$$k_2 = hf(x_0+h, y_0+k_1, z_0+l_1) \\ = 1f(2, 6, 10) \\ = 10$$

$$l_2 = h\phi(2, 6, 10) \\ = 1(6 \times 2) \\ = 12$$

$$\text{So, } y_1 = y_0 + \frac{1}{2}(k_1+k_2) \quad z_1 = z_0 + \frac{1}{2}(l_1+l_2) \\ = 2 + \frac{1}{2}(4+10) \quad = 4 + \frac{1}{2}(6+12) \\ = 9 = B \quad = 13$$

$$\text{So } y(1) = 2, y(2) = 9 \quad z(1) = 4$$

Hence $y(1) = 2, z(1) = 4$ Ans

Solve

$$\frac{d^2y}{dx^2} = 6x, y(1) = 2, y(2) = 9, h = 0.5$$

Soln:

$$\frac{d^2y}{dx^2} = 6x$$

Here Let $\frac{dy}{dx} = z$

$$\text{So, } \frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\text{Then } \frac{dz}{dx} = 6x$$

So Simultaneous eqⁿ is

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = 6x$$

Initial condⁿ

$$y(1)=2 \Rightarrow x_0=1, y_0=2, h=0.5$$

$$\therefore x_1 = x_0 + h = 1 + 0.5 = 1.5$$

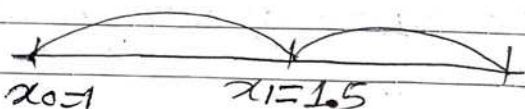
$$x_2 = x_0 + 2h = 1 + 2 \times 0.5 = 2$$

$$y_2 = ?$$

$$z_2 = ?$$

$$\therefore y(2)=9 \Rightarrow y=9 \text{ when } x=2$$
$$b=2, B=9.$$

$$\text{let } z(1)=2=M$$



$$y_0=2$$

$$z_0=2$$

Using Heun's formula

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

where

$$k_1 = hf(x_0, y_0, z_0)$$

$$= 0.5 f(1, 2, 2)$$

$$= 1$$

$$l_1 = h\phi(1, 2, 2)$$

$$= 0.5(6 \times 1)$$

$$= 3$$

$$k_2 = hf(x_0 + h, y_0 + k_1, z_0 + l_1)$$

$$= 0.5 f(1.5, 3, 5)$$

$$= 0.5 \times 5$$

$$= 2.5$$

$$l_2 = h\phi(1.5, 3, 5)$$

$$= 0.5 \times 6 \times 1.5$$

$$= 4.5$$

$$\text{Then } y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 2 + \frac{1}{2}(1 + 2.5)$$

$$= 3.75$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

$$= 2 + \frac{1}{2}(3 + 4.5)$$

$$= 5.75$$

II step

$$\text{where } k_1 = hf(x_1, y_1, z_1)$$

$$= 0.5 f(1.5, 3.75, 5.75)$$

$$= 0.5 \times 5.75$$

$$= 2.8750$$

$$l_1 = h\phi(1.5, 3.75, 5.75)$$

$$= 0.5 \times 6 \times 1.5$$

$$= 4.5$$

$$k_2 = hf(x_1 + h, y_1 + k_1, z_1 + l_1)$$

$$= 0.5 f(2, 6, 10.25)$$

$$= 0.5(10.25)$$

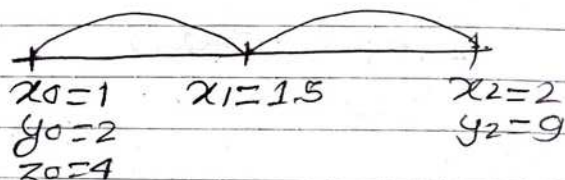
$$= 5.125$$

$$l_2 = h \phi(2, 6.625, 10.25) \\ = 0.5(6 \times 2) \\ = 6$$

So,

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) \\ = 3.75 + \frac{1}{2}(0.875 + 5.125) \\ = 7.75 = B_1$$

This given $B_1 = 7.75$ which is less than $B = 9$. Let us assume another guess $z(1) = 4 = M_2$



$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

where

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) & l_1 &= 0.5 \phi(1, 2, 4) \\ &= 0.5 f(1, 2, 4) & &= 0.5 \times 6 \times 1 \\ &= 0.5 \times 4 & &= 3 \\ &= 2 & & \end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_0 + h, y_0 + k_1, z_0 + l_1) \\ &= 0.5 f(1.5, 4, 7) \\ &= 0.5 \times 7 \\ &= 3.5 \end{aligned}$$

$$\begin{aligned} l_2 &= h \phi(1.5, 4, 7) \\ &= 0.5 \times 6 \times 1.5 \\ &= 4.5 \end{aligned}$$

Then

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \\ &= 2 + \frac{1}{2}(2 + 3.5) \\ &= 4.75 \end{aligned}$$

$$\begin{aligned} z_1 &= z_0 + \frac{1}{2}(l_1 + l_2) \\ &= 4 + \frac{1}{2}(3 + 4.5) \\ &= 7.75 \end{aligned}$$

$$\begin{aligned} \text{Again } y_2 &= y_1 + \frac{1}{2}(k_1 + k_2) \\ z_2 &= z_1 + \frac{1}{2}(l_1 + l_2) \end{aligned}$$

where

$$\begin{aligned} k_1 &= hf(x_1, y_1, z_1) \\ &= 0.5 f(1.5, 4.75, 7.75) \\ &= 0.5 \times 7.75 \\ &= 3.8750 \end{aligned}$$

$$\begin{aligned} l_1 &= h \phi(1.5, 4.75, 7.75) \\ &= 0.5 \times 6 \times 1.5 \\ &= 4.5 \end{aligned}$$

$$\begin{aligned}
 k_2 &= hf(2, 8.6250, 12.25) \\
 &= 0.5 \times 6 \times 12.25 \times 2 \quad 0.5 \times 12.25 \\
 &= 3.875 \quad 6.1250
 \end{aligned}$$

Then

$$\begin{aligned}
 g_2 &= y_1 + \frac{1}{2} (k_1 + k_2) \\
 &= 4.75 + \frac{1}{2} (3.875 + 6.125) \\
 &= 9.0875 = B_2 \neq B
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{M_3 - M_2}{B - B_2} &= \frac{M_2 - M_1}{B_2 - B_1} \\
 \text{or, } \frac{M_3 - 4}{9 - 9.75} &= \frac{4 - 2}{9.75 - 7.75} \\
 \text{or } \frac{M_3 - 4}{-0.75} &= \frac{2}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{or, } M_3 - 4 &= -0.75 \\
 \therefore M_3 &= 3.25 //
 \end{aligned}$$

Chapter - Six

Solution of partial Differential Eqⁿ:



The second order equation involving two independent variable in general form as

$$a \frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial x \partial y} + c \frac{\partial^2 f}{\partial y^2} = f(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \quad \rightarrow (1)$$

where coefficients, a, b, c may be constants or function of x, y . Depending on the value of these coefficients, this equations may be classified as,

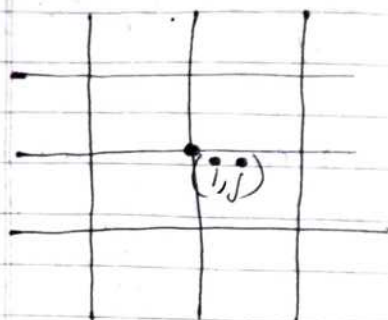
- i) elliptic if $b^2 - 4ac < 0$
- ii) parabolic if $b^2 - 4ac = 0$
- iii) Hyperbolic if $b^2 - 4ac > 0$

Elliptic Equation:-

A differential equation is the elliptic in a region R if $b^2 - 4ac < 0$ at all points of the region. The boundary condition of this type of equation specify the function u at every point of the closed boundary of the region R with in which the solution $u(x, y)$ is to be determined

R
closed Region.

Boundary conditions are given at every or some point of the boundary.



Method I > Laplace Equations

If $a=1, b=0, c=1$ & $F(x, y, f, f_x, f_y) = 0$
in eqⁿ $a \frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial y^2} + c \frac{\partial^2 f}{\partial z^2} = F(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ — (1)

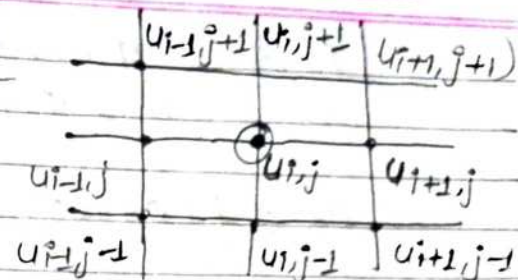
$\frac{\partial f}{\partial y} \rightarrow$ (1)

Then $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

or, $\nabla^2 f = 0$

or, $f_{xx} + f_{yy} = 0$ is called Laplace equation.

Laplace formula



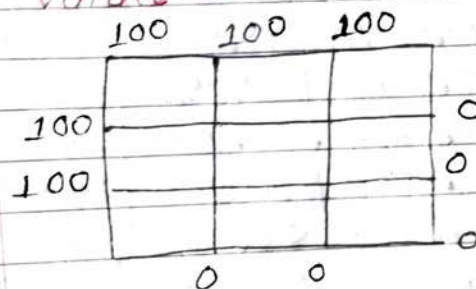
Standard 5-point formula

$$u_{i,j} = \frac{1}{4} [u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}]$$

Diagonal 5-point formula

$$u_{i,j} = \frac{1}{4} [u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}]$$

> Solve the elliptic eqⁿ $u_{xx} + u_{yy} = 0$ for following square mesh with boundary values as



Here Using standard 5-point formula
 $u_1 = \frac{1}{4}(u_2 + 100 + 100 + u_3) = \frac{1}{4}(u_2 + u_3 + 200)$

$$u_2 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(u_1 + u_4 + 100)$$

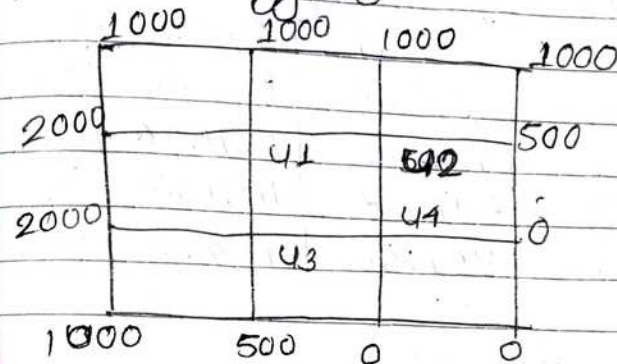
$$u_3 = \frac{1}{4}(u_4 + u_1 + 100 + 0) = \frac{1}{4}(u_1 + u_4 + 100)$$

$$u_4 = \frac{1}{4}(0 + u_2 + u_3 + 0) = \frac{1}{4}(u_2 + u_3)$$

Solving eq^{ns} by Gauss seidel method, we get

it ⁿ	u ₁	u ₂	u ₃	u ₄
Initial	0	0	0	0
	50	37.5	37.5	18.75
	68.75	46.875	46.875	23.4375
	73.4375	49.2188	49.2188	24.6094
	74.6094	49.8047	49.8047	24.9023
	74.9023	49.9512	49.9512	24.9756
	74.9756	49.9878	49.9878	24.9939
	74.9939	49.9969	49.9969	24.9985
	74.9985	49.9992	49.9992	24.9996
	74.9996	49.9998	49.9998	24.9999
	74.9999	50.000	50.000	25.000
	75	50	50	25
	75	50	50	25

$$2) \quad u_{xx} + u_{yy} = 0$$



Here using standard 5-point formula

$$u_1 = \frac{1}{4}(1000 + u_3 + 2000 + u_2) = \frac{1}{4}(3000 + u_2 + u_3)$$

$$u_2 = \frac{1}{4}(1000 + u_1 + u_4 + 500) = \frac{1}{4}(u_1 + u_4 + 1500)$$

$$u_3 = \frac{1}{4}(u_1 + 500 + 2000 + u_4) = \frac{1}{4}(2500 + u_1 + u_4)$$

$$u_4 = \frac{1}{4}(u_2 + u_3 + 0 + 0) = \frac{1}{4}(u_2 + u_3)$$

Then

it ⁿ	u ₁	u ₂	u ₃	u ₄
Int ⁿ	0	0	0	0
1	750	562.5	812.5	343.75
2	1093.75	734.375	984.375	429.6875
3	1179.6875	777.3438	1027.3438	451.17
4	1201.17	777.34	1038.09	456.54
5	1206.54	788.09	1040.77	457.89
6	1206.54	790.77	1040.77	457.89

7)	1207.89	791.44	1041.44	458.22
8)	1208.22	791.61	1041.61	458.31
9)	1208.31	791.65	1041.65	458.33
10	1208.33	791.66	1041.66	458.33
11	1208.33	791.67	1041.67	458.33
12	1208.33	791.67	1041.67	458.33

Hence $u_1 = 1208.33$

$u_2 = 791.67$

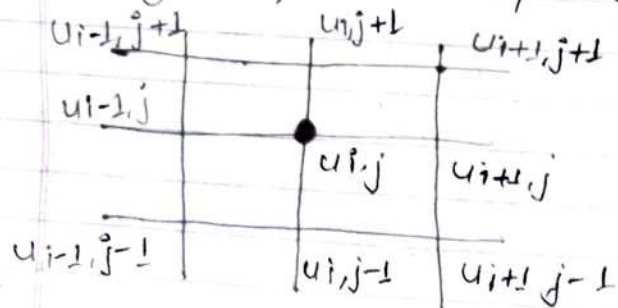
$u_3 = 1041.67$

$u_4 = 458.33 //$

Method II > Poisson Equations

The partial differential equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ is

called poisson equation where $f(x, y)$ is a given function of x & y



Formula

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(x_i, y_j)$$

1) Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square with sides $x=0=y$ & $x=3=y$ with $u=0$ on the boundary and mesh length = 1 unit

Here

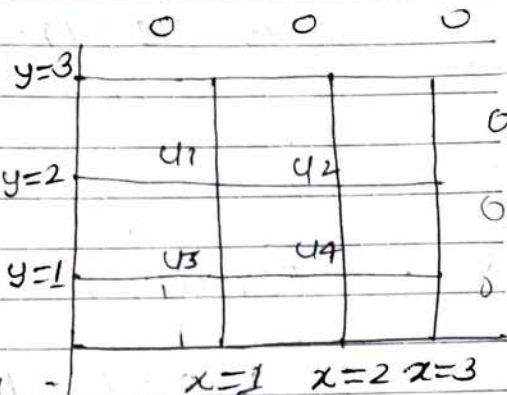
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$$

$$-10(x^2 + y^2 + 10)$$

It is poisson equation.

We know, poisson

standard formula -



$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(x_i, y_j)$$

For u_1 , $x=1$

$y=2$

$$\therefore u_2 + 0 + 0 + u_3 - 4u_1 = 1^2 f(1, 2)$$

$$\text{or, } u_2 + u_3 - 4u_1 = -10(1+4+10)$$

$$\text{or, } u_2 + u_3 - 4u_1 = -150$$

$$\therefore u_1 = \frac{u_2 + u_3 + 150}{4} \quad (1)$$

For u_2 : $i=2, j=2$

$$0+0+u_1+u_4-4u_2 = 1^2 f(2,2)$$

$$\text{or, } u_1+u_4-4u_2 = -10(4+4+10)$$

$$\text{or, } u_1+u_4-4u_2 = -180$$

$$\therefore u_2 = \frac{u_1+u_4+180}{4} \quad \text{--- (ii)}$$

For u_3 : $i=1, j=1$

$$\text{or, } u_1+u_4+0+0-4u_3 = 1^2 f(1,1)$$

$$u_1+u_4-4u_3 = -10(12)$$

$$\therefore u_3 = \frac{u_1+u_4+120}{4}$$

--- (iii)

For u_4 : $i=2, j=1$

$$0+0+u_2+u_3-4u_4 = 1^2 f(2,1)$$

$$\text{or, } u_2+u_3-4u_4 = -10(4+1+10)$$

$$\text{or, } u_2+u_3-4u_4 = -150$$

$$\therefore u_4 = \frac{u_2+u_3+150}{4}$$

--- (iv)

Using Gauss seidel method

$J \backslash I$	u_1	u_2	u_3	u_4
Int ⁿ	0	0	0	0
1	37.50	54.38	39.38	60.94
2	60.94	75.47	60.47	71.48
3	71.48	80.74	65.74	74.12
4	74.12	82.06	67.06	74.78
5	74.78	82.39	67.39	74.95
6	74.95	82.47	67.47	74.99
7	74.99	82.49	67.49	75.00
8	75	82.5	67.5	75.00
9	75	82.5	67.5	75.00

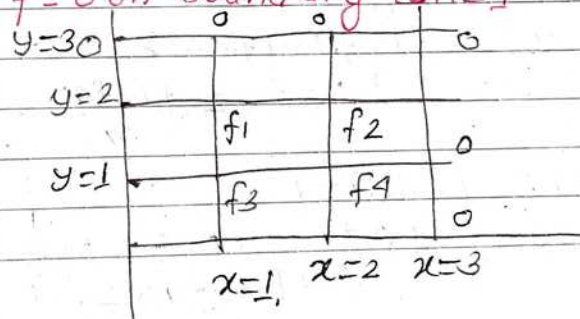
Hence $u_1 = 75$

$u_2 = 82.5$

$u_3 = 67.5$

$u_4 = 75$

2) Solve the poisson eqⁿ $\nabla^2 f = 2xy^2$ over square domain $0 \leq x \leq 3, 0 \leq y \leq 3$ with $f=0$ on boundary & $h=1$



Here

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x^2y^2$$

This is poisson eqⁿ

By using poisson formula

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = h^2 f(x_{ij}, y_{ij})$$

For f_1 $i=1, j=2$

$$\text{Then } 0 + 0 + f_2 + f_3 - 4f_1 = 1^2 f(1,2)$$

$$\text{or } f_2 + f_3 - 4f_1 = 2(1 \times 4)$$

$$\text{or, } f_2 + f_3 - 4f_1 = 8$$

$$\therefore f_1 = \frac{f_2 + f_3 - 8}{4} \quad \text{--- (i)}$$

Again For f_2 , $i=2, j=2$

$$0 + 0 + f_1 + f_4 - 4f_2 = 1^2 f(2,2)$$

$$\text{or, } f_1 + f_4 - 4f_2 = 2 \times 4 \times 4$$

$$\text{or, } f_2 = \frac{f_1 + f_4 - 32}{4} \quad \text{--- (ii)}$$

For f_3 $i=1, j=1$

$$0 + f_1 + 0 + f_4 - 4f_3 = 1^2 f(1,1)$$

$$\text{or, } f_1 + f_4 - 4f_3 = 2$$

$$\therefore f_3 = \frac{f_1 + f_4 - 2}{4} \quad \text{--- (iii)}$$

For f_4 : $i=2, j=1$

$$0 + f_2 + 0 + f_3 - 4f_4 = 1^2 f(2,1)$$

$$f_2 + f_3 - 4f_4 = 2 \times 4$$

$$\therefore f_4 = \frac{f_2 + f_3 - 8}{4}$$

Using Gauss seidel method.

J^n	u_1	u_2	u_3	u_4
Initial	0	0	0	0
1	-2	-8.5	-1	-2.75
2	-4.38	-9.78	-2.28	-3.66
3	-5.02	-10.17	-2.67	-3.92
4	-5.21	-10.28	-2.78	-4.00
5	-5.27	-10.32	-2.82	-4.02
6	-5.28	-10.33	-2.83	-4.03
7	-5.29	-10.33	-2.83	-4.03
8	-5.29	-10.33	-2.83	-4.03

Hence $u_1 = -5.29$

$u_2 = -10.33$

$u_3 = -2.83$

$u_4 = -4.03 //$

Solution of System of Linear Equations.

Relaxation method

Method VI

$$20x + y - 2z = 17$$

$$3x + 20y - z = 18$$

$$2x - 3y + 20z = 25$$

Soln:

Here residuals are given by

$$R_x = 17 - 20x - y + 2z$$

$$R_y = 18 - 3x - 20y + z$$

$$R_z = 25 - 2x + 3y - 20z$$

The operation table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-20	-1	2
δy	-3	-20	1
δz	-2	3	-20

The relation table is

	R_x	R_y	R_z
$x=y=z=0$	17	18	(25)
$\delta z = 1$	(19)	19	5
$\delta x = 1$	-1	(16)	3
$\delta y = 1$	-2	-4	6
$\delta z = 0.3$	-1.4	(-3.7)	0
$\delta y = -0.185$	-1.215	0	-0.555
$\delta x = 0.061$	0.005	0.183	-0.433

$$\sum \delta x = 1 - 0.061 = 0.939$$

$$\sum \delta y = 1 - 0.185 = 0.823$$

$$\sum \delta z = 1 + 0.3 = 1.3$$

Ans