Kritika Baral
- BCT/20/078
- Numerical Method.

Syllabus:

1	⎤	16
2	⎦	
3	–	16
4	–	16
5	–	10
6	–	10
7	–	12
		80

- Reference

1) Numerical methods in Engineering and Science
   - Dr. BS Grewal

2) Numerical methods
   - E Balagurusamy

Chapter - 1

INTRODUCTION

Approximation and errors in computation

imp

error :

deviation from the exact or true value is known as error.

Types of error

(1) Inherent error (Input error)

Those errors which are already present in the statement of problem itself. They arise to limitation of Mathematical table or digital computer.

a) Data error

arise when data for a problem are obtained by some experimental means and therefore of limited accuracy.

b) Conversion error

arise due to limitation of computer to store data easily.

(2) Numerical error (procedural error)

These are introduced during the process of implementation of a numerical method.

a) Round-off error :

Occurs when a fixed number of digits are used to represent exact numbers.

b) Truncation error

here, extra digits are dropped which introduces truncation error.

eg: $\pi = 3.1415$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

example of rounding off

Round off to 3 significant figures

1) 3.567 ⟶ 3.57
2) 7.893 ⟶ 7.89
3) 84767 ⟶ 84800
4) 5.8254 ⟶ 5.82 (because 2 is even)
5) 6.4356 ⟶ 6.44 (because 3 is odd)

If X is the true value and X' is its approximate value, then

i) Absolute error, $E_a = |X - X'|$

ii) Relative error, $E_r = \left|\frac{X - X'}{X}\right|$

iii) Percentage error, $E_p = \left|\frac{X - X'}{X}\right| \times 100\%$

Q. What is significance of relative error?

Chapter-2

SOLUTION OF NON-LINEAR EQUATIONS

imp

1. Bisection method
[8 marks]

"If $f(x)$ is continuous in the interval $[x_1, x_2]$ and $f(x_1)$ and $f(x_2)$ have different signs (ie $f(x_1) * f(x_2) < 0$) then the equation $f(x) = 0$ has at least one root between the interval $[x_1, x_2]$". $\longrightarrow x_0 = \frac{x_1 + x_2}{2}$

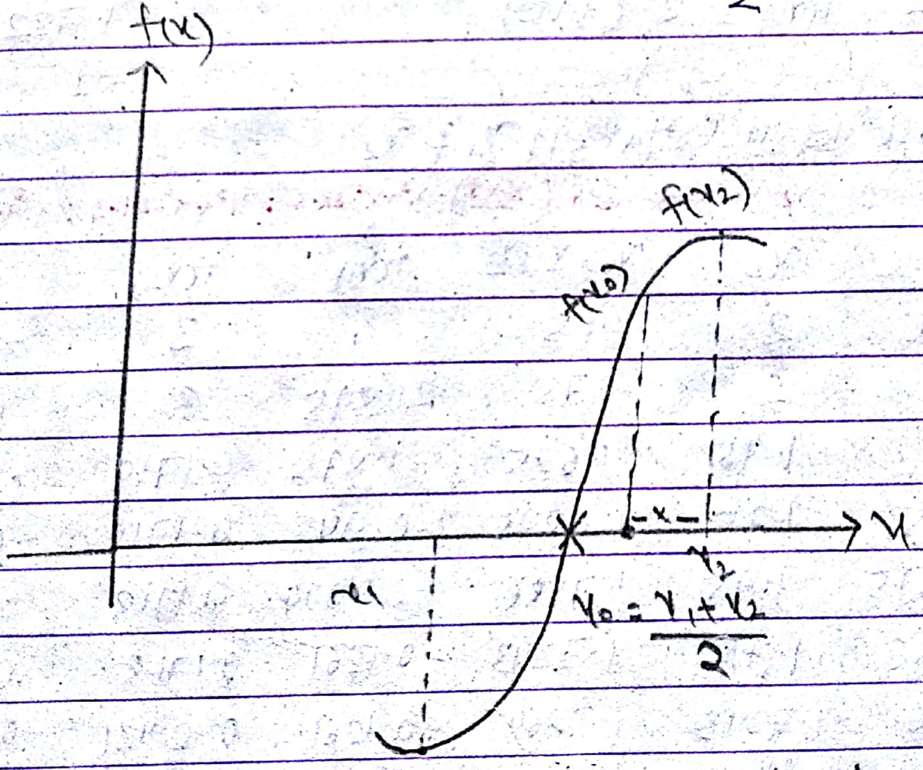

Fig: Illustration of bisection method

There exists three conditions,

i) If $f(x_0) = 0$, then x_0 is the root.

ii) If $f(x_0) * f(x_1) < 0$, the root lies in $[x_1, x_0]$

iii) If $f(x_0) * f(x_2) < 0$, the root lies in $[x_0, x_2]$

Now,

[explain the curve yourself in exam]

Q. Find a real root of $x^3+x^2-3x-3=0$ using Bisection method correct to 3 decimal places. [8 Marks]

⇒ Solution

let $f(x) = x^3+x^2-3x-3$

Taking $x_1 = 1$; $f(x_1) = -4$ (-ve)
$x_2 = 2$; $f(x_2) = 3$ (+ve)

Rough
1.732
-1.732
-1

∴ A root must lie in between x_1 & x_2

CALC

X Y ↳ $C = \left(\frac{X+Y}{2}\right) : X^3+X^2-3X-3 : Y^3+Y^2-3Y-3 : C^3+C^2-3C-3$

Iteration	x_1	x_2	$x_0 = \frac{x_1+x_2}{2}$	$f(x_1)$	$f(x_2)$	$f(x_0)$
1	1	2	1.5	-4	3	-1.875
2	1.5	2	1.75	-1.875	3	0.1718
3	1.5	1.75	1.625	-1.875	0.1718	-0.9433
4	1.625	1.75	1.6875	-0.943	0.1718	-0.409
5	1.6875	1.75	1.7186	-0.4094	0.1718	-0.125
6	1.7186	1.75	1.7343	-0.1261	0.1718	0.02131
7	1.7186	1.7343	1.72645	-0.1261	0.02131	-0.0528
8	1.7264	1.7343	1.7303	-0.0532	0.0213	-0.0160
9	1.7303	1.7343	1.7323	-0.0165	0.0213	2.358
10	1.7303	1.7323	1.7313	-0.0165	2.358	-7.102
11	1.7313	1.7323	1.7318	-7.1022	2.358	-2.373
12	1.7318	1.7323	1.7320	-2.373	2.358	-7.642
13	1.7320	1.7323	$\boxed{1.732}$	-4.808	2.358	9.388
14	1.7320	1.732	$\boxed{1.732}$	-4.808	-4.808	-4.808

$\therefore$ a real root of the given equation correct to 3 decimal places is 1.732.

HW

Q. Find a real root of equation $x - \cos x$ using Bisection Method to correct to 3 decimal places.

[Imp : put calculator in 'Radian Mode' ALLWAYS!]

$\Rightarrow$ Solution

Rough

$f(x) = 0$ [S] [C] = root 0.738

let $f(x) = x - \cos x$

Taking $x_1 = -1$, $f(-1) = -1.54$

$x_2 = 1$, $f(1) = 0.459$

$\therefore$ Root is between x_1 & x_2.

Iteration	x_1	x_2	$x_0 = \frac{x_1 + x_2}{2}$	$f(x_1)$	$f(x_2)$	$f(x_0)$
1	-1	1	0	-1.540	0.459	-1
2	0	1	0.5	-1	0.459	-0.377
3	0.5	1	0.75	-0.377	0.459	0.018
4	0.5	0.75	0.625	-0.377	0.018	-0.188
5	0.625	0.75	0.687	-0.186	0.018	-0.086
6	0.687	0.75	0.7185	-0.086	0.018	-0.034
7	0.7185	0.75	0.73425	-0.034	0.018	-0.008
8	0.73425	0.75	0.742	-0.008	0.018	0.005
9	0.734	0.742	0.738	-0.008	0.004	-0.0081
10	0.7382	0.742	0.740	-0.001	0.001	-0.007
11	0.7382	0.7391	0.7386	-0.001	0.002	-0.007
12	0.7386	0.7391	0.7385	-8.1×10^{-4}	2.4×10^{-4}	-3.9×10^{-4}

$\therefore$ The root is 0.738.

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2. False Position Method [bracketing method]

let us join points A and B. The st line crossing the x-axis at point x_0 is called the false position of the root.

Equation of line AB,

$$y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

at $x = x_0$, $y = 0$

ie $- f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_0 - x_1)$

Solving for x_0

$$\boxed{x_0 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}}$$

is called false position formula

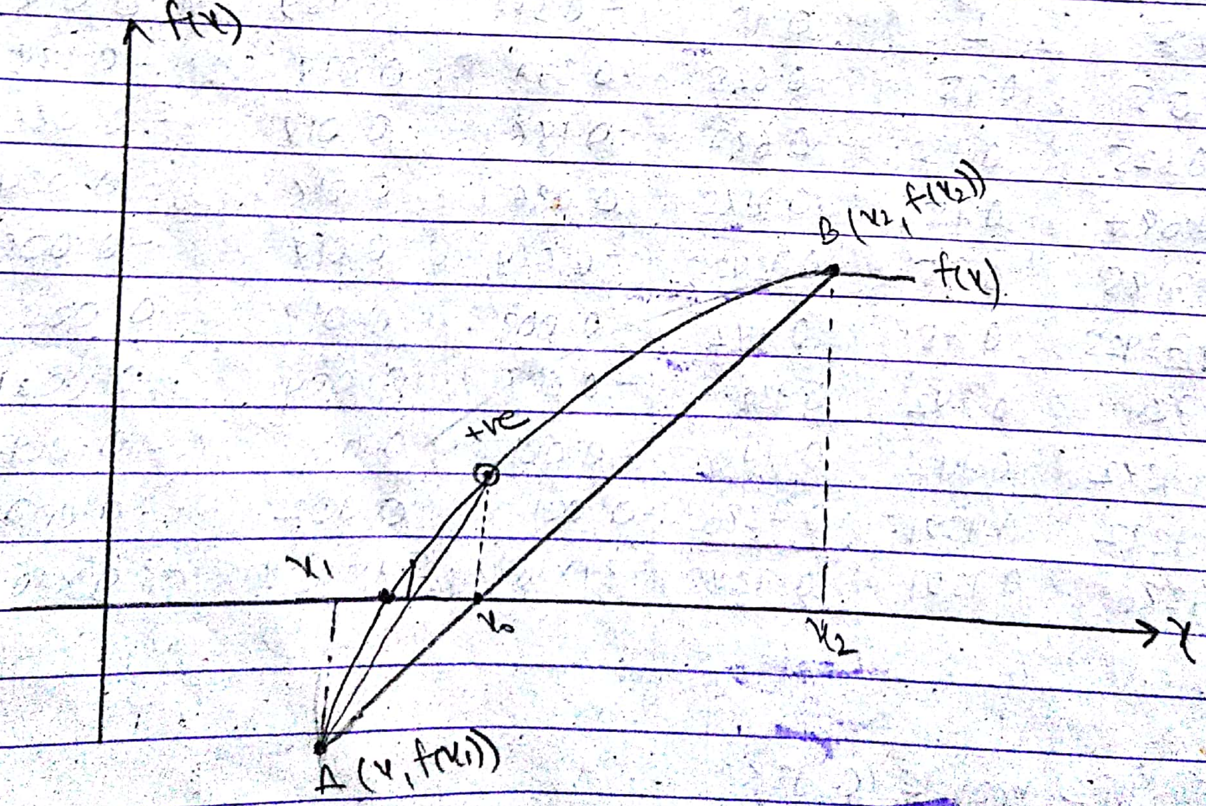

Fig: illustration of false position method

Q. Find a root of equation $x^3 - 2x - 5 = 0$ by method of false position correct to 3 decimal places.

⇒ Solution

let $f(x) = x^3 - 2x - 5$

Taking $x_1 = 2$, $f(x_1) = -1$ (-ve)

$x_2 = 3$, $f(x_2) = 16$ (+ve)

∴ A root must lie in between x_1 & x_2.

Rough

real root

2.0945

(यसलाई कियामा पर्ने गरी value लिने)

$\boxed{\text{CALC}}$

X Y ↳ $A = X^3 - 2X - 5 : B = Y^3 - 2Y - 5 : C = \frac{(XB - YA)}{B - A} : C^3 - 2C - 5$

Iteration	x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
1	2	3	-1	16	2.0588	-0.3907
2	2.0588	3	-0.3910	16	2.0812	-0.1473
3.	2.0812	3	-0.1479	16	2.0896	-0.0549
4.	2.0896	3	-0.0549	16	2.0927	-0.0203
5.	2.0927	3	-0.0203	16	2.0938	-0.0876
6.						
7.						
8.	2.0944	3	-0.0014	16	2.0944	-0.0006

The root of the given equation correct to 3 decimal places is 2.094.

3). Newton Raphson (NR) Method [open end]

Consider graph as shown in figure. The tangent at curve intersects x-axis at x_2.

The slope of the tangent is

$$\tan\theta = \frac{f(x_1)}{x_1 - x_2} = f'(x_1) \leftarrow \text{slope of } f(x) \text{ at } x = x_1$$

Solving for x_2,

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

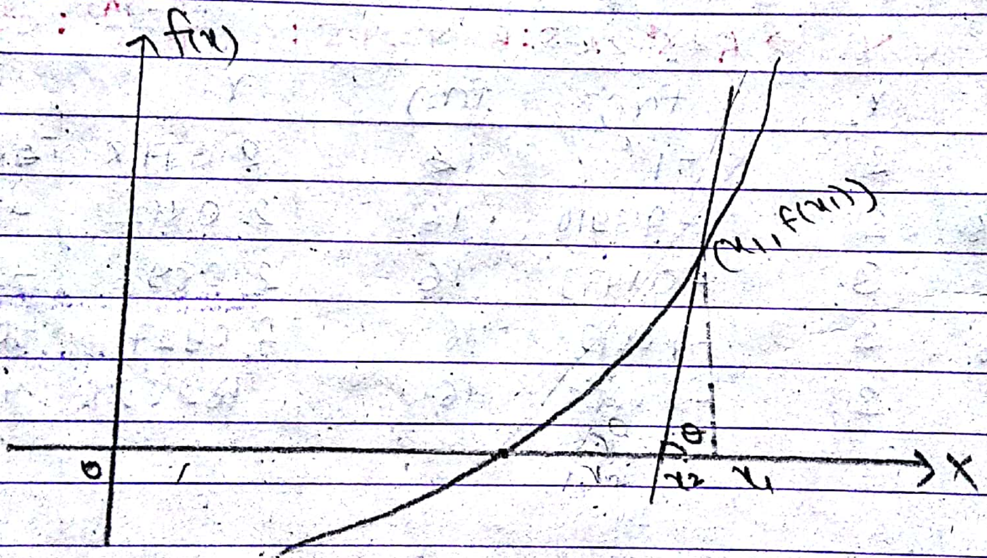

Fig: Illustration of Newton Raphson Method

Q: Find real root of $x \log x - 1.2$ using NR method. Convert to 4 decimal places.

⇒ Solution

let $f(x) = x \log x - 1.2$

Then,

$f'(x) = 1 + \log x$

Rough

real root : 2.74

Taking, $x_1 = 2$

$\hookrightarrow A = x\log x - 12 : B = 1 + \log x : C = x - \frac{A}{B}$

Iteration	x_1	$f(x_1)$	$f'(x_1)$	x_2
1	2	-0.5979	1.3010	2.45958
2.	2.45958	-0.2386	1.3908	2.63116
3.	2.63116	-0.0945	1.4201	2.69772
4.	2.69772	-0.0372	1.4309	2.72377
5.	2.72377	-0.0146	1.4351	2.73400
6.	2.73400	-5.7928×10^{-3}	1.4367	2.73803
7.	2.73803	-2.2810×10^{-3}	1.4374	2.73961
8.	2.73961	-9.0354×10^{-4}	1.4376	2.74023
9.	2.74023	-3.628×10^{-4}	1.4377	2.74048
10.	2.74048	-1.44×10^{-4}	1.4378	2.74058
11.	2.74058	-5.764×10^{-5}	1.4378	2.74062
12.	2.74062	-2.275×10^{-5}	1.4378	2.74063

The root of the given equation correct to 4 decimal method is 2.7406.

Q. Find root of $\sqrt{28}$ using NR method correct to 4 decimal places.

$\Rightarrow$ Solution

$x = \sqrt{28}$

$x^2 - 28 = 0$

Rough: 5.2915

let, $f(x) = x^2 - 28$

$f'(x) = 2x$

taking $x_1 = 5$

Iteration	x_1	$f(x_1)$	$f'(x_1)$	x_2
1	5	-3	10	5.3
2	5.5	0.09	10.6	5.29150
3.	5.29150	-0.00002	10.8	5.29150

∴ The root of the given equation correct to 4 decimal places is 5.2915.

Q.

Find the point of intersection of line $y = x-3$ and $y = \ln(x)$ using bisection method correct to 3 decimal places.

⇒ Given,

$y = x-3$ and $y = \ln(x)$

$f(x) = x - 3 - \ln(x)$

Taking $x_1 = 4$, $f(x_1) = -0.3862$

$x_2 = 5$, $f(x_2) = 0.3905$

Iteration	x_1	x_2	$x_0 = \frac{x_1+x_2}{2}$	$f(x_1)$	$f(x_2)$	$f(x_0)$
1						
:						
12						

The root of $f(x)$ is 4.505 which correct to 3 decimal places

ie $x = 4.505$

Now

$y = x-3 = 4.505 - 3 = 1.505$

∴ The point of intersection is (4.505, 1.505)

4. Secant Method [open end method]

let us take x_1 and x_2 as starting values

Here,

$$\tan\theta = \frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

or, $f(x_1)(x_2 - x_3) = f(x_2)(x_1 - x_3)$

$$\boxed{\therefore \quad x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}}$$

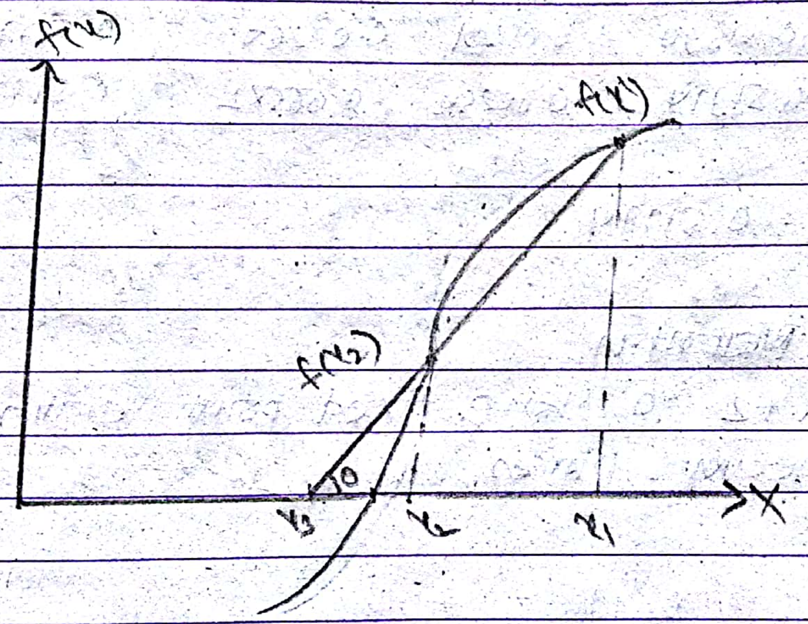

Fig: Illustration of secant Method

Q. Find the root of $xe^x = \cos x$ using secant method. Convert to 4 decimal places.

⟹ Solution

Rough Root: 0.51775

$f(x) = xe^x - \cos x$

Taking $x_1 = 1$, $x_2 = 2$ | $x_1 = 0$ & $x_2 = 1$

$A = xe^x - \cos x$: $B = ye^y - \cos y$: $C = \frac{xA - yB}{x - y}$

Iteration	x_1 (X)	x_2 (Y)	$f(x_1)$	$f(x_2)$	x_3
1	0	1	-1	2.1779	0.3146
2.	1	0.3146	2.1779	-0.51907	0.44672
3	0.314665	0.44612	-0.151987	-0.20356	0.53170
4.	0.44672	0.53170	-0.20356	0.04291	0.51690
	0.53170	0.51690	0.04291	0.00260	0.51774
6.	0.51690	0.51774	0.00260	-0.00582	0.51775

$\therefore$ Root = 0.51774

5) Fixed point Iteration

Q. Solve $x^2 - 3x + 2 = 0$ using fixed point iteration correct to 3 decimal places.

$\Rightarrow$ Soln

Here

$f(x) = x^2 - 3x + 2$

which can be written as

$$\boxed{x = \frac{x^2 + 2}{3}}$$

* Change the arrangement as necessary

eg: $x^2 - 5 = 0$

$2x^2 - x^2 - 5 = 0$

Let us take $x_0 = 0$

$$x_1 = \frac{2}{3} = 0.6666$$

$$x_2 = \frac{(0.6666)^2 + 2}{3} = 0.8147$$

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$$x_3 = \frac{(0.8147)^2+2}{3} = 0.8879$$

$$x_4 = \frac{(0.8879)^2+2}{3} = 0.9294$$

$$x_5 = \frac{(0.9294)^2+2}{3} = 0.95462$$

$$x_6 = \frac{(0.95462)^2+2}{3} = 0.9704$$

$$x_7 = \frac{(0.9704)^2+2}{3} = 0.9805$$

$$x_8 = \frac{(0.9805)^2+2}{3} = 0.9871$$

$$x_9 = \frac{(0.9871)^2+2}{3} = 0.9915$$

$$x_{10} = \frac{(0.9915)^2+2}{3} = 0.9943$$

$$x_{11} = \frac{(0.9943)^2+2}{3} = 0.9962$$

$$x_{12} = \frac{(0.9962)^2+2}{3} = 0.9973$$

$$x_{13} = \frac{(0.9973)^2+2}{3} = \underline{0.9983}$$

$$x_{14} = \frac{(0.9983)^2+2}{3} = \underline{0.99884}$$

Chapter : 3

Solution of System of Linear Algebric Equations

A system of linear equations is represented as

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

In matrix notation, $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

1. Gauss Elimination

Q. Apply Gauss Elimination method to solve following equations.

$$5x_1 - 4x_2 + 3x_3 = 12$$
$$x_1 + x_2 + x_3 = -4$$
$$2x_1 + x_2 + x_3 = 11$$

Solution: The argument matrix is

$$\left[\begin{array}{ccc:c} 5 & -4 & 3 & -12 \\ 1 & -1 & 1 & -4 \\ 2 & 1 & 1 & 11 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 5 & -4 & 3 & 12 \\ 0 & -1/5 & 2/5 & -8/5 \\ 0 & 13/5 & -1/5 & 79/5 \end{array}\right] \quad R_2 \to R_2 - \frac{1}{5}R_1,\ R_3 \to R_3 - \frac{2}{5}R_1$$

$R_3 \to R_3 + 13 R_2$

$$\sim \left[\begin{array}{ccc|c} 5 & -4 & 3 & -12 \\ 0 & -1/5 & 2/5 & -8/5 \\ 0 & 0 & 5 & -5 \end{array}\right]$$

Forward elimination method

Now,
performing backward substitution

$5x_3 = -5$

$\Rightarrow x_3 = -1$

also, $-\frac{1}{5}x_2 + \frac{2}{5}x_3 = -\frac{8}{5}$

$\Rightarrow x_2 = 6$

Finally, $5x_1 - 4x_2 + 3x_3 = -12$

$\Rightarrow x_1 = 3$

2. Gauss Elimination with Partial Pivoting

Q. Solve the following by Gauss Elimination using partial pivoting.

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 - x_2 + x_3 = 0$$

(Max value लाई माथि राख्ने)

Solution:

The argumented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

(Absolute value हेर्ने)

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - \frac{1}{2}R_1$; $R_3 \to R_3 - \frac{1}{4}R_1$

$$\sim \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & 1 & -1/2 & 4 \\ 0 & -3/2 & 1/4 & -1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & -3/2 & 1/4 & -1 \\ 0 & 1 & -1/2 & 4 \end{array}\right]$$

$R_3 \to R_3 + \frac{2}{3}R_2$

$$\sim \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & -3/2 & 1/4 & -1 \\ 0 & 0 & -1/3 & 10/3 \end{array}\right]$$

Now,

performing backward substitution,

$-\frac{1}{3}x_3 = \frac{10}{3}$

$\therefore x_3 = -10$

$-\frac{3}{2}x_2 + \frac{1}{4}x_3 = -1$

or, $\frac{3}{2}x_2 - \frac{1}{4}x_3 = 1$

$\therefore x_2 = -1$

$4x_1 + 2x_2 + 3x_3 = 4$

or, $4x_1 + 2\times(-1) + 3\times(-10) = 4$

$\therefore x_1 = 9$

Q. Solve the following equations using elimination (use partial pivoting)

$\Rightarrow$ Solution:

The argumented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & 2 & 19 \\ 0 & 3 & 2 & 2 & 20 \\ 1 & 4 & 0 & 2 & 17 \\ -2 & 2 & 1 & 1 & 9 \end{array}\right]$$

$\sim R_4 \leftrightarrow R_1$

(Rao मत देखें)

$$\sim \left[\begin{array}{cccc|c} -2 & 2 & 1 & 1 & 9 \\ 0 & 3 & 2 & 2 & 20 \\ 1 & 4 & 0 & 2 & 17 \\ 0 & 1 & 3 & 2 & 19 \end{array}\right]$$

$R_3 \to R_3 + R_1/2$

$$\sim \left[\begin{array}{cccc|c} -2 & 2 & 1 & 1 & 9 \\ 0 & 3 & 2 & 2 & 20 \\ 0 & 5 & 1/2 & 5/2 & 43/2 \\ 0 & 1 & 3 & 2 & 19 \end{array}\right]$$

$\sim \quad R_2 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} -2 & 2 & 1 & 1 & 9 \\ 0 & 5 & 1/2 & 5/2 & 43/2 \\ 0 & 3 & 2 & 2 & 20 \\ 0 & 1 & 3 & 2 & 19 \end{array}\right]$$

$\sim \quad R_3 \to R_3 - \frac{3}{5}R_2, \quad R_4 \to R_4 - 1/5\, R_2$

$$\left[\begin{array}{cccc|c} -2 & 2 & 1 & 1 & 9 \\ 0 & 5 & 1/2 & 5/2 & 43/2 \\ 0 & 0 & 17/10 & 1/2 & 71/10 \\ 0 & 0 & 29/10 & 3/2 & 147/10 \end{array}\right]$$

$\sim \quad R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} -2 & 2 & 1 & 1 & 9 \\ 0 & 5 & 1/2 & 5/2 & 43/2 \\ 0 & 0 & 29/10 & 3/2 & 147/10 \\ 0 & 0 & 17/10 & 1/2 & 71/10 \end{array}\right]$$

$\sim \; R_4 \to R_4 - \frac{17}{29} R_3$

$$\left[\begin{array}{cccc:c} -2 & 2 & 1 & 1 & 9 \\ 0 & 5 & 1/2 & 5/2 & 43/2 \\ 0 & 0 & 29/10 & 3/2 & 147/10 \\ 0 & 0 & 0 & -11/29 & -44/29 \end{array}\right]$$

$x_4 = 4$

$\frac{29}{10} x_3 + \frac{3}{2} x_4 = \frac{174}{10}$

$\therefore x_3 = 3$

$5x_2 + \frac{1}{2} x_3 + \frac{5}{2} x_4 = \frac{43}{2}$

$\therefore x_2 = 2$

$-2x_1 + 4 + 3 + 4 = 9$

$\therefore x_1 = 1$

Q. Solve following equations by Gauss elimination using complete pivoting.

$x_1 - 2x_2 + x_3 = 1$

$2x_2 + 2x_3 = 4$

$-2x_1 + 4x_2 + 2x_3 = 2$

(Row र column दुवै हैन)

Solution

The augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 4 \\ -2 & 4 & 2 & 2 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc:c} -2 & 4 & 2 & 2 \\ 0 & 2 & 2 & 4 \\ 1 & -2 & 1 & 1 \end{array}\right]$$

$C_1 \leftrightarrow C_2$

$$\sim \left[\begin{array}{ccc:c} 4 & -2 & 2 & 2 \\ 2 & 0 & 2 & 4 \\ -2 & 1 & 1 & 1 \end{array}\right]$$

$R_2 \to R_2 - \frac{1}{2}R_1$; $R_3 \to R_3 + \frac{1}{2}R_1$

$$\left[\begin{array}{ccc:c} 4 & -2 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

Using backward substitution

$x_3 = 1$

$x_1 = 2$

$x_2 = 1$

(i) Gauss Jordan Method

Q. Apply Gauss - Jordan Method to solve the following equations

(pivot element लाई 1 बनाउने)

$$2x_1 + 4x_2 - 6x_3 = -8$$
$$x_1 + 3x_2 + x_3 = 10$$
$$2x_1 - 4x_2 - 2x_3 = -12$$

Solution : The argumented Matrix is

$$\left[\begin{array}{ccc:c} 2 & 4 & -6 & -8 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{array}\right]$$

$R_1 \rightarrow R_1/2$ } normalization

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & -4 \\ 0 & 1 & 4 & 14 \\ 0 & -8 & 4 & -4 \end{array}\right]$$

$R_1 \rightarrow R_1 - 2R_2$; $R_3 \rightarrow R_3 + 8R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -11 & -32 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 36 & 108 \end{array}\right]$$

$R_3 \to \frac{R_3}{36}$ by normalization

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -11 & -32 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 \to R_1 + 11R_3$; $R_2 \to R_2 - 4R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\therefore$ $x_1 = 1$
$x_2 = 2$
$x_3 = 3$

HW. Use Gauss Jordan method to solve following equations

$$2x_1 + 2x_2 - 2x_3 + x_4 = 4$$
$$2x_1 - 3x_2 + 4x_3 - 2x_4 = 0$$
$$-3x_1 + 5x_2 - x_3 - x_4 = 0$$
$$x_1 + 2x_2 - x_3 - 2x_4 = -6$$

$$\Rightarrow \left[\begin{array}{cccc:c} 2 & 2 & -2 & 1 & 4 \\ 2 & -3 & 4 & -2 & 0 \\ -3 & 5 & -1 & -1 & 0 \\ 1 & 2 & -1 & -2 & -6 \end{array}\right]$$

$R_1 \to R_1/2$

$$\begin{bmatrix} 1 & 1 & -1 & 1/2 & : & 2 \\ 2 & -3 & 4 & -2 & : & 0 \\ -3 & 5 & -1 & -1 & : & 0 \\ 1 & 2 & -1 & -3 & : & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 3R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & -1 & 1/2 & : & 2 \\ 0 & -5 & 6 & -3 & : & -4 \\ 0 & 8 & -4 & 1/2 & : & 6 \\ 0 & 1 & 0 & -5/2 & : & -8 \end{bmatrix}$$

$R_2 \to R_2/-5$

$$\begin{bmatrix} 1 & 1 & -1 & 1/2 & : & 2 \\ 0 & 1 & -6/5 & 3/5 & : & 4/5 \\ 0 & 8 & -4 & 1/2 & : & 6 \\ 0 & 1 & 0 & -5/2 & : & -8 \end{bmatrix}$$

$R_3 \to R_3 - 8R_2$, $R_1 \to R_1 - R_2$, $R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 1/5 & -1/10 & : & 6/5 \\ 0 & 1 & -6/5 & 3/5 & : & 4/5 \\ 0 & 0 & 28/5 & -43/10 & : & -2/5 \\ 0 & 0 & 6/5 & -31/10 & : & 44/5 \end{bmatrix}$$

$R_3 \to R_3 \times \frac{5}{28}$

$$\begin{bmatrix} 1 & 0 & 1/5 & -1/10 & : & 6/5 \\ 0 & 1 & -6/5 & 3/5 & : & 4/5 \\ 0 & 0 & 1 & -43/56 & : & -1/14 \\ 0 & 0 & 6/5 & -31/10 & : & -44/5 \end{bmatrix}$$

$R_1 \to R_1 - R_3/5$, $R_2 \to R_2 + \frac{6}{5} R_3$, $R_4 \to R_4 - 6/5\, R_3$

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$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 3/56 & 17/14 \\ 0 & 1 & 0 & -9/28 & 5/7 \\ 0 & 0 & 1 & -43/56 & -1/14 \\ 0 & 0 & 0 & -61/28 & -61/7 \end{array}\right]$$

$R_4 \to R_4 \cancel{*} / \left(-61/28\right)$

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 3/56 & 17/14 \\ 0 & 1 & 0 & -9/28 & 5/7 \\ 0 & 0 & 1 & -43/56 & -1/14 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 \to R_1 - \frac{3}{56} R_4$, $R_2 \to R_2 + 9/28 R_4$, $R_3 \to R_3 + 43/56 R_4$

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$\therefore x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$

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* Determination of Inverse of a Matrix using Gauss Jordan Method.

Find inverse of matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{bmatrix}$ using Gauss Jordan.

⇒ Solution:

The Matrix 'A' with Identity matrix I is

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = \frac{R_1}{2}$

$$\left[\begin{array}{ccc:ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = R_2 - R_1$, $R_3 \rightarrow R_3 - 4R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & -3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right]$$

$R_1 = R_1 + \left(\frac{1 \cdot R_2}{3}\right)$

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$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 2/3 & 1/3 & 1/3 & 0 \\ 0 & -3/2 & -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right]$$

$R_1 = R_1 + \frac{1}{10} R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 2/15 \\ 0 & 1 & -1/3 & 1/3 & -2/3 & 0 \\ 0 & 0 & 1 & 0 & 2/5 & -1/15 \end{array}\right]$$

$R_2 \to -\frac{2}{3} R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1/3 & 1/3 & -2/3 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 1/2\, R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 2/3 & 1/3 & -1/3 & 0 \\ 0 & 1 & -1/3 & 1/3 & -2/3 & 0 \\ 0 & 0 & -5/3 & -2 & 0 & 1 \end{array}\right]$$

$R_3 \to \frac{R_3}{-5}$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 2/3 & 1/3 & -1/3 & 0 \\ 0 & 1 & -1/3 & 1/3 & -2/3 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & -1/5 \end{array}\right]$$

$R_1 \to R_1 - 2/3\, R_3$

$R_2 \to R_2 + 1/3\, R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/15 & 1/3 & 2/15 \\ 0 & 1 & 0 & 7/15 & -2/3 & -1/15 \\ 0 & 0 & 1 & 2/5 & 0 & -1/5 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1/15 & 2/3 & 2/15 \\ 7/15 & -2/3 & -1/15 \\ 2/5 & 0 & -1/5 \end{bmatrix}$$

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* LU Decomposition | Method of factorization
(i) Do little method (ii) Crout method

let system of linear equation is $AX = B$ .... (i)
A is factorized into two triangular matrices 'L' and 'U' such that

$$A = LU \quad \text{.....} \quad (ii)$$

where,

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & & \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

Do little = 1 Crout

from (i) & (ii)

$LUX = B$

or, $\boxed{LY = B}$ where $\boxed{Y = UX}$

* Solve the system
$3x_1 + 2x_2 + x_3 = 10$
$2x_1 + 3x_2 + 2x_3 = 14$
$x_1 + 2x_2 + 3x_3 = 14$
by using Do little LU Decomposition.

⇒ Solution
Here,

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = A$$

or,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{11} l_{21} & u_{12} l_{21} + u_{22} & u_{13} l_{21} + u_{23} \\ u_{11} l_{31} & u_{12} l_{31} + u_{22} l_{32} & u_{13} l_{31} + u_{23} l_{32} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Equating both sides,

$u_{11} = 3$ $\quad u_{12} = 2$ $\quad u_{13} = 1$

$l_{21} = \frac{2}{3}$ $\quad u_{22} = 5/3$ $\quad u_{23} = 4/3$

$l_{31} = 1/3$ $\quad u_{32} = 4/5$ $\quad u_{23} = 8/5$

So, the Matrices are

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}$$

or, $\boxed{LY = B}$ where $\boxed{Y = UX}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/5 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$y_1 = 10$ $\qquad \frac{2}{3} y_1 + y_2 = 0$ $\qquad y_3 = 24/5$

$\therefore y_2 = \frac{22}{3}$

Finally, $Ux = Y$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

Solving,

$x_3 = 3$

$x_2 = 2$

$x_1 = 1$

HW Crout method

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \times \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = A$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} l_{11} \Rightarrow 3 & 0 & 0 \\ l_{21} \Rightarrow 2 & l_{32} = 5/3 & 0 \\ l_{31} \Rightarrow 1 & l_{32} = 4/3 & l_{33} = 8/5 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} = 2/3 & u_{13} = 1/2 \\ 0 & 1 & u_{23} = 4/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LY = B$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} y_1 = 10/3 \\ y_2 = 22/5 \\ y_3 = 3 \end{bmatrix}$$

$l_{31} = 2 \quad 2 \times \frac{2}{3} + l_{22} = 5/3 \quad 2 \cdot 1/3 + 5/3 \cdot u_{23} = 2$

Again,

$$Y\cancel{X} = UX$$

$$\begin{pmatrix} 10/3 \\ 22/5 \\ 3 \end{pmatrix} = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 + x_2 \cdot \frac{2}{3} + \frac{1}{3} x_3 = \frac{10}{3}$$

$$x_1 + \frac{2}{3} \cdot 2 + 1/3 \cdot 3 = \frac{10}{3} \Rightarrow \boxed{x_1 = 1}$$

$$\boxed{x_3 = 3}$$

$$x_2 + \frac{4}{5} x_3 = \frac{22}{5}$$

$$\Rightarrow \boxed{x_2 = 2}$$

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Iterative Methods

(i) Gauss Jacobi (ii) Gauss Seidel

* Solve following system of linear equation using Gauss Jacobi method.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

⇒ Solution

We can write the given equation in the form

$$x = (17 - y + 2z)/20$$
$$y = (-18 - 3x + z)/20$$
$$z = (25 - 2x + 3y)/20$$

let, initial values of x, y and z be zero

Iteration	x	y	z
1	0	0	0
2	0.85	-0.9	1.25
3.	1.02	-0.965	1.03
4.	1.0020125	-1.0015	1.00325
5.	1.0004	-1.000025	0.999
6.	"	"	"
	1	-1	1

The solution is $x = 1$, $y = -1$ and $z = 1$

Solve following system of linear equation using Gauss Seidel Method.

⇒ We can write the given equation in the form,

$x = (17 - y + 2z)/20$

$y = (-18 - 3x + z)/20$

$z = (25 - 2x + 3y)/20$

* change the arrangement as necessary

let, initial values of x, y and z be zero.

Table for Gauss seidel.

Iteration	x	y	z
1	0	0	0
2	0.85	−1.0275	1.01087
3.	1.002	−0.9998	0.9997
4.	0.9999	−1	1.0000
	≈	≈	≈
	1	−1	1

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* Determination of Eigen value and Eigen vector by Power Method

Q. Find the largest eigen value and corresponding eigen vector of Matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

⇒ Solution :

let, the initial eigen vector be, $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *any

1st iteration :

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$ highest absolute value

$$X = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix}$$

2nd iteration

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2 \\ 0 \end{bmatrix}$$

$$X = \frac{1}{2.5} \begin{bmatrix} 2.5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.8 \\ 0 \end{bmatrix}$$

3rd iteration

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.8 \\ 0 \end{bmatrix}$$

$$X = \frac{1}{2.8} \begin{bmatrix} 2.6 \\ 2.8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.92 \\ 1 \\ 0 \end{bmatrix}$$

4th iteration

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.92 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.92 \\ 2.84 \\ 0 \end{bmatrix}$$

$$X = \frac{1}{2.92} \begin{bmatrix} 2.92 \\ 2.84 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.96 \\ 0 \end{bmatrix}$$

5th iteration

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.96 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.92 \\ 2.96 \\ 0 \end{bmatrix}$$

$$X = \frac{1}{2.96} \begin{bmatrix} 2.92 \\ 2.96 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}$$

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6th iteration

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \overset{0.98}{\cancel{2.92}} \\ \cancel{0.31} \; 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \overset{2.92}{\cancel{4.90}} \\ \underset{2.96}{\cancel{0.89}} \\ 0 \end{bmatrix}$$

$$X = \frac{1}{2.96} \begin{bmatrix} 2.92 \\ 2.96 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}$$

7th iteration

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.71 \\ 2.42 \\ 0 \end{bmatrix}$$

~~8th iteration~~ $X = \frac{1}{2.71} \begin{bmatrix} 2.71 \\ 2.42 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.89 \\ 0 \end{bmatrix}$

8th iteration

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.89 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.78 \\ 2.67 \\ 0 \end{bmatrix}$$

$X =$

$\therefore 2.90 \approx 2.97$

eigen value largest is 2.96

& corresponding eigen vector is $\begin{bmatrix} 0.98 \\ 1 \\ 0 \end{bmatrix}$

Chapter - 4

Interpolation

1. Lagrange Interpolation

If $y = f(x)$ takes the values $y_1, y_2, y_3 \ldots y_n$ corresponding to $x_1, x_2, x_3, \ldots x_n$ then lagrange interpolation formula is

$$f(x) = \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} y_1 + \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} y_2$$

$$\frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} y_n$$

$$f(x) = \sum_{i=1}^{n} \left[\prod_{\substack{j=1 \\ i \neq j}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \right] y_i$$

* Given the values

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evaluate $f(9)$ using lagrange interpolation.

$\Rightarrow$ Using lagrange interpolation formula,

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$$f(x) = \sum_{i=1}^{5} \left[\prod_{\substack{j=1 \\ i \neq j}}^{5} \frac{(x - x_j)}{(x_i - x_j)} \right] y_i$$

$$= \frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} y_1 + \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} y_2 + \ldots y_3 + \ldots y_4 + \ldots y_5$$

Putting $x = 9$

$$f(9) = -\frac{1}{9} y_1 + \frac{8}{15} y_2 + \frac{8}{9} y_3 + \frac{-1}{3} y_4 + \frac{1}{45} y_5$$

$$= -\frac{1}{9} \times 150 + \frac{8}{15} \times 392 + \frac{8}{9} \times 1452 + \left(\frac{-1}{3}\right) \times 2366 + \frac{1}{45} \times 5202$$

$\therefore f(9) = 810$

* Find lagrange interpolation polynomial to fit following data.

x :	0	1	2	3
$e^x - 1$:	0	1.7183	6.3891	19.0855

Use the polynomial to estimate the value of $e^{1.5}$.

$\Rightarrow$ Solution :

$f(x) =$ in terms of x^3

find the polynomial

$$= \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 +$$

$$\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

$$= 0 + \frac{x \times (x-2)(x-3)}{1 \times (-1) \times (-2)} y_2 + \frac{x \times (x-1)(x-3)}{2 \times 1 \times (-1)} y_3 +$$

$$\frac{x \times (x-1)(x-2)}{3 \times 2 \times 1} y_4$$

$$= \frac{x(x^2-3x-2x+6)}{2} y_2 + \frac{x(x^2-3x-x+3)}{-2} y_3 + \frac{x(x^2-2x-x+2)}{6} y_4$$

$$= \{ \quad \} \quad \{ \quad \}$$

$f(x) = 0.8455x^3 - 1.06025x^2 + 1.9337x$ is the required polynomial,

i.e

$e^x - 1 = 0.8455x^3 - 1.06025x^2 + 1.9337x$

Putting $x = 1.5$

$e^{1.5} = \ldots \ldots \quad 4.3675$

HW

Q. Find out the missing values in the following set of data using language interpolation.

x =	-2	-1	0	1	6
$f(x)$ =	-20	?	2	?	70

② Newtons Interpolation

The newton's form of polynomial is given as

$$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

If $(x_0, f_0), (x_1, f_1) \dots$ are interpolating points,

at $x = x_0$, $P(x_0) = a_0 = f_0$

at $x = x_1$, $P(x_1) = a_0 + a_1(x_1 - x_0) = f_1$

ie $f_0 + a_1(x_1 - x_0) = f_1$

$$\therefore a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

Similarly,

at $x = x_2$,

we get $a_2 = \dfrac{\left(\dfrac{f_2 - f_1}{x_2 - x_1}\right) - \left(\dfrac{f_1 - f_0}{x_1 - x_0}\right)}{x_2 - x_0}$

These quantities can be represented as

$a_0 = f_0 = f[x_0]$

$a_1 = \dfrac{f_1 - f_0}{x_1 - x_0} = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1]$

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$$a_2 = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$= f[x_0, x_1, x_2]$$

Likewise,

$$a_3 = f[x_0, x_1, x_2, x_3]$$

$$= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} - \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}}{x_3 - x_0}$$

$$\therefore P(x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) + \frac{\left(\frac{f_2 - f_1}{x_2 - x_1}\right) - \left(\frac{f_1 - f_0}{x_1 - x_0}\right)}{x_2 - x_0}(x - x_0)(x - x_1) + \dots$$

$$\boxed{\begin{aligned} P_x = f_0 &+ f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots \\ &+ f[x_0, x_1, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1}) \end{aligned}}$$

* Estimate log 2.5 using Newtons Interpolation.

x :	1 x_0	2 x_1	3 x_2
$\log x$:	0 f_0	0.3010 f_1	0.4771 f_2

→ Solution

Here,

$f[x_0] = 0$

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{0.3010 - 0}{2-1} = 0.3010$$

also,

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{[x_2 - x_0]}$$

$$= \frac{\frac{f_2 - f_1}{x_2 - x_1} - 0.3010}{3-1}$$

$$= \frac{\frac{0.4771 - 0.3010}{3-2} - 0.3010}{2}$$

$$= -0.06245$$

The polynomial becomes

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$P(x) = 0 + 0.3010(x-1) - 0.06245(x-1)(x-2)$$

at $x = 2.5$

$$P(2.5) = 0 + 0.3010(2.5-1) - 0.06245(2.5-1)(2.5-2)$$

$$= 0.4046$$

V-IMP

Q. Estimate f(1.732) and f(2.464) from following set of data using Newton's Interpolation.

x :	−2	−1	0	1	2	3
$f(x)$:	64	−5.5	−10	−9.5	56	366.5

Solution : Newton's divided difference table is as below :-

x	$f(x)$ a_0	1st Diffn a_1	2nd Diffn a_2	3rd Diffn a_3	4th Diffn a_4	5th Diffn a_5
−2	64	$\frac{-5.5-64}{-1-(-2)} = -69.5$				
			$\frac{-4.5-(-69.5)}{0-(-2)} = 32.5$			
−1	−5.5	$\frac{-10-(-5.5)}{0-(-1)} = -4.5$		$\frac{2.5-32.5}{1-(-2)} = -10$		
			$\frac{0.5-(-4.5)}{1-(-1)} = 2.5$			
0	−10	$\frac{-9.5-(-5.5)}{1-0} = 0.5$			$\frac{10-(-10)}{2-(-2)} = 5$	
			$\frac{65.5-2.5}{2-0} = 32.5$	$\frac{32.5-2.5}{2-(-1)} = 10$		$\frac{5-5}{3-(-2)} = 0$
1	−9.5	$\frac{56-(-9.5)}{2-1} = 65.5$			$\frac{30-10}{3-(-1)} = 5$	
			$\frac{310.5-65.5}{3-1} = 122.5$	$\frac{122.5-32.5}{3-0} = 30$		
2	56	$\frac{366.5-56}{3-2} = 310.5$				
3	366.5					

Now,

Newton's Interpolation Polynomial is

$$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) + a_5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

$= 64 + (-5.5)x(x+2) + (-10)x(x+2)(x+1) + (-9.5)x(x+2)(x+1)(x-0) + 56(x+2)(x+1)(x-0)(x-1) + 366.5x(x+2)(x+1)(x-0)(x-1)(x-2)$

at $x = 1.732$,
$P(1.732) = 24.0311$

at $x = 2.464$
$P(2.464) = 154.196$

* Newton Gregory Interpolation (only for equidistant points)

Forward

$$P(s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \dots$$

Here, $s = \frac{x - x_0}{h}$, h = step size

Backward

$$P(s) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + s(s+1)(s+2)\nabla^3 f_n + \dots$$

here, $s = \frac{x - x_n}{h}$

Q. Estimate the value of $\sin\theta$ at $\theta = 15°$ using Newton's Interpolation (Gregory) with following data set.

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5	0.6428	0.7660

Solution:

Here, step size 'h' = 10

$x(\theta)$	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
10	0.1736 (f_0)	$0.3420 - 0.1736 = 0.1684$ (Δf_0)	$\Delta^2 f_0$	$\Delta^3 f_0$	
20	0.3420	$0.5 - 0.3420 = 0.158$	-0.0104	-0.0046	$\Delta^4 f_0$
30	0.5	$0.6428 - 0.5 = 0.1428$	-0.0152		0.0004
40	0.6428	$0.766 - 0.6428 = 0.1232$	-0.0196	-0.0044	$\nabla^4 f_n$
50	0.766 (f_n)	∇f_n	$\nabla^2 f_n$	$\nabla^3 f_n$	

Forward

Backward

Forward:

$$P(s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 +$$

$$\frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f_0$$

$$s = \frac{x - x_0}{h} = \frac{15-10}{10} = 0.5$$

$\therefore P(s) = 0.1736 + 0.5 \times 0.1684 + \frac{0.5(0.5-1)}{2} \times (-0.0704) +$

$\frac{0.5(0.5-1)(0.5-2)}{3 \times 2} \times (-0.0048) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4 \times 3 \times 2 \times 1} \times (0.00)$

$= 0.2587$

Backward

$s = \frac{x - x_n}{h} = \frac{15-50}{10} = -3.5$

$P(s) = f_n + s \Delta f_n + \frac{s(s+1)}{2!} \Delta^2 f_0 + \frac{s(s+1)(s+2)}{3!} \times \Delta^3 f_0 +$

$\frac{s(s+1)(s+2)(s+3)}{4!} \times \Delta^4 f_0$

$= 0.2587$

VVI:

* Least square method / Regression

If we fit a straight line $(a+bx)$ for points (x_i, y_i) then the deviation (error) of the point from the line is

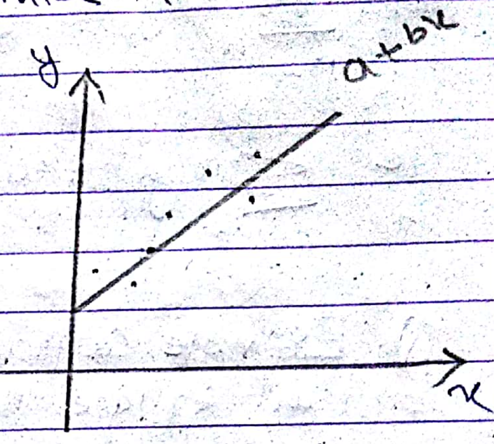
$q_i = y_i - (a + bx_i)$

or, $q_i^2 = [y_i - (a+bx_i)]^2$

The sum of the squares of error is

$Q = \sum_{i=1}^{n} q_i^2 = \sum_{i=1}^{n} [y_i - a - bx_i]^2$

The least square method is used to minimize this error



For minimum value of φ

$\frac{\delta\varphi}{\delta a} = 0$ ~~and~~ $\frac{\delta\varphi}{\delta b} = 0$

ie $\sum_{i=1}^{n} 2(y_i - a - bx_i)(-1) = 0$

or, $\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} 1 - b\sum_{i=1}^{n} x_i = 0$

or, $\sum_{i=1}^{n} y_i - an - b\sum_{i=1}^{n} x_i = 0$

$\therefore \sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$

ie $\sum y_i = na + b\sum x_i$

In Matrix form

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

and $\frac{\delta\varphi}{\delta b} = 0$

ie $\sum_{i=1}^{n} 2(y_i - a - bx_i)(-x_i) = 0$

or, $\sum_{i=1}^{n} x_i y_i - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^2 = 0$

or $\sum_{i=1}^{n} x_i y_i = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2$

ie $\sum x_i y_i = a \sum x_i + b \sum x_i^2$

Q. Estimate coefficients of $y = a + bx$ for following data using least square method.

x	-2	-1	0.5	2	3	5.5
y	-0.4	1.2	3.5	6	7.4	11

Solution:

x_i	y_i	x_i^2	$x_i y_i$
-2	-0.4	4	0.8
-1	1.2	1	-1.2
0.5	3.5	0.25	1.75
2	6	4	12
3	7.4	9	22.2
5.5	11	30.25	60.5
$\sum x_i = 8$	$\sum y_i = 28.7$	$\sum x_i^2 = 48.5$	$\sum x_i y_i = 96.05$

In matrix form

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Now,
we have,

जुन चाहिं x सँग connected छ त्यो चाहिं तल हुनुपर्छ ।

ie $$\begin{bmatrix} 6 & 8 \\ 8 & 48.5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 28.7 \\ 96.05 \end{bmatrix}$$

ie $6a + 8b = 28.7$
$8a + 48.5b = 96.05$

On solving,
$a = 2.74$
$b = 1.527$

$\therefore$ The required straight line is
$$y = 2.74 + 1.527x$$

For second degree polynomial,
$$y = a + bx + cx^2$$

$$\begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma y_i x_i^2 \end{bmatrix}$$

Q. Find best fit curve in the form $y = ax^2 + bx + c$ using least square approximation.

x :	-1.5	-1	0	1	2.5	4	5.5
y :	-0.75	0	0.72	1.12	1.5	1.8	2

→ Solution

x_i	y_i	x_i^2	x_i^3	x_i^4	x_iy_i	$y_ix_i^2$
-1.5	-0.75	2.25	-3.375	5.062	1.125	-1.6875
-1	0	1	-1	1	0	0
0	0.72	0	0	0	0	0
1	1.12	1	1	1	1.12	1.12
2.5	1.5	6.25	15.62	39.062	3.75	9.375
4	1.8	16	64	256	7.2	28.8
5.5	2	30.25	166.37	915.06	4	60.5
Σ = 10.5	6.39	56.75	242.61	1217.18	24.195	98.107

In matrix,

$$\begin{bmatrix} 7 & 10.5 & 56.75 \\ 10.5 & 56.75 & 242.61 \\ 56.75 & 242.61 & 1217.18 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 6.39 \\ 24.19 \\ 98.107 \end{bmatrix}$$

On solving,

$c = 0.529$

$b = 0.604$

$a = -0.564$

∴ The required equation is $y = -0.564x^2 + 0.604x + 0.529$

$Y = ax + b$. $\therefore Y = a + bX$

$y = a + bx^3 \Rightarrow Y = a + bX$.

Cubic Spline

The cubic spline are given by.

$$f(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1} + \frac{(x_{i+1} - x)}{h}\left(y_i - \frac{h^2}{6} M_i\right) + \frac{(x - x_i)}{h}\left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right); \quad i = 0, 1, 2, 3, \ldots n-1$$

so,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) \quad i = 1, 2, \ldots n-1$$

and $M_0 = M_n = 0$.

n = number of splines.

The following values of x and y are

x:	$1 (x_0)$	$2 (x_1)$	$3 (x_2)$	$4 (x_3)$
y:	$1 (y_0)$	$2 (y_1)$	$5 (y_2)$	$11 (y_3)$

Find the cubic splines & evaluate $y(1.5)$ and $y'(3)$

PAGE :
DATE : / /

Here

number of splines $(n) = 3$ $h = 1$ (step size)

we have

$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$;

for $i = 1$:

$\cancel{M_0}^{0} + 4M_1 + M_2 = \frac{6}{1^2}(y_0 - 2y_1 + y_2$

$4M_1 + M_2 = 6(1 - 2\times 2 + 5 = 12$ —— (I)

for $i = 2$.

$M_1 + 4M_2 + \cancel{M_3}^{0} = 6(y_1 - 2y_2 + y_3)$

$M_1 + 4M_2 = 6(2 - 2\times 5 + 11) = 18$ —— (II)

Solving (I) and (II) we get.

$M_1 = 2$ $M_2 = 4$.

The cubic splines are given by.

$f(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1} + \frac{(x_{i+1} - x)}{h}(y_i - \frac{h^2}{6} M_i) + \frac{(x - x_i)}{h}(y_{i+1} - \frac{h^2}{6} M_{i+1})$; $i = 0, 1, 2 \dots n-1$.

for $i = 0$.

$f(x) = \cancel{\frac{(x_1 - x)^3}{6} M_0}^{0} + \frac{(x - x_0)^3}{6} M_1 + \frac{(x_1 - x)}{1}(y_0 - \frac{1}{6} M_0) + \frac{(x - x_0)}{1}(y_1 - \frac{1}{6} M_1)$

$= \frac{(x-1)^3}{6} \times 2 + (2 - x)(1 - \frac{1}{6}\times 0) + \frac{(x-1)}{1}(2 - \frac{1}{6}\times 2)$

$= \frac{x^3 - 3x^2 + \cancel{3x} - \cancel{1} + \cancel{6} - \cancel{3x} + 5x - \cancel{6}}{3}$

a certain increment is

PAGE

DATE: / /

$= \frac{1}{3}(x^3 - 3x^2 + 5x) \; ; \quad 1 \le x \le 2.$

$y(1.5) = f(1.5) = \frac{1}{3}(1.5^3 + 3(1.5)^2 + 5 \times 1.5).$

$= \frac{11}{8}$ ans.

for i = 1.

$f(x) = \frac{(x_2 - x)^3}{6} M_1 + \frac{(x - x_1)^3}{6} M_2 + (x_2 - x)(y_1 - \frac{1}{6} M_1) +$

$(x - x_1)(y_2 - \frac{1}{6} M_2).$

$= \frac{(3-x)^3}{6} \times 2 + \frac{(x-2)^3}{6} \times 4 + (3-x)(2 - \frac{1}{6} \times 2) + (x-2)$

$(5 - \frac{1}{6} \times 4)$

$= \frac{27 - 27x + 9x^2 - x^3}{3} + \frac{(x^3 - 6x^2 + 12x - 8) \times 2}{3} + 5 - \frac{5}{3}x$

$+ \frac{13}{3}x - \frac{26}{3}.$

$= \frac{27 - 27x + 9x^2 - x^3 + 2x^3 - 12x^2 + 24x - 16 + 15 - 5x + 13x - 26}{3}$

$= \frac{1}{3}(x^3 - 3x^2 + 5x) \; ; \quad 2 \le x \le 3.$

$f'(x) = \frac{1}{3}(3x^2 - 6x + 5.$

$f(3) = \frac{1}{3}(3 \times 9 - 6 \times 3 + 5)$

$= \frac{14}{3}$ ans

Chapter-5

Numerical Differentiation and Integration

(1) Numerical Differentiation

Newton's Interpolating formula for equal interval for equal interval

$$f(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f_0$$

where, $\boxed{s = \frac{x - x_0}{h}}$ ie $\frac{ds}{dx} = \frac{1-0}{h} = \frac{1}{h}$

Now,

$$\frac{df(x)}{ds} = \Delta f_0 + \frac{(2s-1)}{2}\Delta^2 f_0 + \frac{3s^2-6s+2}{3!}\Delta^3 f_0 + \frac{4s^3-18s^2+22s-6}{4!}\Delta^4 f_0 + \dots$$

$$\frac{df(x)}{dx} = \frac{df(x)}{ds} \times \frac{ds}{dx}$$

$$= \frac{1}{h}\left[\Delta f_0 + \frac{2s-1}{2}\Delta^2 f_0 + \frac{3s^2-6s+2}{3!}\Delta^3 f_0 + \frac{4s^3-18s^2+22s-6}{4!}\Delta^4 f_0 + \dots\right.$$

at $x = x_0$, $s = 0$

$$\boxed{\therefore \left(\frac{df(x)}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0 - \frac{1}{6}\Delta^6 f_0\right.}$$

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Again,
Differenciating wrt 'x'

$$\frac{d^2f(x)}{dx^2} = \frac{1}{h^2}\left[\Delta^2 f_0 + (s-1)\Delta^3 f_0 + \frac{6s^2-18s+11}{12}\Delta^4 f_0 + \dots\right]$$

at $x = x_0$, $s = 0$

$$\boxed{\left(\frac{d^2f(x)}{dx^2}\right) = \frac{1}{h^2}\left[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\Delta^4 f_0 - \frac{5}{6}\Delta^5 f_0 + \frac{137}{180}\Delta^6 f_0 + \dots\right]} \dots (ii)$$

Similarly, from Newton's Backward difference formula

$$f(x) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \dots$$

we get similar formulas,

$$\boxed{\left(\frac{df(x)}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla f_n + \frac{1}{2}\nabla^2 f_n + \frac{1}{3}\nabla^3 f_n + \frac{1}{4}\nabla^4 f_n + \frac{1}{5}\nabla^5 f_n + \dots\right]}$$

And,

$$\boxed{\left(\frac{d^2f(x)}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 f_n + \nabla^3 f_n + \frac{11}{12}\nabla^4 f_n + \frac{5}{6}\nabla^5 f_n + \frac{137}{180}\nabla^6 f_n + \dots\right]}$$

Given:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.457	9.75	10.31

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (i) $x = 1.1$ and (ii) $x = 1.6$

Solution:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
1.0	7.989	8.403 − 7.989 = 0.414	−0.036	0.006	−0.002		
1.1	8.403					0.001	0.002
		0.378	−0.03	0.004	−0.001		
1.2	8.781						
1.3	9.129	0.348	−0.026	0.003	0.002	0.003	
1.4	9.457	0.328	−0.023	0.005			
1.5	9.75	0.299	−0.018				
1.6	10.31	0.281					

We have,

$$h = 1.1 - 1.0 = 0.1$$

$$\frac{df(x)}{dx} = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0 - \frac{1}{6}\Delta^6 f_0\right]$$

Now at $x = 1.1$

$$\left(\frac{df(x)}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}\times(-0.03) + \frac{1}{3}\times(0.004) - \frac{1}{4}\times(-0.001) + \frac{1}{5}\times(0.003) - \frac{1}{6}\times(0.002)\right]$$

$$= \simeq 3.9485$$

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again,

$$\frac{d^2 f(x)}{dx^2} = \frac{1}{h^2}\left[\Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\nabla^4 f_0 - \frac{5}{6}\Delta^5 f_0 + \frac{137}{180}\Delta^6 f_0 + \ldots\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.03 - 0.004 + \frac{11}{12}\times(-0.001) - \frac{5}{6}\times(0.003) + \frac{137\times(0.00}{180}\right.$$

$$= \frac{1}{(0.1)^2} \times \left[-0.03 - 0.004 + (-9.16\times10^{-4}) - 2.5\times10^{-3} + 1.522\times10^{-3}\right]$$

$$\simeq -3.58$$

Now, for $x = 1.6$, we use backward formula,

ie $$\left(\frac{df(x)}{dx}\right) = \frac{1}{h}\left[\nabla f_n + \frac{1}{2}\nabla^2 f_n + \frac{1}{3}\nabla^3 f_n + \frac{1}{4}\nabla^4 f_n + \frac{1}{5}\nabla^5 f_n + \frac{1}{6}\right.$$

$$\simeq 2.7$$

Again,

$$\left(\frac{d^2 f(x)}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 f_n + \nabla^3 f_n + \frac{11}{12}\nabla^4 f_n + \frac{5}{6}\nabla^5 f_n + \frac{137}{180}\nabla^6 f_n\right]$$

$$= -0.714$$

Find $f'(10)$ from following data.

	x_0	x_1			
x :	3	5	11	27	34
$f(x)$:	-13	23	899	17315	35606

Soln: Since x is unequispaced, we use Newton's divided difference table :-

x	$f(x)$ (a_0)	1st diff	2nd diff	3rd diff	4th diff
3	-13				
		$\frac{23+13}{5-3} = 18$ a_1			
			$= 16$ a_2		
5	23			a_3 1	
		$\frac{899-23}{11-5} = 146$			
			40		a_4 0
11	899				
		$= 1026$		1	
			69		
27	17315				
		$= 2613$	69		
34	35606				

We have, Newton's interpolation formula,

$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$

$+ a_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)$

$= -13 + 18(x - 3) + 16(x - 3)(x - 5) + 1(x - 3)(x - 5)(x - 11)$

$+ 0(x - 3)(x - 5)(x - 11)(x - 27)$

$= -13 + 18x - 54 + 16(x^2 - 5x - 3x + 15) + (x - 1)(x^2 - 5x - 3x + 15)$
$+ 0$

$= -13 + 18x - 54 + 16x^2 - 128x + 240 + (x^3 - 8x^2 + 15x - x^2 + 8x - 15)$

$= -13 + 18x - 54 + 16x^2 - 128x + 240 + x^3 - 8x^2 + 15x - x^2 + 8x - 15$

$= 158 - 87x + 7x^2 + x^3$ ×

$f(x) = x^3 - 3x^2 - 7x + 8$

$\therefore f'(x) = 3x^2 - 6x - 7$

$\therefore f'(10) = 233$

Q. Using the following data, find 'x' for which y is minimum also find the minimum value of y.

			↓ min	
x :	0.6	0.65	0.7	0.75
y :	0.6221	0.6155	0.6138	0.6170

Solution : The difference table is

x	y	Δf	$\Delta^2 f$	$\Delta^3 f$
0.6	0.6221			
		$0.6155 - 0.6221 = -6.6 \times 10^{-3}$		
0.65	0.6155		4.9×10^{-3}	
		-1.7×10^{-3}		0
0.7	0.6138			
		3.2×10^{-3}	4.9×10^{-3}	
0.75	0.6170			

From Newton's formula, we have

$$y = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \cancel{\frac{s(s-1)(s-2)}{3!}\Delta^3 f_0}^{\,0}$$

$$= 0.6221 + s(-0.0066) + \frac{(s^2-s)}{2!} \times 0.0049$$

$$y = 0.6221 - 0.0066s + (s^2 - s) \times 0.00245 \quad \ldots (1)$$

$$\frac{dy}{ds} = -0.0066 + (2s-1)(0.00245)$$

For y to be minimum

$$\frac{dy}{ds} = 0$$

ie $2s - 1 = \frac{0.0066}{0.00245}$

$\therefore s = 1.84693$

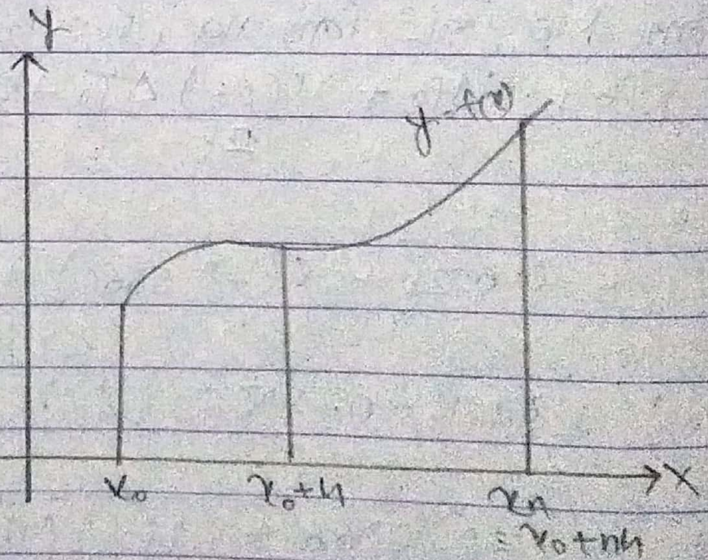
$s = \frac{x - x_0}{h}$ $\therefore x = x_0 + sh = 0.6923$

Putting value of s in (1), to find minimum value of y' in equation (1)

ie $y_{min} = 0.6221 - 0.0066(1.84693) + \ldots$
$= 0.6137$

* Numerical integration

$$I = \int_{x_0}^{x_n} f(x)\,dx$$



we have Newton gregory forward interpolation formula

$f(s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 +$

$\frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \ldots$ (1)

where $s = \frac{x - x_0}{h}$ ... (ii)

when $x = x_0$, $s = 0$

$x = x_n$, $s = n$

also,

$$\frac{ds}{ds} = \frac{dx/ds - 0}{h}$$

or, $dx = h\,ds$

Then (i) becomes

$$I = \int_{s=0}^{s=n} f(s)\, ds$$

$$= \int_0^n \left[f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \cdots\right]$$

$$= h\left[s f_0 + \frac{s^2}{2}\Delta f_0 + \left(\frac{s^3}{3} - \frac{s^2}{2}\right)\frac{\Delta^2 f_0}{2!} + \left(\frac{s^4}{4} - s^3 + s^2\right)\frac{\Delta^3 f_0}{3!} + \right]_0^n$$

$$= h\left[n f_0 + \frac{n^2}{2}\Delta f_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f_0}{3!} + \right]$$

$$= T_0 + T_1 + T_2 + \cdots$$

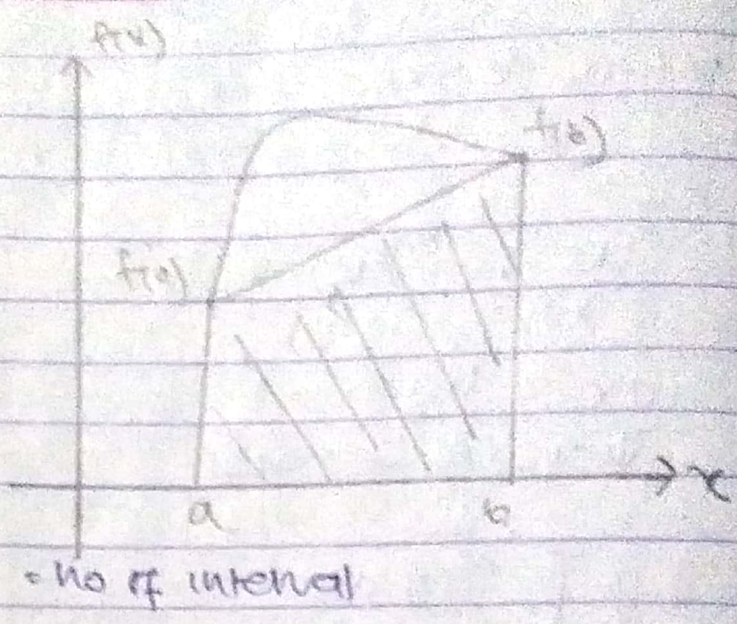
This is $\boxed{\text{Newtons Cotes formula.}}$

(1) a) Trapezoidal Rule (Two-point)

The integration consist of 1st two terms of Newton cotes formula

$$I_t = T_0 + T_1$$

$$= h\left[n f_0 + \frac{n^2}{2}\Delta f_0\right]$$



here,

$$h = \frac{b-a}{n} = b-a$$

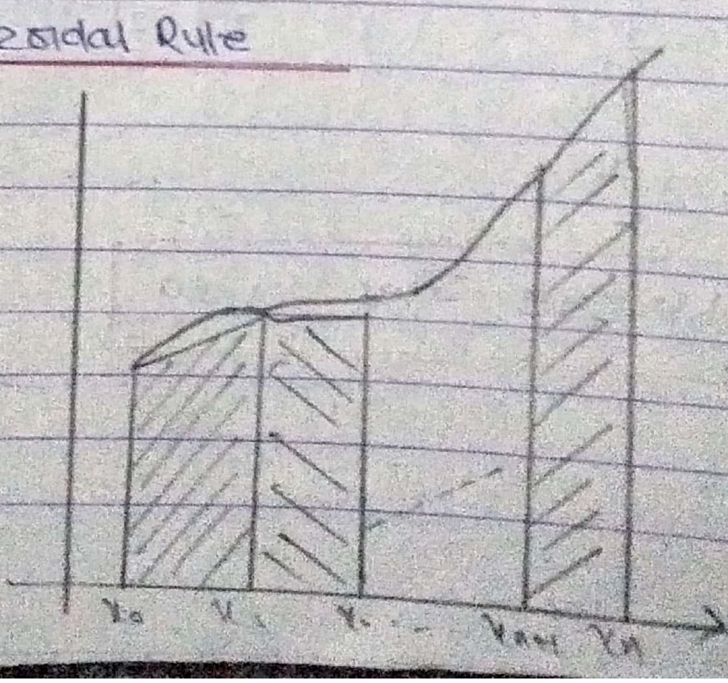
; n = no of interval

$$I_t = (b-a)\left[f_0 + \frac{1}{2}(f_1 - f_0)\right]$$

$$= (b-a)\left[\frac{f_0 + f_1}{2}\right]$$

$$\boxed{I_t = \frac{h}{2}(f_0 + f_1)}$$

b) Composite Trapezoidal Rule



Here,

$$I_{ct} = \int_{x_0}^{x_n} f(x)\,dx$$

$$I_{ct} = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \dots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \dots + \frac{h}{2}(f_{n-1} + f_n)$$

$$\boxed{I_{ct} = \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + f_3 + \dots + f_{n-1}) + f_n\right]}$$

Q. Evaluate the integral $I = \int_1^2 (x^3+1)\,dx$ using

(i) Trapezoidal rule (ii) Composite trapezoidal, $n=4$ and find absolute error.

Solution

(i) Here,

$f(x) = x^3+1$

$a = 1$, $b = 2$

We have trapezoidal rule

$$I_t = \frac{h}{2}(f_0 + f_1)$$

Here,

$$h = \frac{b-a}{1} = 2-1 = 1$$

$f_0 = f(1) = x^3+1 = 1^3+1 = 2$

$f_1 = f(2) = x^3+1 = 2^3+1 = 9$

$\therefore I_t = \frac{1}{2}(2+9)$

(ii) $h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$

$f_0 = f(1) = x^3+1 = (1)^3+1 = 2$

$f_1 = f(1+h) = f(1+0.25) = 2.95$

$f_2 = f(1+2h) = 4.375$

$f_3 = f(1+3h) = 6.3593$

$f_4 = f(1+4h) = 9$

$$I_{ct} = \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + f_3 + \ldots + f_{n-1}) + f_n\right]$$

$$= \frac{0.25}{2}\left[2 + 2[2.95 + 4.375 + 6.3593 + 9]\right]$$

$= 4.796$

$$\int_1^2 (x^3+1)\,dx$$

$$= \left[\frac{x^4}{4} + x\right]_1^2$$

$= 4.75$

Absolute error $= |I_{exact} - I_{calculated}|$

$= |4.75 - 4.796|$

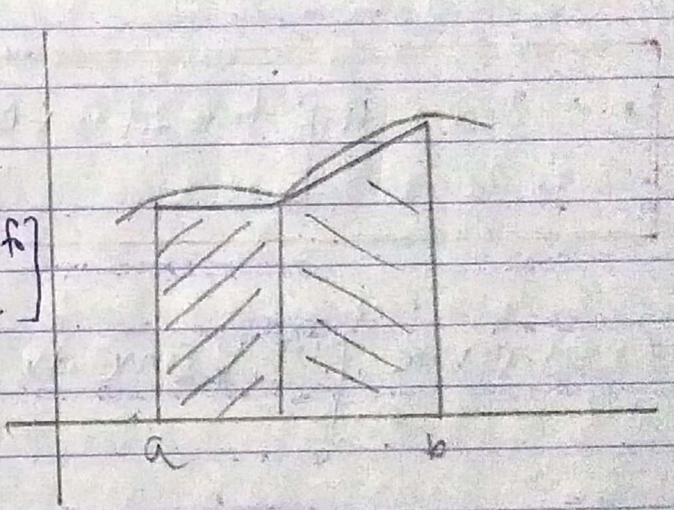
$=$

(2)

a) Simpson's 1/3 Rule (Three - point)

Here, we use three terms of Newton's - cotes formula

here, no of interval 'n' = 2



$$I_{S1} = h\left[nf_0 + \frac{n^2}{2}\Delta f_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f_0}{2!}\right]$$

$$= h\left[2f_0 + \frac{4}{2}(f_1 - f_0) + \left(\frac{8}{3} - \frac{4}{2}\right)\frac{(\Delta f_1 - \Delta f_0)}{2!}\right]$$

$=$

$$\boxed{\therefore I_{S1} = \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]}$$

b) Composite Simpson's 1/3

$$I_{CS1} = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \dots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

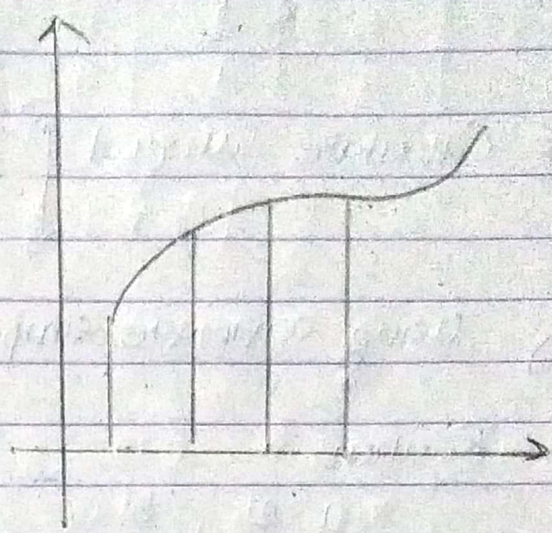
$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right] + \dots + \frac{h}{3}\left[f_{n-2} + 4f_{n-1} + f_n\right]$$

$$\boxed{\therefore I_{CS1} = \frac{h}{3}\left[f_0 + 2(f_2 + f_4 + \dots f_{n-2}) + 4(f_1 + f_3 + \dots + f_{n-1}) + f_n\right]}$$

Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\,dx$ using (a) Simpson's 1/3 Rule
(b) Composite Simpson's 1/3 with $n=6$

Note: Calculate in "Radian" mode

3) a) Simpson's 3/8 Rule (four-point formula)



Here $n = 3$

From Newton-cotes formula, we use four terms,

ie

$$I_{S_2} = h\left[nf_0 + \frac{n^2}{2}\Delta f_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f_0}{3!}\right]$$

Since $n = 3$

$$I_{S_2} = h\left[3f_0 + 9/2\,\Delta f_0 + (9 - 9/2)\frac{\Delta^2 f_0}{2!} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 f_0}{3!}\right]$$

On simplification

$$I_{S_2} = \frac{3h}{8}\left[f_0 + 3(f_1 + f_2) + f_3\right]$$

b) Composite Simpson 3/8 Rule

$$I_{S_2} = \int_{x_0}^{x_n} f(x)\,dx$$

$$= \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \dots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$= \frac{3h}{8}\left[f_0 + 3(f_1 + f_2) + f_3\right] + \frac{3h}{8}\left[f_3 + 3(f_4 + f_5) + f_6\right] + \dots$$

$\therefore I_{cs_2} = \frac{3h}{8}\left[f_0 + 2(f_3 + f_6 + f_9 + \cdots) + 3(f_1 + f_2 + f_4 + f_5 + \cdots) + f_n \right]$

* Calculate integral $\int_2^3 \left(x^2 + \frac{\sin x}{x}\right) dx$

→ Using composite simpson's 3/8 Rule, $n = 6$

Solution:

$a = 2, \quad b = 3, \quad n = 6$

$h = \frac{b-a}{n} = \frac{1}{6}$

$f(x) = x^2 + \frac{\sin x}{x}$

$f_0 = f(2) = 4 + \frac{\sin(2)}{2} = 4.9546$

$f_1 = f(2 + 1/6) = 5.0764$

$f_2 = f(2 + 2 \times 1/6) = \left(\frac{7}{3}\right)^2 + \frac{\sin(7/3)}{7/3} = 5.754$

$f_3 = f(2 + 3 \times 1/6) = 6.4893$

$f_4 = f(2 + 4 \times 1/6) = 7.2825$

$f_5 = f(2 + 5 \times 1/6) = 8.1348$

$f_6 = f(2 + 6 \times 1/6) = 9.047$

Now,

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$$I_{CS_2} = \frac{3 \times 1/6}{8}\left[4.4546 + 2(6.4893) + 3(5.0764 + 5.754 + 7.2925 + 8.1348) + 9.047\right]$$

$$= 6.57645$$

* Romberg Integration

Use Romberg integration to evaluate integral $\int_1^2 \frac{dx}{x}$ correct to 3 decimal places.

⇒ Solution

Here,

$$h = \frac{b-a}{2} = \frac{2-1}{2} = 0.5$$

Therefore, taking $h = 0.5$, $\frac{0.5}{2} = 0.25$ and $\frac{0.25}{2} = 0.125$ respectively.

$$f(x) = 1/x = y_i$$

(i) when $h = 0.5$

x_i	1	1.5	2
y	1	0.6667	0.5

$$\therefore I_1 = \frac{h}{2}\left[y_0 + 2y_1 + y_2\right]$$

$$= 0.70835$$

(ii) when $h = 0.25$

x_i	1	1.25	1.5	1.75	1
y_i	1	0.8	0.867	0.57	0.5

$$I_2 = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$= 0.89675$

(iii) when $h = 0.125$

x_i	1	1.125	1.25	1.375	1.5	1.625	1.75	1.87	2
y_i	1	0.89	0.8	0.72	0.67	0.62	0.57	0.53	0.5

$$I_3 = h/2\left[y_0 + 2(\ldots) + y_8\right]$$

$= 0.69408$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.6931$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.69319$$

Finally, $I = I_5 + \dfrac{I_5 - I_4}{3} = 0.69318$

* Gaussian Integration / Gauss legendre Formula

$$I_g = \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

Parameters for Gaussian integration

numbers of points (n)	w_i	x_i or z_i
$n = 2$	$w_1 = w_2 = 1$	$x_1 = -1/\sqrt{3}$ $x_2 = 1/\sqrt{3}$
$n = 3$	$w_1 = w_3 = 5/9$ $w_2 = 8/9$	$x_1 = -\sqrt{3/5}$ $x_2 = 0$ $x_3 = \sqrt{3/5}$

* Compute $\int_{-1}^{1} e^x dx$ using 1-point Gaussian legendre formula.

⇒ Solution

$n = 2$

$$I = \int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{2} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2)$$

$$= f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$= e^{(-1/\sqrt{2})} + e^{(2/\sqrt{3})}$

$= 2.342$

* Compute integral $I = \int_{-2}^{2} e^{-x/2}\,dx$ using 2-point Gaussian integration.

⇒ Solution

$a = -2,\ b = 2$

We have,

$$\boxed{x = \left(\frac{b-a}{2}\right)z + \frac{b+a}{2}}$$

$$= \left(\frac{2+2}{2}\right)z + \frac{2-2}{2}$$

$x = 2z$ also, $g(z) = e^{-z}$

Now,

$$I = \frac{b-a}{2}\int_{-1}^{1} g(z)\,dz$$

$$= \frac{b-a}{2}\sum_{i=1}^{2} w_i\, g(z_i)$$

$$= 2[g(z_1) + g(z_2)]$$

$$= 2[g(-1/\sqrt{3}) + g(1/\sqrt{3})]$$

$$= 2[e^{-(-1/\sqrt{3})} + e^{-(1/\sqrt{3})}]$$

$= 4.685$

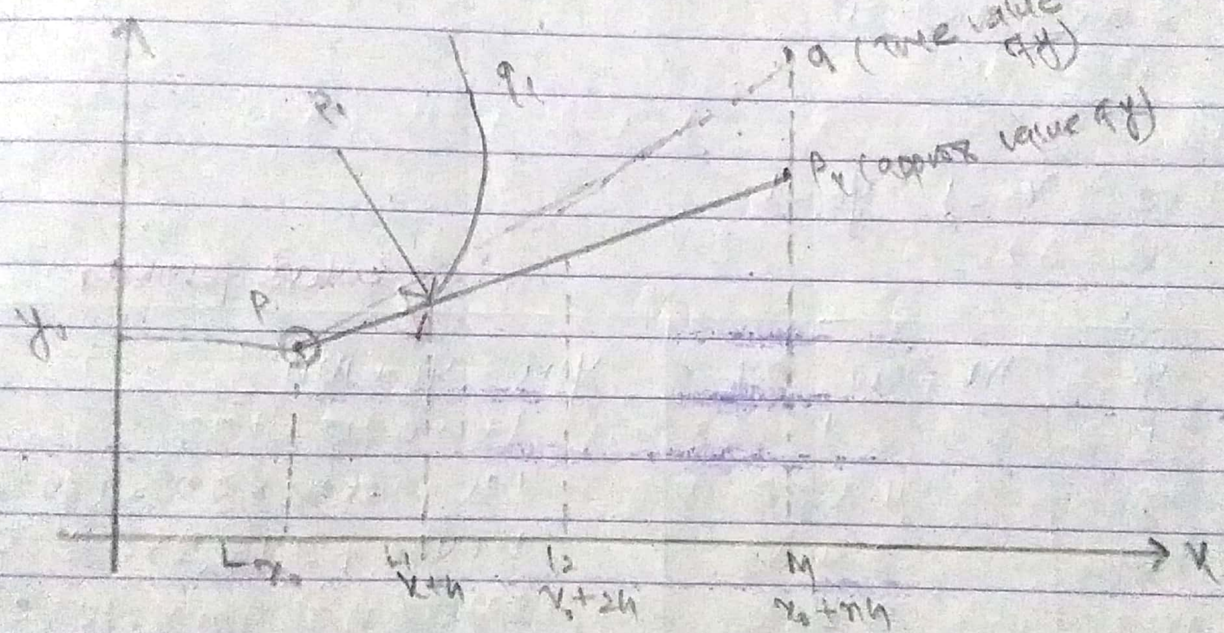
* Chapter 6

Solution of Ordinary Differential Equation

(1) Euler's method

Consider differential equation $dy/dx = f(x,y)$
given $y(x_0) = y_0$ [initial condition]

Its curve of solution through $P(x_0, y_0)$ is shown dotted.



let us divide LM into 'n' sub intervals each of width 'h'.
In interval LL_1, we approximate curve by tangent at P. If the ordinate through L_1 meets to this tangent at $P_1(x_0+h, y_1)$ then

$$y_1 = L_1P_1 = LP + h\tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P$$

$$= y_0 + hf(x_0, y_0)$$

Continuing this way, $y_2 = y_1 + hf(x_0+h, y_1)$
Repeating this process 'n' times,
we reach mP_n

ie $\boxed{y_n = y_{n-1} + hf(x_0 + (n-1)h, y_{n-1})}$

This is Euler's method.

* For $dy/dx = xy$ and given $y(1) = 1$, find $y(2)$ with $h = 0.25$ using Euler's method.

⟹ Solution:

Here $x_0 = 1$, $y_0 = 1$
$h = 0.25$

Euler's formula

x_i	y_i	$m = \frac{dy}{dx} = xy$	$y_{i+1} = y_i + mh$
1	1	1	$1 + 1 \times 0.25 = 1.25$
1.25	1.25	1.5625	$1.25 + 1.5625 \times 0.25 = 1.6406$
1.5	1.6406	2.4609	$1.6406 + 2.4609 \times 0.25 = 2.2558$
1.75	2.2558	3.9476	$2.2558 + 3.9476 \times 0.25 = 3.2427$

$\therefore y(2) = 3.2427$

~~Heun's~~

* Heun's method (modified Euler's method)

$y_{i+1} = y_i + mh$; $m = \frac{m_1 + m_2}{2}$

Where,

$m_1 = f(x_i, y_i)$

$M_2 = f(x_{i+1}, y_{i+1}) = f(x_{i+1}, y_i + m_1 h)$

* Solve $y' = 2y/x$ with $y(1) = 2$ to calculate $y(1.5)$ given 'h' = 0.25 using modified Euler's method.

⇒ Solution

Given, $\frac{dy}{dx} = \frac{2y}{x} = f(x, y)$

$x_0 = 1$, $y_0 = 2$

$h = 0.25$

$y(1.5) = ?$

1st iteration:

$m_1 = f(x_0, y_0) = f(1, 2) = 2 \times \frac{2}{1} = 4$

$M_2 = f(x, y)$

$= f(x_0 + h, y_0 + M_1 h) = f(1.25, 3) = 4.8$

$M = \frac{M_1 + M_2}{2} = 4.4$

$\therefore y_1 = y(1.25) = y_0 + mh = 2 + 4.4 \times 0.25 = 3.1$

2nd iteration $x_1 = 1.25$, $y_1 = 3.1$

$M_1 = f(x_1, y_1) = f(1.25, 3.1) = 4.96$

$M_2 = f(x_1 + h, y_1 + h) = f(1.5, 4.34) = 5.7867$

$M = \frac{M_1 + M_2}{2} = 5.37$

$y_2 = y(1.5) = y_1 + mh = 3.1 + 5.37 \times 0.25 = 4.44$

(2)

Runge - Kutta Method (RK Method)

(1) 1st order RK

$y_{i+1} = y_0 + mh$ (same as Eulers)

(2) 2nd order RK

$y_{i+1} = y_i + mh =$ (same as Heuns)

$m = \frac{m_1 + m_2}{2}$

VVI

(3) 4th order RK method

VVIMP $y_{i+1} = y_i + mh$, $m = \frac{(m_1 + 2m_2 + 2m_3 + m_4)}{6}$

where,

$m_1 = f(x_i, y_i)$

$m_2 = f(x_i + h/2, y_i + m_1 h/2)$

$m_3 = f(x_i + h/2, y_i + m_2 h/2)$

$m_4 = f(x_i + h, y_i + m_3 h)$

* Solve $y' = \frac{y^2 - x^2}{y^2 + x^2}$ using RK-4 method given $y(0) = 1$ to find $y(0.4)$ taken $h = 0.1$

⇒ Solution

Here, $y' = \frac{y^2 - x^2}{y^2 + x^2} = f(x, y)$

$h = 0.2$

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1st iteration

$x_0 = 0$, $y_0 = 1$

$m_1 = f(x_0, y_0) = 1$

$m_2 = f(x_0 + h/2, y_0 + m_1 h/2) = f(0.1, 1.1) = 0.9836$

$m_3 = f(0.1, 1.0983) = 0.9835$

$m_4 = f(0.2, 1.1967) = f(x_0 + h, y_0 + m_3 h) = 0.9456$

$m = 0.9744$

$y_1 = y(0.2) = y_0 + mh$

$= 1 + 0.9799 \times 0.1$

$= 1.19598$

2nd iteration

$x_1 = 0.2$, $y_1 = 1.1959$

$m_1 = f(x_1, y_1) = 0.9455$

$m_2 = f(0.3, 1.2904) = 0.8974$

$m_3 = f(0.3, 1.2856) = 0.8967$

$m_4 = f(0.4, 1.3752) = 0.8439$

$$m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6} = 0.8962$$

$y_2 = y(0.4) = y_1 + mh$

$= 1.1959 + 0.8962 \times 0.2$

$= 1.3751$

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V.V.imp

\# Solve the equation $\frac{d^2y}{dx^2} + \frac{2dy}{dx} - 3y = 6x$ for $y(0.2)$ and $y'(0.2)$, given $y(0) = 0$, $y'(0) = 1$ taking $h = 0.2$. using RK-2 method.

$\Rightarrow$ Solution

Sometimes given in the question $\rightarrow$ let $\frac{dy}{dx} = z = f(x,y,z)$

Then,

from given, $\frac{d^2y}{dx^2} = \frac{dz}{dx} = 6x - 2z + 3y = \phi(x,y,z)$

$x_0 = 0, \quad y_0 = 0, \quad z_0 = 1$

1st iteration

$m_1 = f(x_0, y_0, z_0)$
$= f(0,0,1) = 1$

$l_1 = \phi(x_0, y_0, z_0)$
$= \phi(0,0,1) = -2$

$m_2 = f(x_0 + h, y_0 + m_1 h, z_0 + l_1 h)$

$= f(0.2, 0.2, 0.6) = 0.6$

$l_2 = \phi(x_0 + h, y_0 + m_1 h, z_0 + l_1 h)$
$= \phi(0.2, 0.2, 0.2) = 0.6$

$m = (m_1 + m_2)/2$
$= (1 + 0.6)/2$
$= 0.8$

$l = (l_1 + l_2)/2$
$= (-2 + 0.6)/2$
$= -0.7$

$y_1 = y(0.2) = y_0 + mh = 0 + 0.8 \times 0.2$
$= 0.16$

$z_1 = z(0.2) = y'(0.2)$
$= z_0 + lh$
$= 0.86$

* for $y(0.4)$ and $y'(0.4)$

2nd iteration

$x_1 = 0.2$, $y_1 = 0.16$, $z_1 = 0.86$

$m_1 = f(x_1, y_1, z_1)$ $\qquad$ $l_1 = \phi(x_1, y_1, z_1)$

y_2 $\qquad$ z_2

* Boundary value problem
Shooting Method

Use shooting method to solve the equation $\frac{d^2y}{dx^2} = 6x$
using (RK-2) method given
$y(1) = 2$ and $y(2) = 9$, take $h = 0.5$

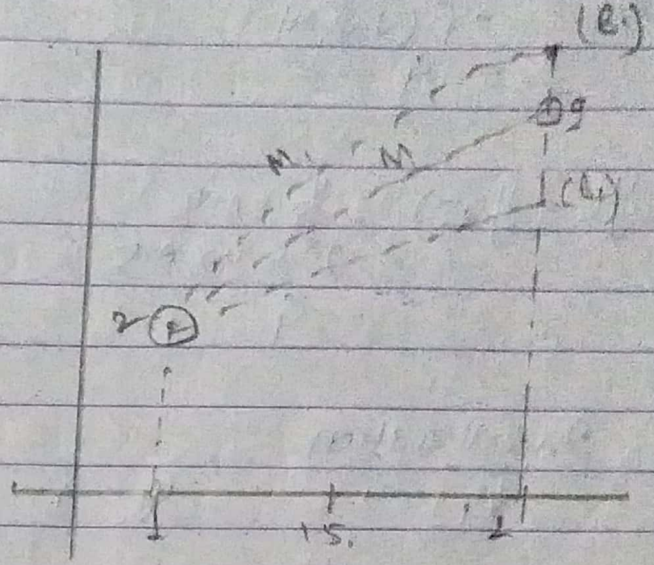
$\Rightarrow$ Solution

let $\frac{dy}{dx} = z = f(x, y, z)$

Then,

$\frac{dz}{dy} = 6x = \phi(x, y, z)$

$x_0 = 1$, $y_0 = 2$, $z_0 = 2\ (M_1)$ suppose



1st iteration

$m = f(x_0, y_0, z_0)$
$= f(1,2,2) = 2$

$y_1 = y(1.5) = y_0 + mh$
$= 3$

$l = \phi(x_0, y_0, z_0) = \phi(1,2,2)$
$= 6$

$z_1 = z(1.5) = z_0 + lh = 5$

2nd iteration

$x_1 = 1.5$, $y_1 = 3$, $z_1 = 5$

$m = f(1.5, 3, 5)$
$= 5$

$l = \phi(1.5, 3, 5)$
$= 9$

$y_2 = y(2) = y_1 + mh$
$= 5.5\ (B_1)$

Again,

$x_0 = 1$, $y_0 = 2$, $z_0 = 4\ (M_2)$. suppose

1st iteration .

$\Rightarrow$ $M = f(x_0, y_0, z_0)$ $\qquad$ $l = \phi(x_0, y_0, z_0)$

$= f(1, 2, 4)$ $\qquad$ $= \phi(1, 2, 4)$

$= 4$ $\qquad$ $= 6$

$y_1 = y(1.5) = y_0 + mh$ $\qquad$ $z = z(1.5) = z_0 + lh$

$= 2 + 4 \times 0.5$ $\qquad$ $= 4 + 6 \times 0.5$

$= 4$ $\qquad$ $= 7$

<u>2nd iteration</u>

$x_1 = 1.5$ $\qquad$ $y_1 = 4$ $\qquad$ $z_1 = 7$

$M = f(x, y, z)$ $\qquad$ $l = \phi(x_1, y_1, z)$

$= f(1.5, 4, 7)$ $\qquad$ $= \phi(1.5, 4, 7)$

$= 7$ $\qquad$ $= 9$

$y_1 = y(2) = y_1 + mh$

$= 4 + 7 \times 0.5$

$= 7.5 \ (B_2)$

Now, we have

$$\boxed{M - m_1 = \frac{m_2 - m_1}{B_2 - B_1}(B - B_1)}$$

or, $M - 2 = \dfrac{4 - 2}{75 - 55}(9 - 5.5)$

$\therefore M = 5.5$

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$x_0 = 1$, $y_0 = 2$, $z_0 = 5.5$

1st Iteration

$m = f(x_0, y_0, z_0)$
$= f(1, 2, 5.5)$
$= 5.5$

$l = \phi(x_0, y_0, z_0)$
$= \phi(1, 2, 5.5)$
$= 6$

$y_1 = y(1.5) = y_0 + mh$
$= 2 + 5.5 \times 0.5$
$= 4.75$

$z = z(1.5) = z_0 + lh$
$= 5.5 + 6 \times 0.5$
$= 8.5$

2nd Iteration

$x_1 = 1.5$, $y_1 = 4.75$, $z_1 = 8.5$

$m_1 = f(1.5, 4.75, 8.5)$
$= 8.5$

$l_1 = f(1.5, 4.75, 8.5)$
$= 8.5$

$y_1 = y(1.5) = y_1 + mh$
$= 4.75 + 8.5 \times 0.5$
$= 9$

* Finite Difference Method

We have

$$\boxed{y_i' = \frac{y_{i+1} - y_{i-1}}{2h}}$$

And

$$\boxed{y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}}$$

* Solve $d^2y/dx^2 = e^{x^2}$ with $y(0) = 0$ and $y(1) = 0$, using finite Difference method taking $h = 0.25$.

⇒ Solution

Here,

$y_0 = y(0) = 0$

$y_1 = y(0.25) = ?$

$y_2 = y(0.5) = ?$

$y_3 = y(0.75) = ?$

$y_4 = y(1) = 0$

We have,

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = e^{x_i^2}$$

ie $y_{i+1} - 2y_i + y_{i-1} = e^{x_i^2} \times (0.25)^2$

putting $i = 1$, $x_1 = 0.25$

$y_2 - 2y_1 + y_0 = e^{(0.25)^2} \times (0.25)^2$

$\Rightarrow y_2 - 2y_1 + y_0 = 0.0665$

ie $y_2 - 2y_1 = 0.0665$ ... (1)

put $i = 2$, $x_2 = 0.5$

$$\frac{y_1 - 2y_2 + y_{2-1}}{(0.25)^2} = e^{(0.5)^2}$$

or, $y_3 - 2y_2 + \cancel{y_0} = 1.58 \quad \dots (2)$

Put $i = 3$, $x_3 = 0.75$

$$y_4 - 2y_3 + y_{3-1} = e^{(0.75)^2} \times (0.75)^2$$

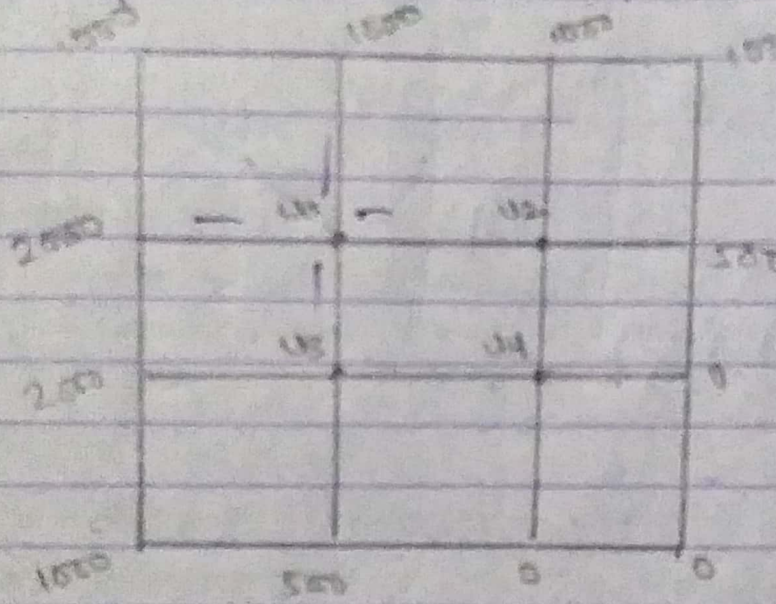
or, $y_4 - y_3 = 1.98 \quad \dots (3)$

Solving eq (1), (2) & (3)

Chapter - 7

Solution of Partial Differential Equations

* Given the values of $u(x,y)$ at the ~~begin~~ boundary of square in fig, evaluate function $u(x,y)$ satisfying laplace equation $(\nabla^2 u = 0)$ at pivotal points.



$\Rightarrow$ Using standard 5-point formula,

$$u_1 = \frac{1}{4}\left[2000 + 1000 + u_2 + u_3\right]$$

$$u_2 = \frac{1}{4}\left[u_1 + 1000 + 500 + u_4\right]$$

$$u_3 = \frac{1}{4}\left[2000 + u_1 + u_4 + 500\right]$$

$$u_4 = \frac{1}{4}\left[u_3 + u_2 + 0 + 0\right]$$

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assuming all the initial values zero, we get

$u_1 = 750$

$u_2 = 375$

$u_3 = 625$

$u_4 = 0$

Now, using gauss jacobi iteration,

Iteration	u_1	u_2	u_3	u_4
1.	750	375	625	0
2.	1000	562.5	812.5	250
3.	1093.75	687.5	937.5	343.75
4.	1156.25	734.37 ~~621.87~~	984.375 ~~1000~~	406.25 ~~320.312~~
5.	1179.68	765.625	1015.62	429.68
6.	1195.31	777.34	1027.34	445.31
7.	1201.17	785.15	1035.15	451.17
8.	1205.5	788.08	1038.08	455.07
9.	1206.543	790.839	1040.039	456.543
10.	1207.519	790.7715	1040.771	457.519
11.				

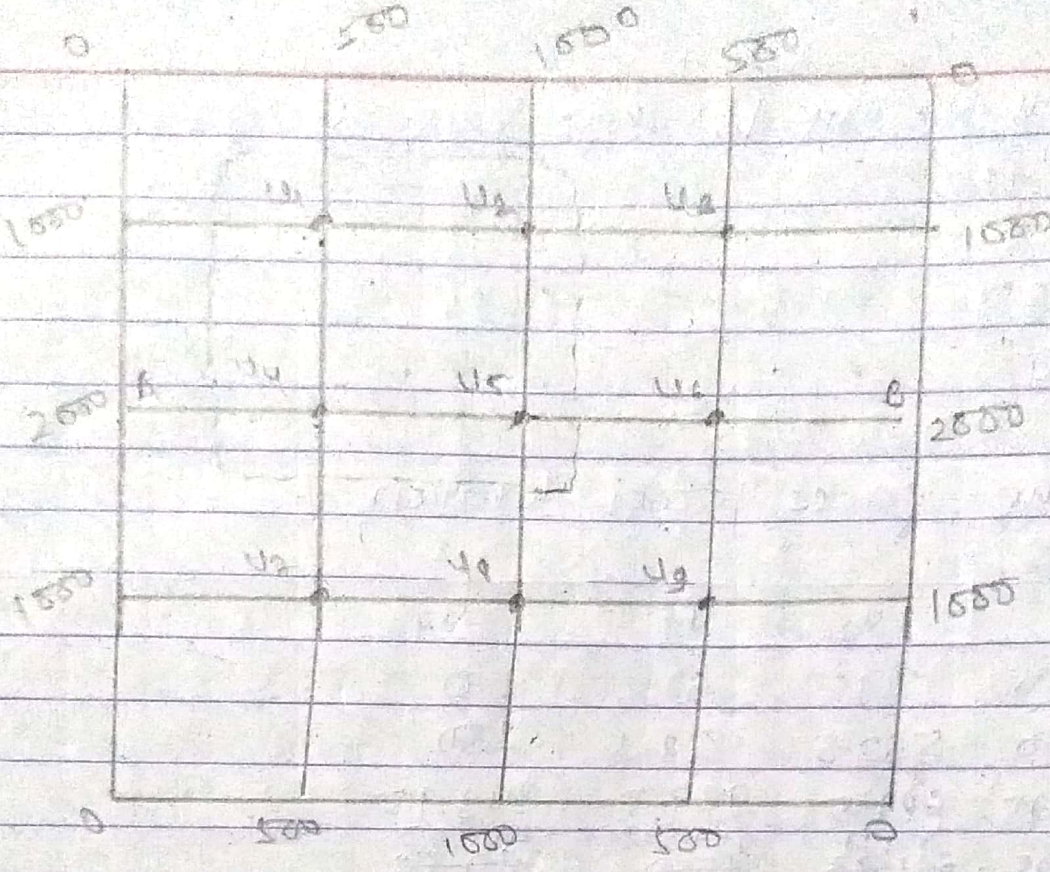
$u_1 = \ldots \quad u_2 = \quad u_3 = \quad u_4 =$

* Given the values of $u(x,y)$ at the boundary of square in figure, evaluate function $u(x,y)$ satisfying laplace equation ($\nabla^2 u = 0$) at pivotal points.

$$\downarrow$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

$$u_{xx} + u_{yy} = 0$$



Since, boundary values are symmetrical across AB and CD, we know $u_1 = u_3 = u_7 = u_9$, $u_4 = u_6$ and $u_2 = u_8$

∴ finding values of u_1, u_2, u_4 and u_5 is sufficient to deduce all nodal values.

To find the initial values,

$$u_5 = \frac{1}{4}[2000 + 1000 + 2000 + 1000] = 1500 \text{ [Standard]}$$

$$u_1 = \frac{1}{4}[2000 + 0 + 1000 + \overset{1500}{\cancel{u_5}}] = 1125 \text{ (diagonal)}$$

$$u_2 = \frac{1}{4}[\overset{1125}{\cancel{1000}\,u_1} + 1000 + \overset{1125}{\cancel{1000}\,u_3} + \overset{1500}{\cancel{u_5}}] \simeq 1188 \text{ [Standard]}$$

$$u_4 = \frac{1}{4}\left[2000 + \overset{1125}{\cancel{u_1}} + \overset{1500}{\cancel{u_5}} + \overset{1125}{\cancel{u_7}}\right] = 1438 \text{ [standard]}$$

Now, using standard formula for all, ie

$$u_1 = \frac{1}{4}\left[1000 + 500 + u_2 + u_4\right]$$

$$u_2 = \frac{1}{4}\left[u_1 + 1000 + u_3 + u_5\right] = \frac{1}{4}\left[2u_1 + 1000 + u_5\right]$$

$$u_4 = \frac{1}{4}\left[2000 + u_1 + u_5 + u_7\right] = \frac{1}{4}\left[2000 + 2u_1 + u_5\right]$$

$$u_5 = \frac{1}{4}\left[u_4 + u_2 + u_6 + u_8\right] = \frac{1}{4}\left[2u_2 + 2u_4\right]$$

Now,

using gauss siedal method

Iteration	u_1	u_2	u_4	u_5
1	1125	1188	1438	1500
2.	1021.5	1140.75	1390.75	1265.75
3.	1007.87	1070.37	1320.37	1195.37
4.	972.66	1035.15	1285.16	1160.16
5	955.07	1017.58	1267.58	1142.58
6.	946.88	1008.79	1258.79	1133.79

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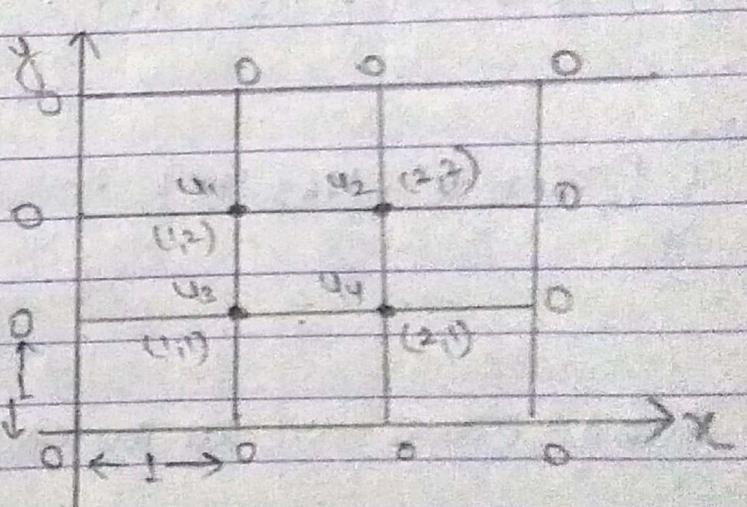
7					
8					
9					
10	938.54	1000.54	1250.54	1125.54	

* Solve the equation $\nabla^2 u = -10(x^2+y^2+10)$ over square Mesh with sides $x=0=y$ and $x=3=y$ with $u=0$ on the boundary and mesh length = 1



Solution

Here,

Mesh-length 'h' = 1

Using standard 5-point formula, for u_1

$u_1 = (i=1, j=2)$

$$\boxed{0+0+u_2+u_3-4u_1 = h^2 f(ih, jh)}$$

or, $0+0+u_2+u_3-4u_1 = -10(1^2+2^2+10)$

$u_2+u_3-4u_1 = -150$

$\therefore u_1 = \frac{1}{4}(u_2+u_3+150)$ ..... (1)

for u_2 $(i=2, j=2)$

$u_1 + 0 + 0 + u_4 - 4u_2 = -10(4+4+10)$

or $u_2 = \frac{1}{4}(u_1 + u_4 + 180)$ ...... (ii)

for u_3 $(i=1, j=1)$

$u_1 + 0 + 0 + u_4 - 4u_3 = -10(1+1+10)$

or $u_3 = \frac{1}{4}(u_1 + u_4 + 120)$ ...... (iii)

for (u_4 $i=2, j=1$)

$u_3 + u_2 + 0 + 0 - 4u_4 = -10(4+1+10)$

or $u_4 = \frac{1}{4}(u_2 + u_3 + 150)$ ...... (iv)

Since, $u_1 = u_4$, The equations reduce to

$u_1 = \frac{1}{4}(u_2 + u_3 + 150)$

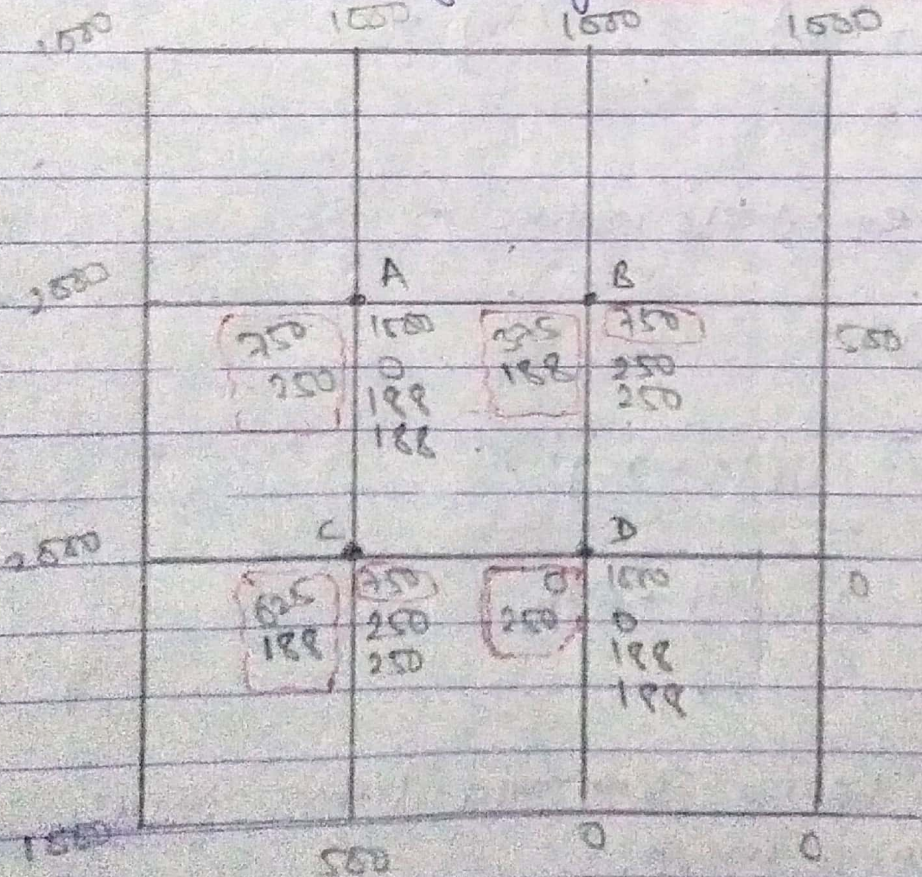
$u_2 = \frac{1}{4}(2u_1 + 180) = \frac{1}{2}(u_1 + 90)$

$u_3 = \frac{1}{4}(2u_1 + 120) = \frac{1}{2}(u_1 + 60)$

Assuming all initial values zero and using gauss siedal iteration method we get,

Iteration	u_1	u_2	u_3	u_4
1.	37.5	63.7	48.7	37.5
2.	65.65	77.181	62.81	65.62
3.	72.65	81.32	66.22	72.65
4.	74.41	82.20	67.20	74.41
5.	74.85	82.42	67.42	74.85
6.	74.96	82.46	67.48	74.96
7.	74.99	82.49	67.49	74.99
8.	74.99	82.49	67.49	74.99

* Solve the following by Relaxation process.



Solution

The initial values of A, B, C and D are found out to be 750, 375, 625, 0 respectively.

Now,

the residues at nodal points are calculated as

$r_A = 2000 + 1000 + 375 + 625 - 4\times750 = 1000$

$r_B = 1000 + 750 + 0 + 500 - 4\times375 = 750$

$r_C = 750 + 2000 + 500 + 0 - 4\times625 = 750$

$r_D = 375 + 625 + 0 + 0 - 4\times0 = 1000$

To liquidate r_A, increase u_A by 250

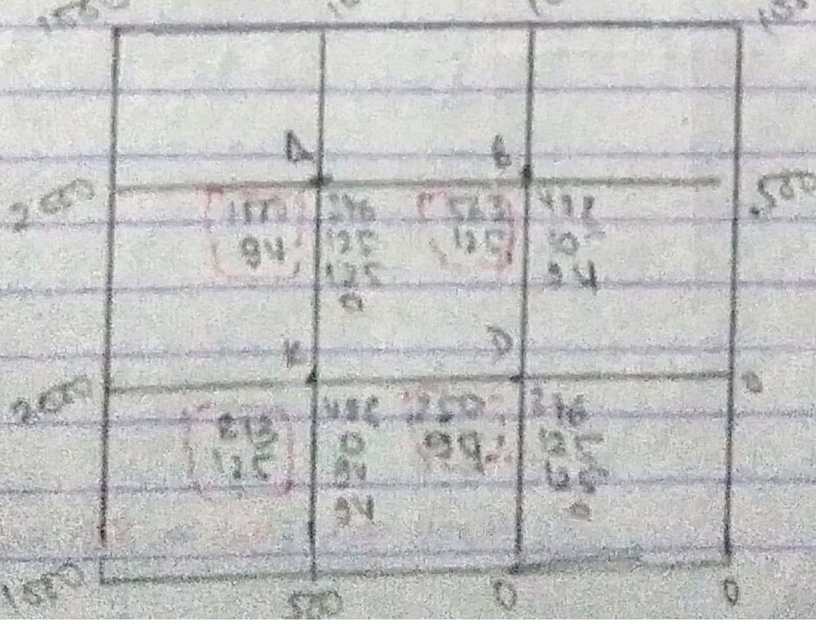
To liquidate r_D, increase u_D by 250

To liquidate r_B, increase u_B by ≈ 188

To liquidate r_C, increase u_C by ≈ 188

Modified values at A, B, C & D are 1000, 563, 813 and 250 respectively.

Corresponding residues are



$r_A = 2080 + 1080 + 563 + 813 - 4 \times 1080 = 376$

$r_B = \quad\quad = 498$

$r_C = \quad\quad = 498$

$r_D = \quad\quad = 376$

To liquidate r_A, increase U_B by $\simeq 125$

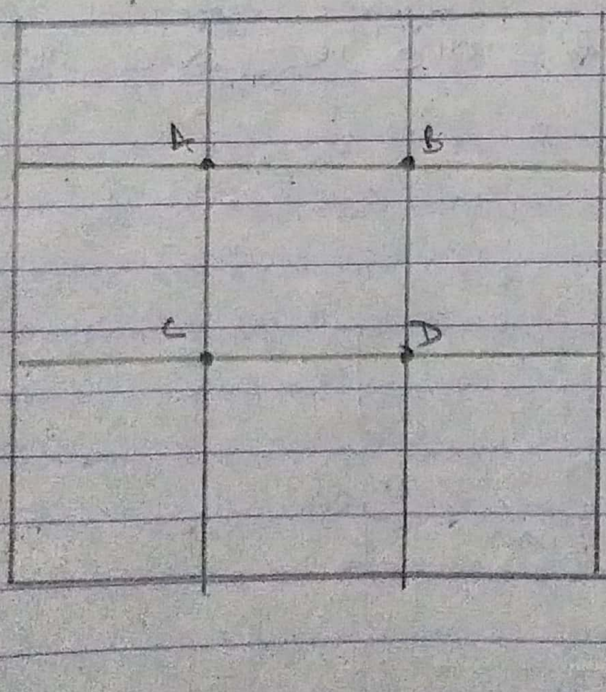
To liquidate r_C, increase U_C by $\simeq 125$

To liquidate r_A, increase U_A by 94

To liquidate r_D, increase U_D by 94

Modified values at A, B, C and D are 1094, 688, 938, 344

Corresponding residues are



...

We'll get value in 8th step ....

* Solution of one dimensional Heat equation by schmidt Method

$$\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2};$$

Here,

$$\frac{\delta u}{\delta t} = \frac{u_{i,j+1} - u_{i,j}}{K} \quad \text{and} \quad \frac{\delta^2 u}{\delta x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

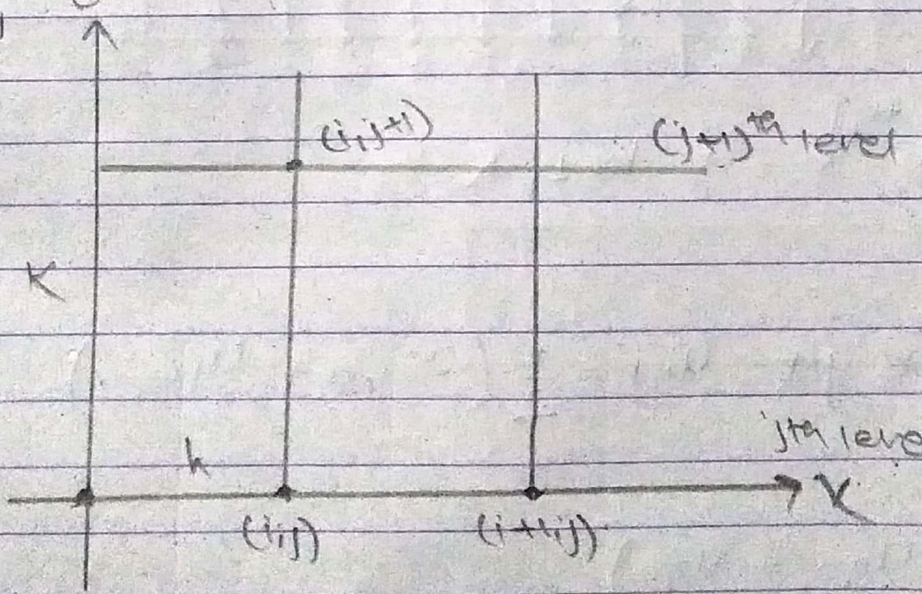
Then,

$$u_{i,j+1} - u_{i,j} = \frac{Kc^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

or,

$$\boxed{u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha) u_{i,j} + \alpha u_{i+1,j}}$$

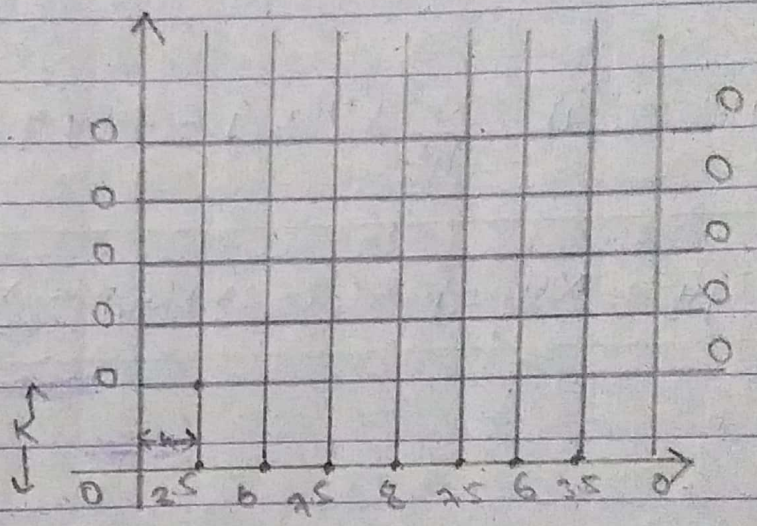
where,



* Find the values of $u(x,t)$ satisfying the parabolic eqn. $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0 = u(8,t)$ and $u(x,0) = 4x - \frac{1}{2}x^2$ at points $x = i,\ i = 0,1,2,\dots 7$

and $t = \frac{1}{8}j,\ j = 0,1,2,\dots 5$.

$\Rightarrow$ Solution



Here,

$c^2 = 4,\ h = 1,\ k = 1/8$

Then,

$\alpha = \frac{c^2 k}{h^2} = 1/2$

for $\alpha = 1/2$

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}$$

$h = 1,\ k = 1/8$

Now,

put $j = 0$, we get, $u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$

we get,

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0})$$

$$= \frac{1}{2}(0 + 6)$$

$$= 3$$

$u_{1,2} = \frac{1}{2}($

Continuing this way for every level upto 5
we can tabulate the nodal values as below:-

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5						0
2	0								0
3	0								0
4	0								0
5	0								0

* - * - * - * - * - *

THE END