Fluid Mechanics

B.E. (Civil Engineering)

Lecture notes

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By Dr. K. N. Dulal

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CHAPTER 1. FLUID AND ITS PROPERTIES

1.1 General introduction

1.1.1 Basics

States of matter: Solid, Liquid, Gas

Fluid

The basic definition of fluid is that it is a substance which is capable of flowing. Liquids and gases come under the category of fluid.

Mechanics
Mechanics is the study of force and motion.

Fluid Mechanics
Fluid Mechanics is the science which deals with the behaviour of fluids at rest and in motion.

Hydraulics
Hydraulics is the science which deals with the behaviour of water at rest and in motion.

Branches of Fluid Mechanics

I. Fluid Statics: Fluid statics is the study of fluids at rest.
II. Fluid Dynamics: Fluid statistics is the study of fluids in motion. It is classified into two branches.

a) Fluid Kinematics: Fluid Kinematics is the study of fluid motion without considering the causes of motion (forces).

b) Fluid Kinetics: Fluid Kinetics is the study of fluid motion by considering the causes of motion (forces).

Application of Fluid Mechanics

- Water distribution and sanitation
- Dams
- Irrigation
- Pumps and Turbines
- Water retaining structures
- Flood flow analysis

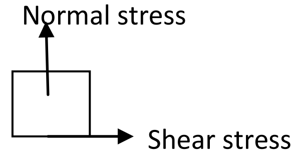
- Flow of air in and around buildings
- Bridge piers in rivers
- Ground-water flow

<u>Stress</u>

A stress is a force per unit area over which it acts. Stresses have both magnitude and direction, and the direction is relative to the surface on which the stress acts. There are two types of stresses:

I. Normal stress: The stress which acts perpendicular to the surface is normal stress.

II. Tangential stress: The stress which acts along the surface is tangential stress. Shear stress is tangential stress.



<u>Strain</u>

Strain is the measurement of deformation. In case of fluid, the deformation caused by shear stress is measured in terms of angle, which is known as shear strain.

<u>Formal definition of fluid</u>

A Fluid is a substance which deforms continuously or flows under the application of shearing forces, however small they may be.

For a fluid at rest, there are no shearing forces acting on it, and any force must be acting perpendicular to the fluid.

<u>Difference between fluids and solids regarding stress</u>

- Fluids lack the ability of solids to resist deformation.
- For a solid, strain is a function of applied stress, provided the elastic limit is not exceeded. For a fluid, the rate of strain is proportional to applied stress.
- In a fluid shear strain increases for as long as shear stress is applied. This means the fluid flows as long as the forces acts and will not recover its original position when the force is removed. In a solid shear strain is constant for a fixed shear stress, and if the elastic limit is not exceeded, the deformation disappears when the force is removed.

<u>Shear stress in a moving fluid</u>

For a fluid at rest, there is no shear stress. There is also no shear stress if the velocity of fluid is same at each point as the fluid particles are at rest relative to each other. When one layer of the fluid moves relative to an adjacent layer with different velocities, transfer of molecular momentum sets up shear stress which resists the relative motion. The measure of the motion of one layer relative to an adjacent layer is velocity gradient, du/dy. According to Newton's law of viscosity, shear stress varies linearly with the velocity gradient.

Forces

Point force: Single concentrated force

Line force: Force which acts along a line

Body force: Body force is the force which acts throughout the volume of body. e.g. gravity (weight), magnetic force, centrifugal force

Surface force: Surface force is the force which acts on the surface of the body. e.g. Pressure, shear force

Liquid and Gas

Liquid: incompressible, fixed volume, forms free surface

Gas: compressible, no fixed volume, expand continuously until restrained by a containing vessel, no free surface, fill the vessel in which it is placed.

1.1.2 System and Control volume

System: A system is a fixed identifiable quantity of matter. The system boundary separates the boundary from its surroundings.

Control volume: A control volume is a fixed region in space through which fluid flows. The region is usually at a fixed location and fixed size. The boundary of the system is its control surface and its shape does not change with time. The element within the control volume obeys the physical laws. This approach makes mathematical analysis simpler.

As the fluid flows continuously, only a part of it is considered for analysis. The control volume is chosen arbitrarily for reasons of convenience of analysis. Control surface follows solid boundaries if present.

Differential and integral approach

- Differential approach: If the control volume is of infinitesimal size, differential equations are used. This approach gives value of variable at a point.
- Integral approach: If the control volume is of finite size, integral equations are used. This approach gives global or overall values.

1.1.3 Continuum concept in Fluid Mechanics

In Fluid Mechanics, a fluid is considered as a continuous substance. This concept is called continuum concept. In this concept, molecular structure of the fluid is not considered and the separation between molecules is neglected. The fluid properties such as velocity and pressure are a continuous function of space and time. The fluid properties can be considered to be constant at any point in space, which is average of large number of molecules surrounding that point within a characteristic distance. Using continuum concept, the mathematical equations relating the physical laws can be derived easily as we don't need to consider the motion of individual molecule. This concept is not valid if the mean free path of molecules is greater than the characteristic dimension of fluid considered for analysis. The ratio of mean free path length to the characteristic length is known as Knudesn number (Kn). The continuum hypothesis is valid for Kn<0.01.

1.1.4 Velocity profile

No slip condition: The velocity of fluid particle immediately in contact with the boundary is same as that of the boundary. This is called no-slip condition.

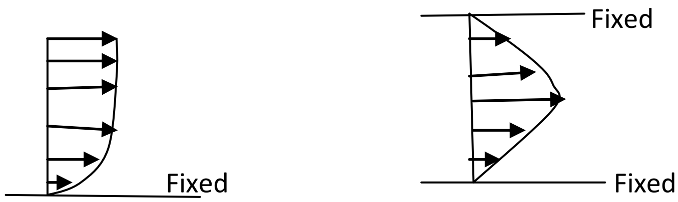


Diagram for velocity profile

Velocity of the fluid at the fixed boundary is zero and increases away from the boundary until it reaches a maximum value.

1.1.5 Basic laws used in Fluid Mechanics

Newton's laws of motion
I. A body will remain at rest or in a state of uniform motion in a straight line unless acted upon by an external force.
II. The rate of change of momentum of a body is proportional to the force applied and takes place in the direction of action of that force. (Force = mass x acceleration)
III. Action and reaction are equal and opposite.
Conservation of mass: Mass remains constant.
Conservation of momentum (Newton's second law of motion): Force = rate of change of momentum or F=ma
Conservation of energy: Energy remains constant.

1.2 Fluid properties

1.2.1 Density

The density of a fluid is defined as its mass per unit volume.

$$\rho = \frac{m}{V}$$

Where ρ = density, m = mass, V = Volume
Unit: kg/m^3
Dimension: ML^{-3}

ρ decreases with increase of temperature and increases with increase of pressure. As the temperature increases, molecular activity increases and spacing between molecules increases, thus increasing volume and reducing density. If pressure is increased, large number of molecules can be forced into a given volume, thus reducing volume and increasing density.

1.2.2 Specific weight

The specific weight of a fluid is defined as its weight per unit volume.

$$\gamma = \frac{W}{V} = \frac{mg}{V} = \rho g$$

Where γ = specific weight, W = weight, V = Volume, m = mass, ρ = density and g= acceleration due to gravity
γ varies from point to point according to the value of g.
Unit: N/m^3
Dimension: ML^{-2}T^{-2}

1.2.3 Specific gravity (or relative density)

Specific gravity (or relative density) is the ratio of specific weight (or density) of a fluid to that of standard fluid. In case of liquid, the standard fluid is water at 4^{0}C.

$$Sp\ gr = \frac{Density\ \ of\ fluid}{Density\ of\ water}\ or \frac{Sp\ wt\ \ of\ fluid}{Sp\ wt\ of\ water}$$

Unit: As it is ratio, it does not have unit.

1.2.4 Specific volume

The specific volume of a fluid is defined as its volume per unit mass.

$$v_s = \frac{V}{m} = \frac{1}{\rho}$$

Where v_s = specific volume, V= Volume, m = mass and ρ = density

Unit: m^3/kg
Dimension: $M^{-1}L^3$

Variation of temperature and pressure has little effect on density, specific weight and specific volume of liquids as the molecules of liquids are packed together, whereas the impact on these properties in case of gases is significant.

1.2.5 Compressibility and Bulk modulus

Compressibility is the change in the volume of fluid under the action of external force. When temperature changes are involved or velocity of flow is very high, the compressibility of a fluid becomes important. It is expressed by Bulk modulus of elasticity

If pressure increases from P to P+dP, then the volume V of a given mass will be reduced to V-dV.

$$Bulk\ modulus = \frac{Change\ in\ pressure}{Volumetric\ strain}$$

$$K = \frac{dP}{-dV/V} \qquad \text{(a)}$$

Where K= Bulk modulus of elasticity , dv/v = Volumetric strain
-ve sign means decrease in volume with pressure.
Unit of K: N/m^2 (Pa)
Dimension: $ML^{-1}T^{-2}$

Compressibility is the inverse of the Bulk modulus of elasticity.

Considering unit mass of substance, $V = \frac{1}{\rho}$
Differentiating w.r.t. ρ

$$\frac{dV}{d\rho} = \frac{d(1/\rho)}{d\rho}$$

$$\frac{dV}{d\rho} = -\frac{1}{\rho^2}$$

$$dV = -\frac{d\rho}{\rho^2} = -\frac{d\rho}{\rho}.\frac{1}{\rho}$$

$$dV = -\frac{d\rho}{\rho}.V \qquad \text{(b)}$$

From a and b,

$$K = \rho\frac{dP}{d\rho}$$

This shows that the value of K depends on the relationship between the pressure and density. Since density is also affected by temperature, it will depend on how the temperature changes during compression. K increases with increase in pressure and decreases with increase in temperature in case of liquids. This relationship is opposite in case of gases.

For practical purpose, liquid is considered to be incompressible because the change in density of liquid due to change in pressure is not significant.

1.2.6 Surface tension

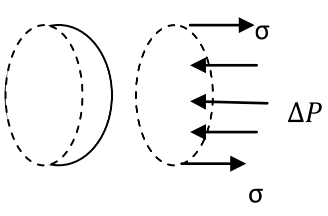
Surface tension is defined as the tensile force per unit length acting on a line lying in the interface of two fluids. The force is normal to imaginary line in the surface, tangent to the free surface and is same at all points. Surface tension is constant at any given temperature for the surface of the separation of two particular substances but it decreases with increase in temperature because intermolecular cohesive force decreases with rise in temperature.

Intermolecular attraction is the cause of surface tension. In other words, it is due to cohesion between liquid particles at the surface. A molecule within the body of a liquid is equally attracted in all directions by the other molecules surrounding it. At the interface between two fluids, the upward and downward attractions are unbalanced, and the surface molecules are pulled inward making the surface like an elastic membrane. The effect of surface tension is to reduce the surface of a free body of a liquid to a minimum (formation of spherical drop).

Example of phenomenon of surface tension: raindrops, rise of sap in tree, capillary rise and capillary siphoning, collection of dust particles on water surface

Symbol: σ
Unit: N/m
Dimension: MT^{-2}

a. Pressure intensity inside a droplet



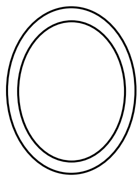
Consider a small spherical droplet of radius r. Let ΔP be pressure inside a droplet in excess of external pressure and σ be the surface tension.

Force due to internal pressure = Force due to surface tension around perimeter

$$\Delta P x \pi r^2 = \sigma x 2\pi r$$

$$\Delta P = \frac{2\sigma}{r}$$

Pressure inside soap bubble
(contribution of inside and outside interface)

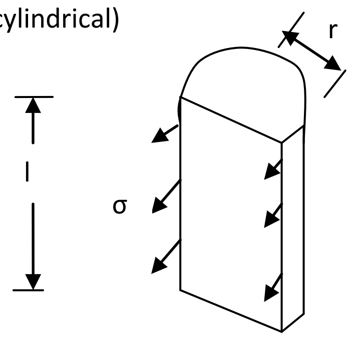


Force due to internal pressure = Force due to surface tension around perimeter

$$\Delta P x \pi r^2 = 2(\sigma x 2\pi r)$$

$$\Delta P = \frac{4\sigma}{r}$$

Liquid jet (cylindrical)



Force due to internal pressure = Force due to surface tension around perimeter

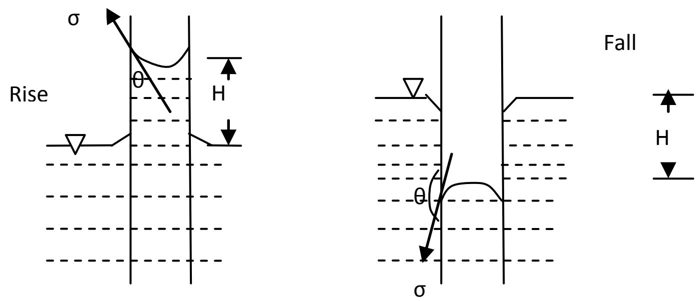
$$\Delta P x l x 2r = \sigma x 2l$$

$$\Delta P = \frac{\sigma}{r}$$

1.2.7 Capillarity

Capillarity is the rise or fall of liquid in a column of very small diameter when the latter is dipped in it. It is caused by surface tension as well as adhesion (attraction between molecules of different substances) and cohesion (attraction between molecules of same liquid).

If adhesion is greater than cohesion, the liquid wets the solid and the liquid will rise. If cohesion is greater than adhesion, the liquid does not wet the solid and the liquid will fall. The contact angle is less than 90^0 for capillary rise and greater than 90^0 for fall.



Let θ is the angle of contact between liquid and solid, d is the diameter of the cylindrical tube, σ is the surface tension and H is capillary rise. As the liquid is at rest, there is no shear stress and therefore no vertical shear forces acting. Weight of the fluid and the vertical component of the surface tension are the only forces acting.

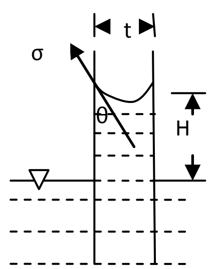
Upward pull due to surface tension force= Weight of column contained in height H

$$\sigma cos\theta x\pi d = \rho g \frac{\pi d^2}{4} H$$

$H = \frac{4\sigma cos\theta}{\rho g d}$ or $H = \frac{4\sigma cos\theta}{\gamma d}$ or $H = \frac{2\sigma cos\theta}{\gamma r}$ where r = radius of tube

For water and glass: $\theta = 0$

Capillary rise of fluid contained between parallel plates at a distance t



H = Capillary rise
t = distance between plates
σ = Surface tension
θ = Angle of contact
L= width of plate

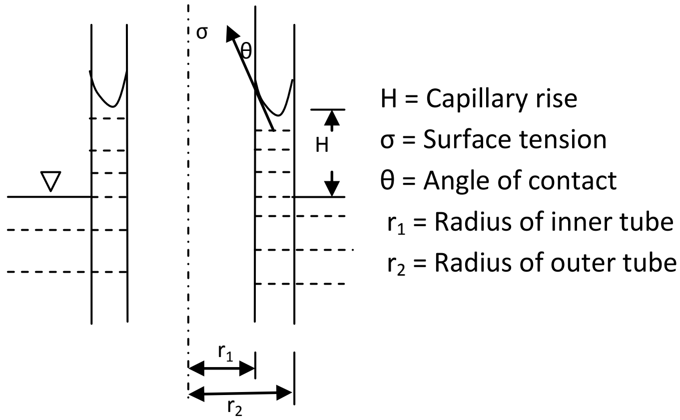
Upward force due to surface tension = weight of fluid

$$(\sigma cos\theta xL)x2 = \rho gHtxL$$

(surface tension force acting on both sides)

$$H = \frac{2\sigma cos\theta}{\rho g t}$$

Capillary rise between two concentric glass tubes



Force due to surface tension = $\sigma cos\theta x 2\pi r_1 + \sigma cos\theta x 2\pi r_2 = 2\pi\sigma cos\theta(r_1 + r_2)$
Weight of fluid contained in height H = $\rho g(Volume\ of\ fluid\ contained\ in\ between\ tubes)$

$$= \rho g(\pi r_2^2 H - \pi r_1^2 H) = \rho g \pi H(r_2^2 - r_1^2)$$

Equating

$$2\pi\sigma cos\theta(r_1 + r_2) = \rho g \pi H(r_2^2 - r_1^2)$$

$$H = \frac{2\sigma cos\theta}{\rho g(r_2 - r_1)}$$

1.2.8 Vapor pressure and cavitation

Liquid evaporate because of molecules with sufficient kinetic energy escaping from the liquid surface. The vapor molecules exert a partial pressure in the space, which is called vapor pressure. Vapor pressure depends on temperature and increases with it. In equilibrium, the number of molecules striking the surface and condensing is equal to the number of escaping molecules. When the pressure above a liquid equals the vapor pressure of the liquid, boiling occurs.

When flow of liquid passes through a region having pressure less than vapor pressure, there will be local boiling and a cloud of vapor bubbles will form. This phenomenon is known as cavitation. The bubbles of low pressure zone move towards the high pressure zone and collapse under that pressure. If this occurs in contact with a solid surface, serious damage can result. Cavitation can affect the performance of hydraulic machinery such as propellers, turbines and pumps and the impact of collapsing bubbles can cause local erosion of metal surface.

1.2.9 Viscosity

Viscosity is the property of a fluid due to which it offers resistance to shear. It is a measure of internal friction which causes resistance to flow.

The molecules of gas are not rigidly constrained and cohesive forces are small. So, the molecular mass interchange (momentum) is the cause of viscosity in a gas. As cohesive forces are significant in a liquid, both mass interchange and cohesion contribute to the viscosity of the liquid.

Viscosity is practically independent upon pressure and depends on temperature only. If the temperature increases, the molecular interchange will increase. Therefore, the viscosity of a gas will increase with increase in temperature. Cohesion is the predominant cause of viscosity in liquid and since cohesion decreases with temperature, the viscosity of a liquid decreases with increase in temperature.

Newton's law of viscosity

Newton's law of viscosity states that the shear stress is proportional to the rate of deformation or velocity gradient.

$$\tau \propto \frac{du}{dy}$$

$$\tau = \mu \frac{du}{dy}$$

Where τ = shear stress, du/dy = velocity gradient and the constant of proportionality (μ) = coefficient of viscosity. The constant is also called dynamic viscosity or absolute viscosity.

Unit of μ: Ns/m^2 or Kg/ms or Pa S (in SI)

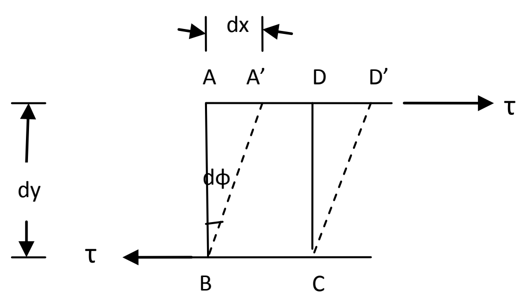
Poise or dyn S/ cm^2 (in CGS)

$1N = 1\ kg\ m/s^2 = 10^5$ Dyn ($1Dyn = 1gm\ cm/s^2$)

$1\ NS/m^2$ = 10 Poise

Dimension: $ML^{-1}T^{-1}$

Derivation



Let us consider a fluid confined between two plates, where the bottom plate is stationary and the upper plate is moving. Let ABCD is the fluid at any time t. Due to the application of shear force τ, the fluid deforms to A'BCD' at time t+dt. Let dy = distance between two layers, AA' = dx and shear strain = dϕ.

For small angle, dx = dϕ. dy

Also dx = du. dt

Equating

$$d\phi . dy = du . dt$$

$$\frac{d\phi}{dt} = \frac{du}{dy} \qquad \text{(a)}$$

Shear stress is proportional to rate of shear strain.

$$\tau \propto \frac{d\phi}{dt} \qquad \text{(b)}$$

From a and b

$$\tau \propto \frac{du}{dy}$$

$$\tau = \mu \frac{du}{dy}$$

Kinematic viscosity is defined as the ratio of dynamic viscosity to density.

$$\upsilon = \frac{\mu}{\rho}$$

Where υ = Kinematic viscosity and ρ = density.
Unit of υ: m^2/s (SI)
Stokes or cm^2/s (CGS)
1 $m^2/s = 10^4$ Stokes
Dimension: L^2T^{-1}
Relationship between viscosity and temperature
Liquids
$\mu = \mu_0 \left(\frac{1}{1+\alpha t+\beta t^2}\right)$ where μ = viscosity of liquid at t^0C in poise, μ_0 = viscosity of liquid at 0^0C in poise
α, β = constants
For gas
$\mu = \mu_0 + \alpha t - \beta t^2$ where μ = viscosity of gas at t^0C in poise, μ_0 = viscosity of gas at 0^0C in poise
α, β = constants

1.3 Classification of fluids

a. Newtonian and non-Newtonian fluids

Fluids which obey Newton's law of viscosity are called Newtonian fluids. E.g. water, light oil, air, milk, glycerin, kerosene, benzene, mercury, ethanol. For Newtonian fluids, viscosity is constant i.e. viscosity depends on temperature only. Fluids which do not obey Newton's law of viscosity are called Non-Newtonian fluids. E.g. paint, sewage sludge, crude oil, mud flow, blood, paste. Viscosity is not constant for Non-Newtonian fluids i.e. viscosity depends on temperature, rate of strain and time.

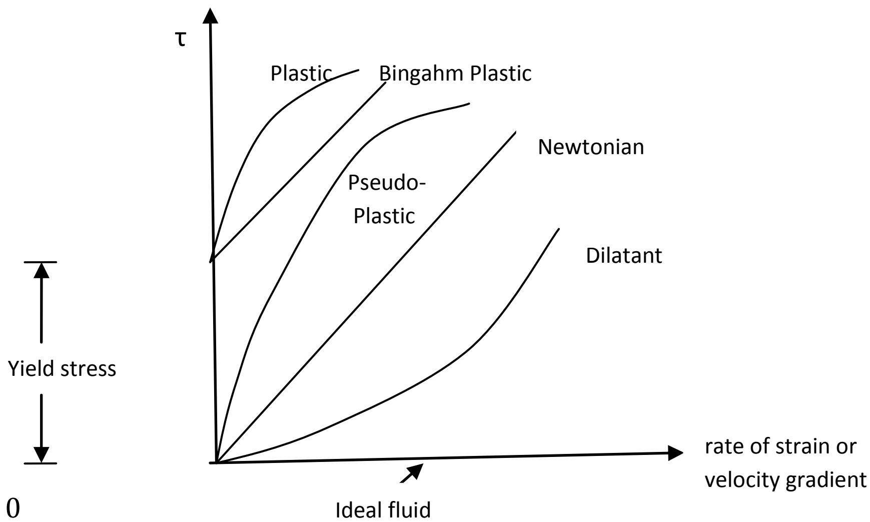
b. Compressible and incompressible fluid

Fluids whose density changes due to change in pressure are called compressible fluids, e.g. air. Fluids whose density remains constant are called incompressible fluids, e.g. water.

b. Ideal and real fluid

The fluid which is incompressible and has no viscosity is called ideal fluid (non-viscous or inviscid). It is an imaginary fluid. The fluid which has viscosity is called real fluid (viscous). All the fluids that exist in nature are real fluids.

1.4 Shear stress-rate of strain (velocity gradient) diagram



Ideal fluid: $\tau = 0$

Newtonian fluid: $\tau = \mu \frac{du}{dy}$

Non-Newtonian fluid: $\tau \neq \mu \frac{du}{dy}$. The relationship is $\tau = \mu \left(\frac{du}{dy}\right)^n$

Classification of Non-Newtonian fluid

- Pseudo-plastic: Viscosity decreases with rate of strain. $\tau = \mu \left(\frac{du}{dy}\right)^n$, $n < 1$,e.g. paint, shampoo, slurries, ketch up, milk, blood
- Dilatant: Viscosity increases with rate of strain. $\tau = \mu \left(\frac{du}{dy}\right)^n$, $n > 1$,e.g. printing ink
- Bingahm plastic: After yield stress is reached, the flow commences and thereafter shear stress is linear with rate of strain. $\tau = \tau_0 + \mu \frac{du}{dy}$,e.g. sewage sludge, drilling mud
- Plastic fluid: After shear stress reaches a certain minimum value (yield stress), the flow commences and thereafter shear stress is non-linear with rate of strain. $\tau = \tau_0 + \mu \left(\frac{du}{dy}\right)^n$,e.g. tooth paste, hand cream, grease.

Time dependent non-Newtonian fluid: $\tau = \mu \left(\frac{du}{dy}\right)^n + f(t)$

- Thixotropic substances: Viscosity decreases with time, e.g. paints, enamels, yoghurt, crude oil.
- Rheopectic substances: Viscosity increases with time, e.g. gypsum suspension

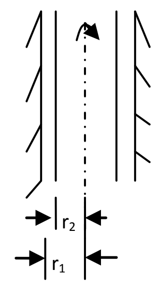
The behavior of Newtonian fluid is studied In Fluid Mechanics. The study of non-Newtonian fluid is called Rheology.

1.5 Determination of viscosity

a. Rotating cylinder viscometer

A cylinder of radius r_2 is rotated coaxially inside a fixed cylinder of radius r_1 both cylinders having a length of l. The annular space between the two cylinders is filled with a liquid of viscosity μ. A torque T is required to maintain an angular velocity ω of the inner cylinder. The torque is transmitted from the inner to the outer cylinder through the liquid which consists of layers.

For small space between cylinders, the velocity gradient may be assumed to be a straight line.



Frictional force (F) = Shear stress(τ) x surface area (A)

$T = F\, r_2 = \mu \frac{du}{dy}\,(2\pi r_2 L) r_2$

Here, Angular velocity (ω) = $\frac{2N\pi}{60}$ and Tangential velocity of inner cylinder (u) = $r_2\omega$ =du

dy = dr = r_1-r_2

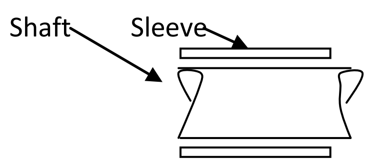
Power = T ω or F u

(Instead of r_2, average radius can also be taken to compute T).

b. Bearing

I. Journal bearing

A shaft of radius r_1 is fixed axially and rotated at N rpm inside a sleeve of radius r_2. The clearance is filled with fluid of viscosity μ and the torque T is measured.



Angular velocity (ω) = $\frac{2N\pi}{60}$, Tangential velocity (u) = $r_1\omega$

du = u-0 (Assuming linear variation of velocity)

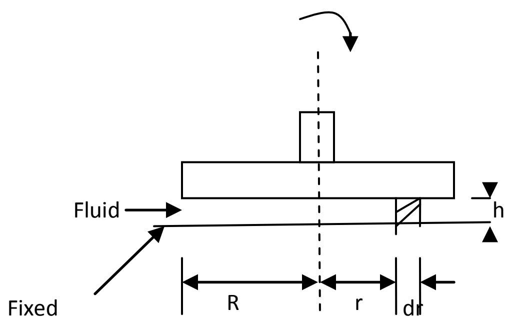
dy = r_2- r_1

Frictional force (F) = Shear stress(τ) at the shaft x surface area of shaft = $\mu \frac{du}{dy}\,(2\pi r_1 L)$

$Torque = Fr_1$

II. Foot step bearing

A circular disc of radius R rotates on a table separated by fluid of thickness t. By measuring the torque required to rotate the disc, the viscosity of the fluid is found out.



Angular velocity (ω) = $\frac{2N\pi}{60}$

Consider an elementary ring or disc at radius r and having a width dr.

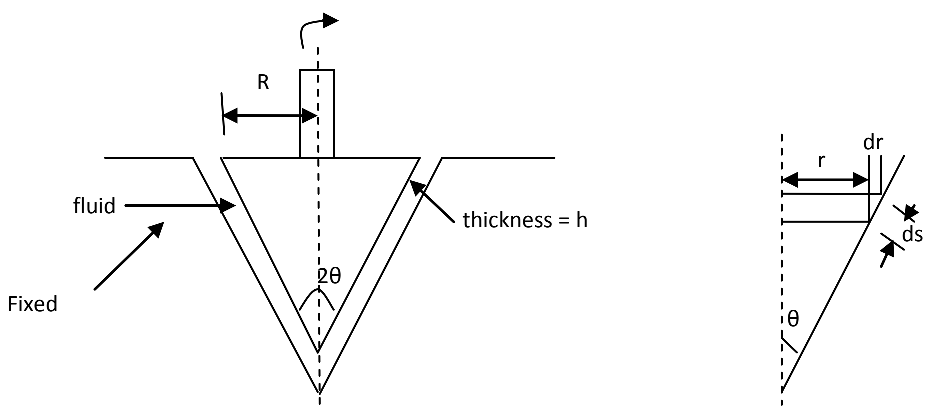
Torque on the element (dT) = Shear force x r

$= \tau dAr = \mu \frac{du}{dy}(2\pi r dr)r = \mu \frac{u}{h}(2\pi r dr)r = \mu \frac{r\omega}{h}(2\pi r dr)r = \frac{2\pi\mu\omega}{h} r^3 dr$

Total torque (T) = $\int_o^R \frac{2\pi\mu\omega}{h} r^3 dr = \frac{\pi\mu\omega R^4}{2h}$

III. Conical thrust bearing

A solid cone of maximum radius R and vertex angle rotates at an angular velocity ω . A fluid of viscosity μ and thickness h fills the gap between the cone and the housing.



Angular velocity = (ω) = $\frac{2N\pi}{60}$

Consider an elementary area dA at radius r of the cone.

Torque on the element (dT) = Shear force x r

$= \tau dAr = \mu \frac{du}{dy}(2\pi r ds)r = \mu \frac{u}{h}(2\pi r ds)r = \mu \frac{r\omega}{h}\left(2\pi r \frac{dr}{sin\theta}\right) r = \frac{2\pi\mu\omega}{hsin\theta} r^3 dr$

Total torque (T) = $\int_o^R \frac{2\pi\mu\omega}{hsin\theta} r^3 dr = \frac{\pi\mu\omega R^4}{2hsin\theta}$

CHAPTER 2. FLUID PRESSURE

2.1 Introduction

Pressure is defined as the force per unit area.

$$Pressure\ (P) = \frac{Force\ (F)}{Area\ over\ which\ the\ force\ is\ applied\ (A)}$$

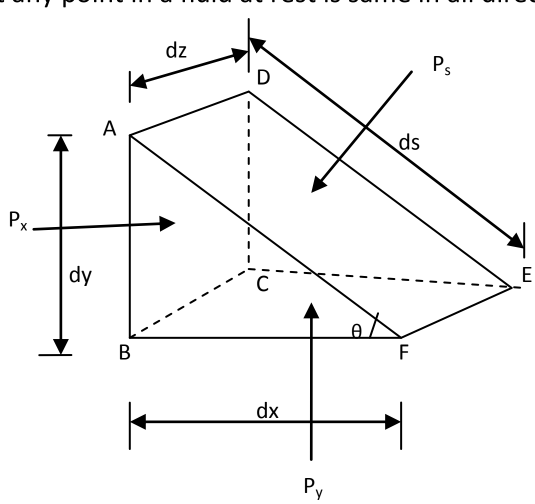
As the fluid is at rest, there are no shear stresses in it. The pressure at a point on a plane surface always acts normal to the surface, and all forces are independent of viscosity. The pressure variation is due only to the weight of the fluid. The fluid and the container in which it contains are at rest. The fluid mass is in absolute or static equilibrium as there is no relative velocity. In this case, the fluid mass is in equilibrium under gravity (body force) and pressure (normal surface force).

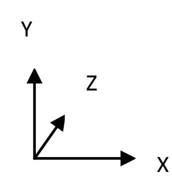
Unit of P: N/m^2 or Pa
Frequently used alternative unit: bar
1bar = 10^5 N/m^2
Standard atmospheric = 101.325 Kpa

Application of pressure concept in civil engineering
Dams
Gates in hydraulic structures
Water tanks

2.2 Pascal's law for pressure at a point

Pressure at any point in a fluid at rest is same in all directions. This is known as Pascal's law.
Proof





Consider a small fluid element at rest in the triangular shape. Let P_x is the pressure in X-direction, P_y is the pressure in Y-direction and P_s is the pressure normal to any plane inclined at an angle θ to the horizontal. All these pressure act at right angles to the plane.

Considering X-direction

$$\sum F_x = 0$$

$$P_x\, dy\, dz - P_s\, ds\, dz\, Sin\theta = 0$$

$$P_x\, dy\, dz - P_s\, ds\, dz\, \frac{dy}{ds} = 0$$

$$P_x = P_s \qquad \text{(a)}$$

Considering Y-direction

$$\sum F_y = 0$$

$$P_y\, dx\, dz - P_s\, ds\, dz\, Cos\theta - Weight\ of\ element = 0$$

$$P_y\, dx\, dz - P_s\, ds\, dz\, \frac{dx}{ds} - \rho g \frac{1}{2} dx\, dy\, dz = 0$$

Since dx, dy and dz are very small, the product dx dy dz is negligible in comparison with other two terms.

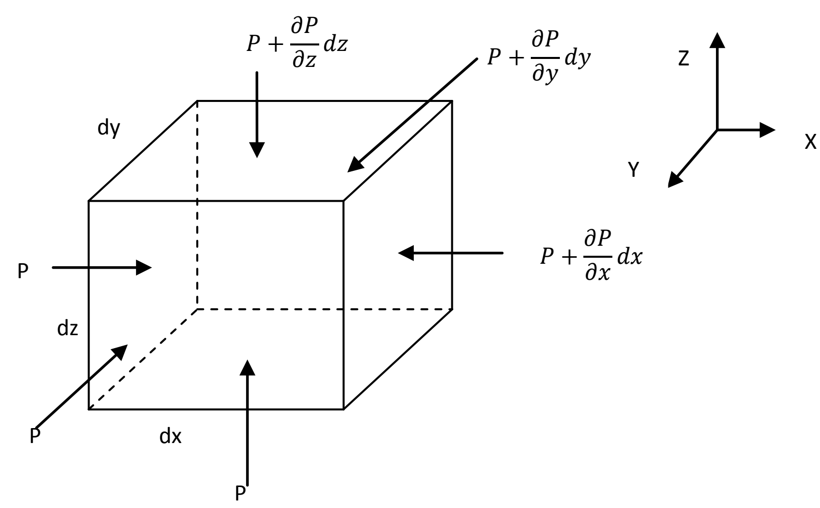
$$P_y = P_s \qquad \text{(b)}$$

From a and b

$$P_x = P_y = P_s$$

The element chosen is so small that it can be considered a point. Hence, pressure at a point is equal in all directions.

2.3 Pressure-depth relationship (General equations for pressure)



Consider an element of fluid at rest. Let P is the pressure at point (x, y, z).

Weight of fluid element = $\gamma\ volume = \gamma\ dx\ dy\ dz$ where γ = specific gravity of fluid.

Considering X-direction

$\sum F_x = 0$

$P_{x1}\, dy\, dz - P_{x2}\, dy\, dz = 0$

$Pdydz - \left(P + \frac{\partial P}{\partial x} dx\right) dydz = 0$

$\frac{\partial P}{\partial x} = 0$ (a)

Considering Y-direction

$\sum F_y = 0$

$P_{y1}\, dx\, dz - P_{y2}\, dx\, dz = 0$

$Pdxdz - \left(P + \frac{\partial P}{\partial y} dy\right) dxdz = 0$

$\frac{\partial P}{\partial y} = 0$ (b)

Considering Z-direction

$\sum F_z = 0$

$P_{z1}\, dx\, dy - P_{z2}\, dx\, dy - Weight\ of\ element = 0$

$Pdxdy - \left(P + \frac{\partial P}{\partial z} dz\right) dxdy - \gamma dxdydz = 0$

$\frac{\partial P}{\partial z} = -\gamma$ (c)

From a, b and c, it is clear that pressure at any point in a static mass of fluid does not vary in x and y directions and it varies in only z direction.

Since P is a function of Z only,

$$dP = -\gamma dz$$

For incompressible fluid, γ = constant. After integration,

$$P = -\gamma z + c$$

z is measured vertically downward from free surface so that z = -h. As surface pressure is atmospheric, c = P_{atm} at z=0. Taking P_{atm} as datum

$P = \gamma h$ or $P = \rho g h$. This is the relationship for hydrostatic pressure law.

Conclusions

Pressure is same at all points on a horizontal plane.

Pressure varies in vertical direction only.

Pressure is independent of the shape of container.

Pressure increases with depth in the fluid.

2.4 Some terminologies and points for pressure computation

Atmospheric Pressure

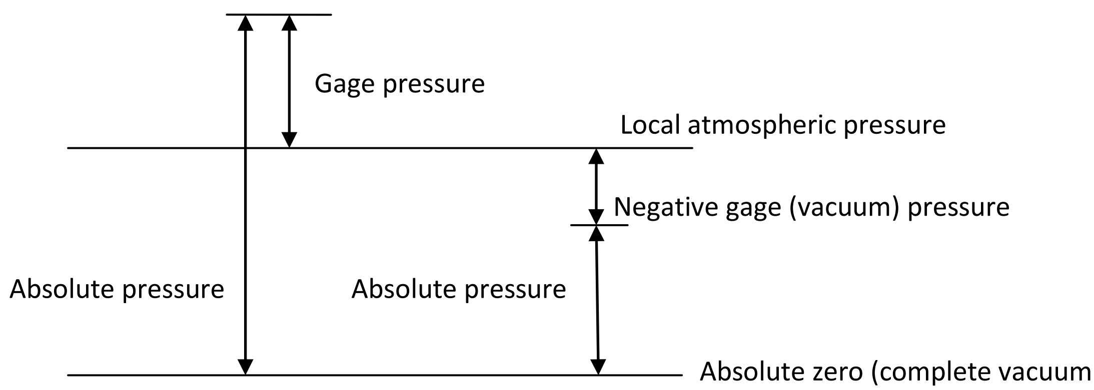
The pressure exerted by the atmospheric air on the surface with which it is in contact is known as atmospheric pressure. The atmospheric pressure varies with the altitude and can be measured by means of Barometer. It is also called barometric pressure.

Atmospheric pressure = 760mm Hg or 10.3 m of water or 101.325 Kpa

Gauge pressure
If the pressure is quoted by taking atmospheric pressure as a datum, it is called gauge pressure. Pressure measuring instrument measures the gauge pressure. In such instrument, atmospheric pressure on the scale is marked zero.

Vacuum Pressure
The pressure below the atmospheric pressure is known as vacuum pressure. It is also called negative gage pressure or suction pressure.

Absolute Pressure
When there is perfect vacuum, the pressure is zero. The pressure measured above a perfect vacuum is called absolute pressure.



Relationships
Absolute Pressure = Atmospheric Pressure+ Gauge pressure
Absolute Pressure = Atmospheric Pressure- Vacuum pressure

Head
The vertical height of a column of given fluid of above any point at rest is called head. The gage pressure can be expressed in head.

$h = \frac{P}{\rho_{fluid} x g} = \frac{P}{\gamma_{fluid}}$

Points to be considered for pressure computation

- Any surface open to the atmosphere has atmospheric pressure. In terms of gage pressure, this is taken as zero.
- The shape of container does not matter in the value of pressure as it is dependent on head and density and not on weight of fluid.
- Pressures at the same depths in a continuous mass of fluid are equal.

- $P_{next\ level} = P_{previous\ level} \pm P_d$
  Or, $P_2 = P_1 + \gamma h$, use + if 2 is below 1 and – if 2 is above 1)
- For different fluids, pressure is continuous at interface.

Effect of capillarity: capillary correction
For capillary rise: true static height = given height of fluid-capillary rise
For capillary fall: true static height = given height of fluid+ capillary fall

2.5 Pressure measurement

2.5.1 Manometer

Manometers are the instruments which use the relationship between pressure and head to measure pressure.

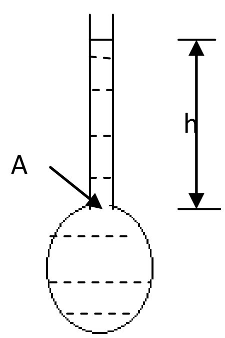
Types

I. Simple manometer

a. Piezometer

Piezometer is the simplest form of manometer. It is an open tube, which is attached to the top of a container with liquid at pressure. The liquid rises to a height depending on the pressure. As the top end is open to the atmosphere, the pressure measured is gage pressure.

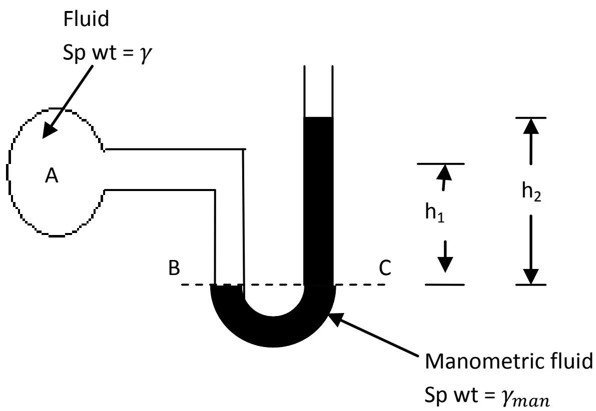
Pressure at A = γh



Problems with the Piezometer
- It can only be used for liquids.
- Pressure must be above atmospheric
- Liquid height must be convenient, i.e. not be too small or too large.

b. U-tube manometer

This device consists of a glass tube bent into the shape of a U, which is filled with manometric fluid (e.g. mercury) and connected to pipe or tank. This manometer can be used to measure the pressure of both liquids and gases. In this type of manometer, the manometric fluid density should be greater than that of the fluid measured and the two fluids should not be able to mix.



Pressure at B = Pressure at C

$$P_A + \gamma h_1 = \gamma_{man} h_2$$
$$P_A = \gamma_{man} h_2 - \gamma h_1$$

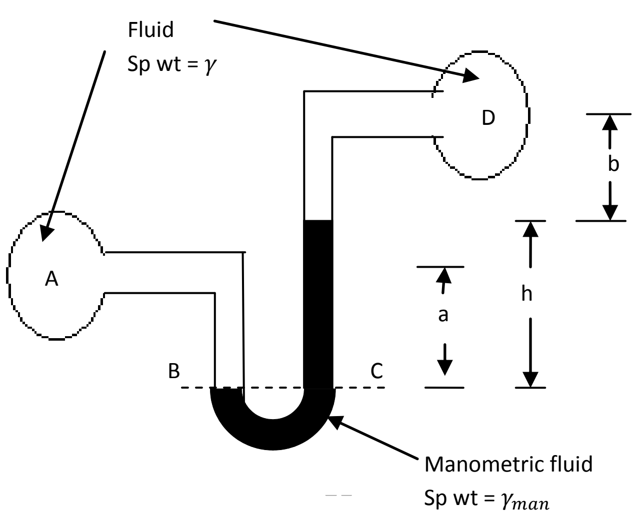
In case of air as the fluid, we can generally neglect γh_1 (very small).

II. Differential U-tube manometer

The U-tube manometer can be connected at both ends to measure pressure difference between these two points.

Pressure at B = Pressure at C

$$P_A + \gamma a = P_D + \gamma b + \gamma_{man} h$$
$$P_A - P_D = \gamma_{man} h + \gamma b - \gamma a$$



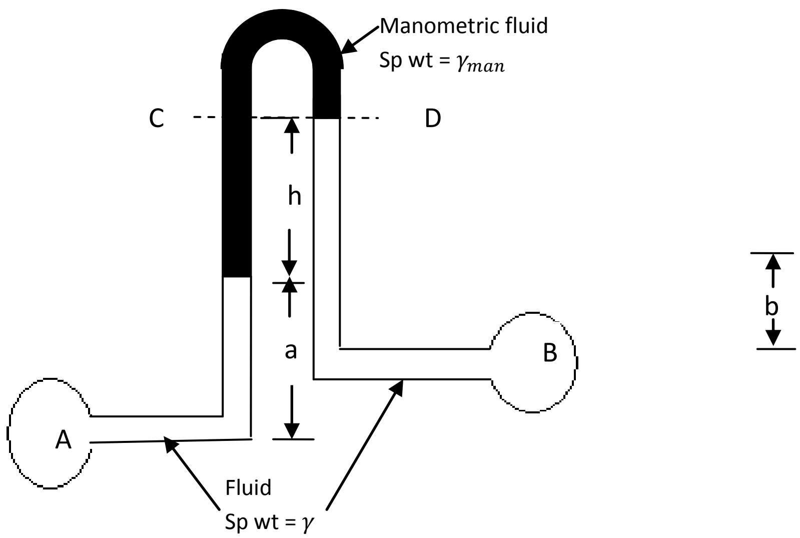
Another method for computing pressure difference

The pressure difference can be computed by starting from one end, using $P_{next\ level} = P_{previous\ level} + P_d$, and equating pressure at same level in a continuous fluid. The total value will be pressure at the other end.

Starting from A,

$$P_A + \gamma a - \gamma_{man} h - \gamma b = P_D$$
$$P_A - P_D = \gamma_{man} h + \gamma b - \gamma a$$

The U-tube can also be inverted. Manometric liquid of lighter density than that of the fluid can be used in such manometer. In such manometer, large deflection of manometric liquid will occur even for small pressure difference between two points and pressure difference can be measured more accurately.



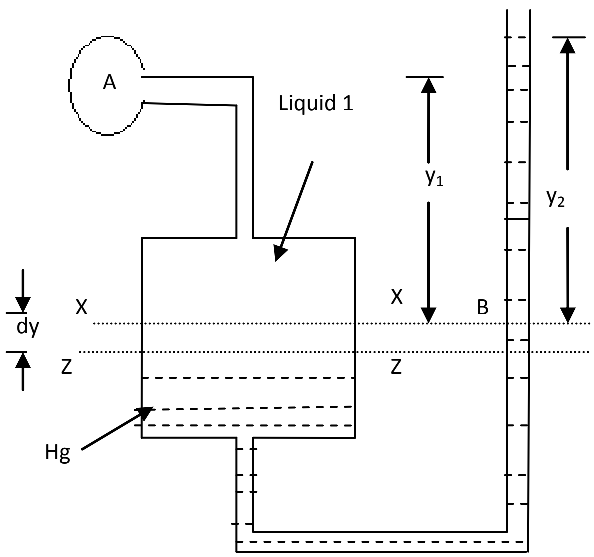
Pressure at C = Pressure at D

$$P_A - \gamma a - \gamma_{man} h = P_B - \gamma(b + h)$$
$$P_A - P_B = \gamma_{man} h + \gamma a - \gamma(b + h)$$

III. Advances to U-tube manometer

a. Making diameter of one leg very large

In the simple U-tube manometer, movements of liquids in both limbs must be read. This limitation can be overcome by making diameter of one leg very large as compared with the other. The result is that the level of fluid in the narrow leg is much higher than the other, which is the only reading required for computing pressure.



When the manometer is not connected to the container, the mercury in the reservoir is at original level (XX) and at level B in the tube.

Due to pressure, manometric liquid in the reservoir drops by dy and it will travel a distance of y_2 in the tube.

Volume of fluid fallen = Volume of fluid risen

$Ady=ay_2$

A = Area of reservoir and a = area of tube

$dy = \frac{a}{A} y_2$

Equating pressure at new level (ZZ)

$P_A + \gamma_1(y_1 + dy) = \gamma_{mercury}(y_2 + dy)$

From a and b

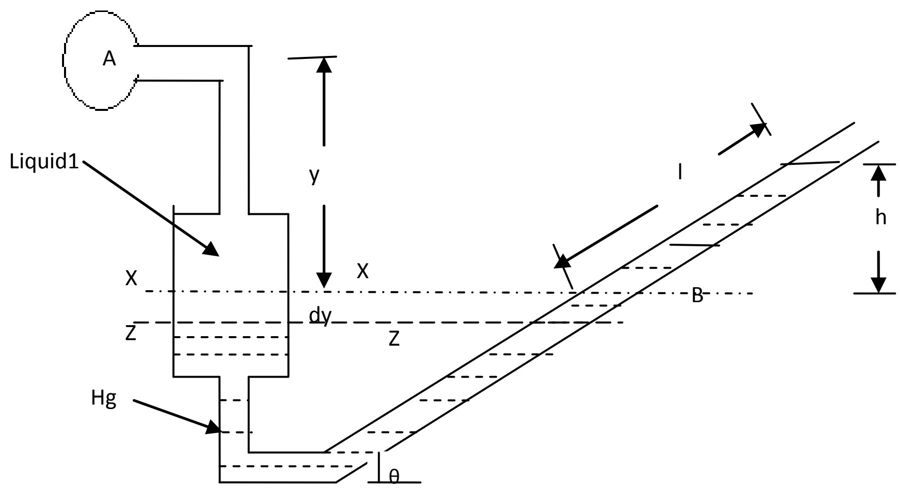
$P_A = \gamma_{mercury} y_2 - \gamma_1 y_1 + (\gamma_{mercury} - \gamma_1)\frac{a}{A} y_2$

If A is very large compared to a, a/A is very small and dy can be neglected. Then,

$P_A = \gamma_{mercury} y_2 - \gamma_1 y_1$

b. Inclined manometer (Tilting one arm)

If the pressure difference to be measured is small, movement cannot be read in the simple U-tube manometer. This limitation can be overcome by tilting one arm of manometer. In case of inclined manometer, the movement along the manometer arm is much higher than the change in level, which is easier to read.



When the manometer is not connected to the container, the mercury in the reservoir is at original level and at level B in the tube.
Due to pressure, manometric liquid in the reservoir drops by dy and it will travel a distance of h in the tube.
Volume of fluid fallen = Volume of fluid risen
Ady=ah
A = Area of reservoir and a = area of tube
$dy = \frac{a}{A}h$
Equating pressure at new level (ZZ)
$P_A + \gamma_1(y + dy) = \gamma_{mercury}(h + dy)$ (b)
$P_A = \gamma_{mercury} l sin\theta - \gamma_1 y + (\gamma_{mercury} - \gamma_1)\frac{a}{A} l sin\theta$
If A is very large compared to a, a/A is very small and dy can be neglected. Then,
$P_A = \gamma_{mercury} l sin\theta - \gamma_1 y$

The sensitivity to pressure change can be increased further by a greater inclination.

Advantages of manometers

- They are very simple and cheap.
- No calibration is required - the pressure can be calculated from first principles.

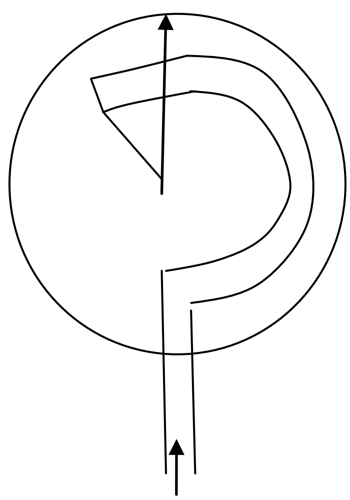
Disadvantages of manometers

- Slow response: It is only really useful for very slowly varying pressures and cannot be used at all for fluctuating pressures.
- For the U tube manometer, two measurements must be taken simultaneously to get the h value.
- It is often difficult to measure small variations in pressure.
- It cannot be used for very large pressures unless several manometers are connected in series.
- For very accurate work the relationship between temperature and density must be known.

2.5.2 Mechanical pressure gage

Bourdon tube gage

Bourdon tube gage is a simple mechanical device for measuring pressure. It consists of a bent tube of elliptical cross-section fixed at one end through which fluid enters. The other end is linked to a pointer which moves through the scale. When fluid pressure is made to enter the tube, its cross-section tends to become circular, causing the tube to straighten and move the pointer.



CHAPTER 3. FLUID STATICS

3.1 Pressure force (Hydrostatic force)

3.1.1 Introduction

The force exerted by fluid pressure on a solid boundary or across any plane is called pressure force. It is also called total pressure or hydrostatic force or resultant force. The point of application of the resultant force is called the center of pressure. Since the fluid is at rest, the force will act at right-angle to the surface.

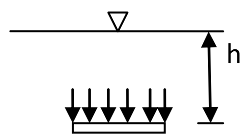
Considering the forces on each element of area,

Resultant force = sum of forces on all elements of area = $\sum PdA$

Where P = pressure on element and dA = area of element

The above formula is valid if the direction of pressure on each element is parallel.

3.1.2 Pressure force (hydrostatic force) on horizontal submerged plane



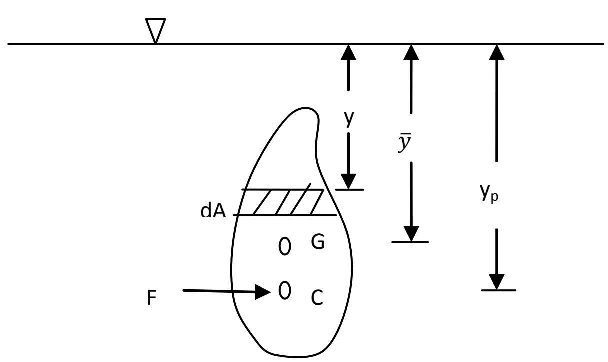
On a horizontal plane, the pressure, p, will be equal at all points of the surface.

Resultant force (F) = $\sum PdA = P\sum dA$ = PA

= γhA

where A= Area of plane

3.1.3 Pressure force (Hydrostatic force) on vertical submerged plane



Consider a vertical plane surface of area A totally immersed in a liquid of specific weight γ. Consider an element of area dA at a vertical depth y from free surface. P is the pressure on each element, $\bar{y}$ is the position of CG and y_p is the position of CP from the free surface. F is the resultant force.

Finding resultant force
Force on element (dF) = $PdA = \gamma y dA$
Summing the forces,

$$F = \sum \gamma y dA$$

Assuming γ to be constant

$$F = \gamma \sum y dA$$

Here, $\sum ydA$ = first moment of area about free surface = $A\bar{y}$
Hence, $F = \gamma A \bar{y}$

Finding the position center of pressure

Taking moment of force on element (dM) about free surface
dM= dF x y = $\gamma y dA x y = \gamma y^2 dA$
Total moment of all forces (M) = $\sum dM = \sum \gamma y^2 dA = \gamma \sum y^2 dA$
$\sum y^2 dA$ = second moment of area about an axis through free surface = moment of inertia = I_0
$M = \gamma I_0$ (a)

Moment of resultant force F about free surface = $F\ y_p = \gamma A \bar{y} y_p$ (b)
Equating a and b

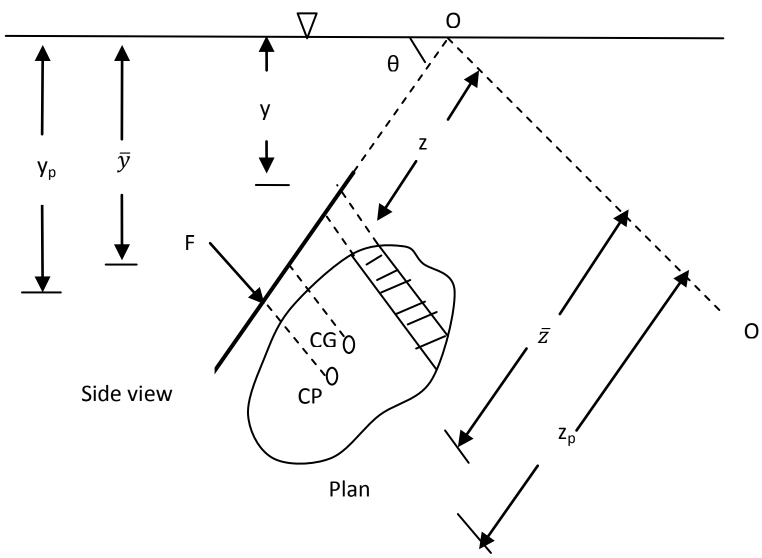
$$\gamma I_0 = \gamma A \bar{y} y_p$$

$$y_p = \frac{I_0}{A\bar{y}}$$

From parallel axis theorem
$I_0 = I_G + A\bar{y}^2$ where I_G = M.I. about an axis through CG. Substituting for I_0

$$y_p = \bar{y} + \frac{I_G}{A\bar{y}}$$

3.1.4 Pressure force (Hydrostatic force) on inclined submerged plane



Consider a plane surface of area A totally immersed in a liquid of specific weight γ and inclined an angle θ to the free surface. Consider an element of area dA at a vertical distance y from the free surface.
P = pressure on each element, $\bar{y}$ = vertical distance of CG from free surface, y_p = vertical distance of CP from the free surface, F = resultant force, z = distance of element from O-O, $\bar{z}$ = distance of CG from O-O and z_p = distance of CP from O-O.

Finding resultant force
Force on element (dF) = $PdA = \gamma ydA = \gamma zsin\theta dA$
Summing the forces,

$$F = \sum \gamma zsin\theta dA$$

Assuming γ to be constant

$$F = \gamma sin\theta \sum zdA$$

Here, $\sum zdA$ = first moment of area about an axis through O-O = $A\bar{z}$
Hence, $F = \gamma sin\theta A\bar{z} = \gamma sin\theta A\frac{\bar{y}}{sin\theta} = \gamma A\bar{y}$

Finding the position center of pressure

Taking moment of force on element (dM) about O
dM= dF x z = $\gamma zSin\theta dAxz = \gamma Sin\theta z^2dA$
Total moment of all forces (M) = $\sum dM = \sum \gamma Sin\theta z^2dA = \gamma Sin\theta \sum z^2dA$
$\sum z^2dA$ = second moment of area about an axis through O-O = moment of inertia = I_0
$M = \gamma Sin\theta I_0$ (a)

Moment of resultant force F about O = $F\, z_p = \gamma A\bar{y}z_p$ (b)

Equating a and b

$$\gamma Sin\theta I_0 = \gamma A\bar{y}z_p$$
$$z_p = \frac{I_0 Sin\theta}{A\bar{y}}$$

From parallel axis theorem

$I_0 = I_G + A\bar{z}^2$ where I_G = M.I. about an axis through CG. Substituting for I_0

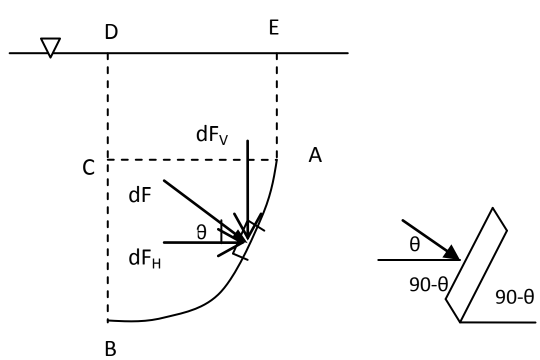
$$z_p = \frac{(I_G + A\bar{z}^2)Sin\theta}{A\bar{y}}$$
$$\frac{y_p}{Sin\theta} = \frac{[I_G + A(\bar{y}/Sin\theta)^2]Sin\theta}{A\bar{y}}$$

$$y_p = \bar{y} + \frac{I_G Sin^2\theta}{A\bar{y}}$$

3.1.5 Pressure force (hydrostatic force) on submerged curved surface

The pressure on the curved surface varies from point to point. Each elemental force on curved surface is a different magnitude and in a different direction, but acts normal to the surface. It is, in general, not easy to calculate the resultant force for a curved surface by combining all elemental forces. Therefore, the resultant force is found by combining the elemental forces using some vectorial method. In this approach, the horizontal and vertical components of force are calculated and these are combined to obtain the resultant force and direction.

Finding resultant force



Consider a curved surface AB totally immersed in a liquid of specific weight γ. Let dF is the resultant force acting on an element of area dA, which makes an angle θ with the horizontal. dF_H and dF_V are the horizontal and vertical components of the resultant force. If P is the pressure on dA, then

dF = PdA

$$dF_H = PdACos\theta$$
$$dF_V = PdASin\theta$$

Horizontal component for the whole curve

$$F_H = \int PdACos\theta = \int \gamma ydACos\theta$$

$dACos\theta$ = projection of element on vertical plane

F_H = Resultant force on the projection of the curve on a vertical plane

R_H acts through the center of pressure of the projection of the whole curve on a vertical plane.
If A is the projected area on vertical plane and $\overline{y}$ is the distance of CG of this plane, then,

$$F_H = \gamma A\overline{y}$$

Vertical component for the whole curve

$$F_V = \int PdASin\theta = \int \gamma ydASin\theta$$

$dASin\theta$ = projected area of element on horizontal plane

$$F_V = \int \gamma dV = \gamma V = Weight\ of\ fluid$$

Where dv = volume of fluid above element
Resultant vertical force= Weight of fluid vertically above the curved surface and extending up to the free surface
In case of fluid below the curve, imaginary weight of same fluid above the curve is considered. (In such case, vertical force on the curve acts upward.)

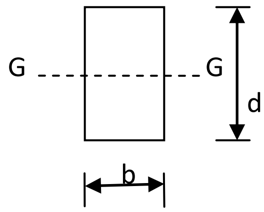
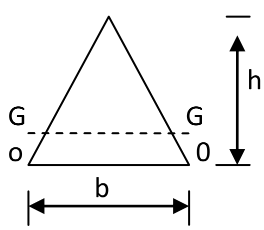
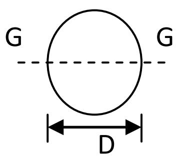
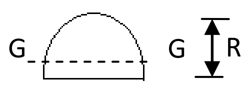
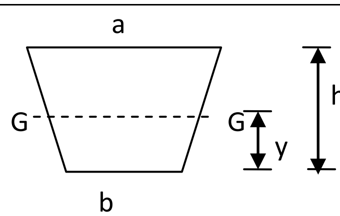
Resultant force

$$F = \sqrt{F_H^2 + F_V^2}$$

Direction of resultant force

$$\theta = tan^{-1}\left(\frac{F_V}{F_H}\right)$$

Geometrical properties of some common figures

Shape	Position of CG	Area	I_G
	d/2 from bottom	bd	$\frac{bd^3}{12}$
	h/3 from bottom	$\frac{1}{2}bh$	$\frac{bh^3}{36}$ $\left[I_{00} = \frac{bh^3}{12}\right]$
	D/2 from bottom	$\frac{\pi D^2}{4}$	$\frac{\pi D^4}{64}$
	$\frac{4R}{3\pi}$ from bottom	$\frac{\pi R^2}{2}$	$0.11R^4$
	$y = \frac{a+2b}{a+b}\frac{h}{3}$	0.5(a+b)h	$\left(\frac{a^2+4ab+b^2}{36(a+b)}\right)h^3$

CG of quarter circle: $\frac{4R}{3\pi}$ from horizontal and vertical line passing through center, $IG = 0.055R^4$

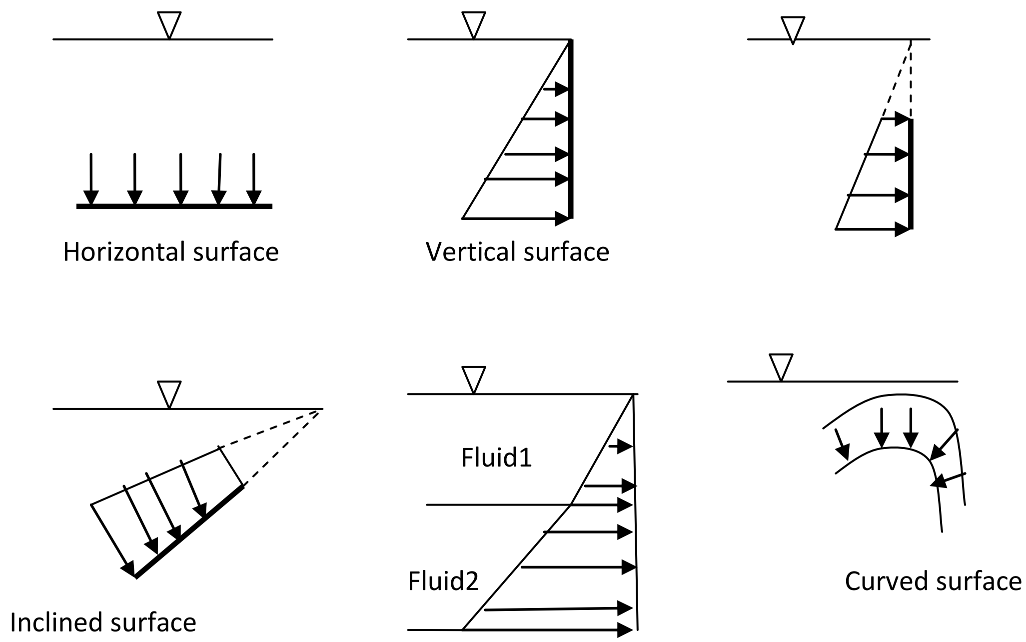
CG of parabola: 2h/5 from bottom (h = height of parabola), IG =$\frac{8bh^3}{175}$

CG of cone (vertex downward): 3H/4 from bottom (H = height of cone)

CG of sector = $\frac{2rSin\alpha}{3\alpha} = \frac{2r}{\pi}$

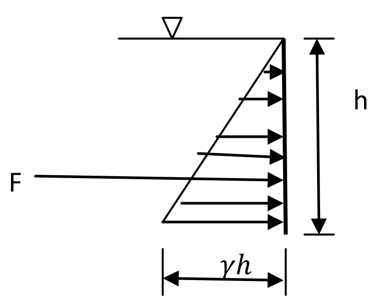
3.1.6 Pressure diagram

The graphical representation of the distribution of fluid pressure over a surface is called pressure diagram.



The resultant force on vertical surface of constant width can also be computed by using pressure diagram.
Resultant force = Area of pressure diagram x width
The resultant force acts through the CG of pressure diagram.
If the pressure diagram contains more than one area, then compute force on each area and compute resultant force. To compute the point of application of resultant force, take moment of all forces about top or bottom.

Example:



F = Area of pressure diagram x width
$= \frac{1}{2}\gamma h x h x b = \frac{1}{2}\gamma h^2 b$
F acts at a distance of 2h/3 from top.

3.2 Buoyancy and floatation

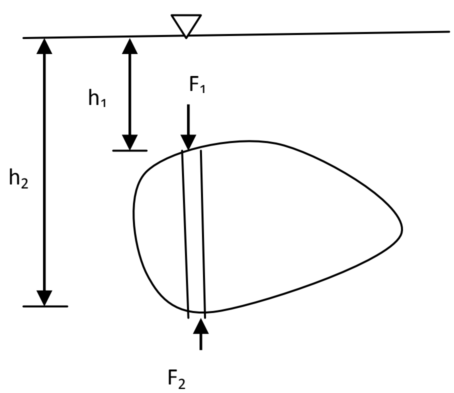
3.2.1 Introduction

Buoyancy

When a body is immersed in a fluid either wholly or partially, the resultant horizontal force is zero as the pressure force on a plane is equal and opposite. So, only vertical upward force acts on the body. The tendency of a submerged body to be lifted up in the fluid due to upward force opposite to the force of gravity is called buoyancy. The vertical upward force on a floating or submerged body is called buoyant force or upthrust or force of buoyancy. The point of application of the buoyant force is called the center of buoyancy. This is the center of gravity of the volume of fluid displaced.

Archimedes' principle of buoyancy

It states that when a body is totally or partially immersed in a fluid, it is lifted up by a force which is equal to the weight of the fluid displaced by the body.
Buoyant force (F_B) = weight of the fluid displaced by the body

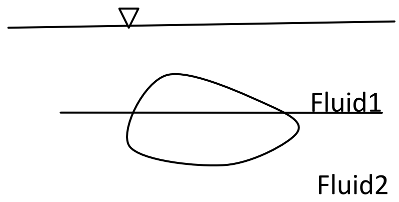


Consider a body immersed in a fluid of specific weight γ. Consider an elementary vertical prism of cross-sectional area dA. Let P_1 and P_2 be the pressure acting at top and bottom of the prism.
Net upward force on the prism (dF_B) = $F_2 - F_1 = P_2\,dA - P_1\,dA$

$$= \gamma h_2 dA - \gamma h_1 dA = \gamma(h_2 - h_1)dA = \gamma dV$$

Total upward force on entire submerged body is
$F_B = \int \gamma dV = \gamma V$
where V = volume of immersed portion of body
Hence, buoyant force (F_B) = weight of the fluid displaced by the body = γV or $\rho g V$

Body immersed in fluids of different specific weights

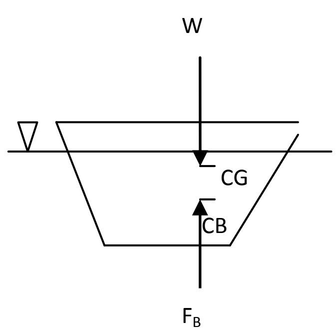


Consider a body immersed in two immiscible fluids of specific weight γ_1 and γ_2 respectively.
Total buoyant force = Buoyant force on upper part + Buoyant force on lower part
= $\gamma_1 V_1 + \gamma_2 V_2$ = total weight of the fluids displaced by the body
Center of buoyancy is CG of corresponding volume of fluid displaced.

Partially submerged body

For a partially submerged body in a liquid with the top portion exposed to air, the weight of the air displaced by the top portion is neglected as the specific weight of air is negligible.

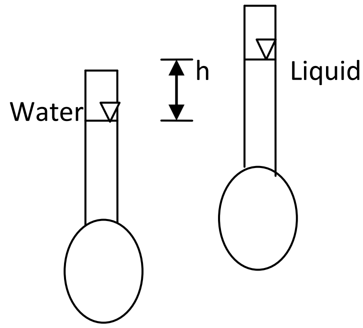
3.2.2 Equilibrium of floating bodies



Two forces act on an immersed body: Buoyant force (F_B) acting upward and weight (W) of the body acting downward. For the equilibrium of floating bodies, F_B = W. This is known as principle of floatation, which states that weight of body floating in a fluid is equal to the buoyant force which in turn is equal to the weight of the fluid displaced by the body.
If W> F_B, the body will sink.
If W = F_B, the body will float.
If W< F_B, the body will rise until W = F_B.

3.2.3 Hydrometer

Hydrometer is an instrument used to measure specific gravity of a liquid. The working of hydrometer is based on Archimedes' principle. The level at which hydrometer floats depends only on the density of the liquid. A body will sink more in a lighter fluid than heavier one. It consists of a glass tube closed at both ends, with one end enlarged into a bulb. Calibrated scale is marked in the glass tube. The reading of the hydrometer at the free liquid surface indicates the specific gravity of the liquid.



Initially, the hydrometer is dipped in water.

Buoyant force (F_{B1}) = $\gamma_{water}V$ (a)

where γ_{water} = specific weight of water, V = Volume of water displaced by hydrometer.

Next, the hydrometer is dipped in a fluid whose specific gravity S is to be determined. If the liquid is heavier, then the stem of the hydrometer would go up. Let A is the cross-sectional area of stem and h is the difference in reading in two cases.

Buoyant force (F_{B2}) = $\gamma V_1 = S\gamma_{water}V_1$

where γ = specific weight of fluid, V_1 = Volume of fluid displaced.

$F_{B2} = S\gamma_{water}(V - Ah)$ (b)

Equating a and b (as weight of body is same for both cases, which is equal to buoyant force)

$$\gamma_{water}V = S\gamma_{water}(V - Ah)$$

$$S = \frac{V}{V - Ah}$$

3.2.4 Three conditions of equilibrium

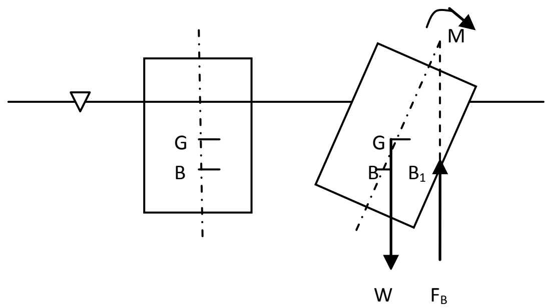
Stable equilibrium: If the body comes to original position after applying small angular displacement, the body is said to be in a state of stable equilibrium.

Unstable equilibrium: If the body moves away after applying small angular displacement and does not return to the original position, the body is said to be in a state of unstable equilibrium.

Neutral equilibrium: If the body moves to new position and remains at rest in that position, the body is said to be in a state of neutral equilibrium.

3.2.5 Metacenter and Metacentric height

Metacenter is the point about which body starts oscillating when it is tilted by a small angle.

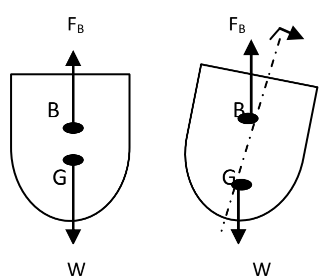


Consider a body floating in a fluid. Let the body is in equilibrium and G be the CG of the body and B be the CB. If the body is subjected to a small angular displacement, then the CB will shift from B to a new position B_1, but the position CG does not change. The point M, which is the point of intersection of a line passing through G and B, and a vertical line passing through new CB B_1, is called the metacenter. The distance between the CG of the floating body and the metacenter (GM) is called metacentric height.

3.2.6 Stability of submerged bodies

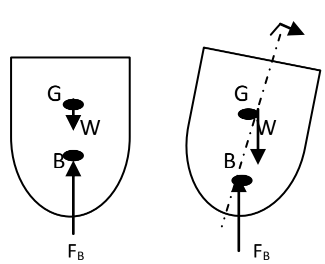
The forces acting on submerged bodies are: weight (W) of body acting through CG (G) and buoyant force (F_B) acting through CB (B). The position of B and G determines the stability of submerged bodies.

Stable equilibrium



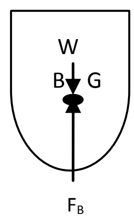
For stable equilibrium, W= F_B and B is above G. If a small angular displacement is given in clockwise direction, the couple due to W and F_B produces moment in anticlockwise direction, thus bringing the body in stable condition.

Unstable equilibrium



For unstable equilibrium, W= F_B and G is above B. If a small angular displacement is given in clockwise direction, the couple constituting W and F_B produces moment in clockwise direction (overturning moment), thus moving the body further away.

Neutral equilibrium

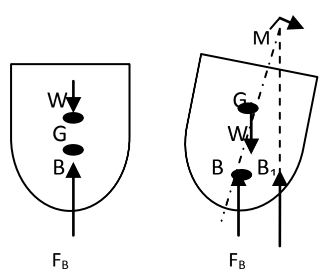


For neutral equilibrium, W= F_B and B and G are at the same point. If a small angular displacement is given, no any moment is produced in such equilibrium.

3.2.7 Stability of floating bodies

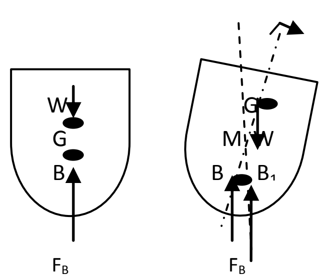
The location of metacenter (M) and the CG (G) governs the stability of floating bodies. The weight of body is equal to the weight of the fluid displaced.

Stable equilibrium



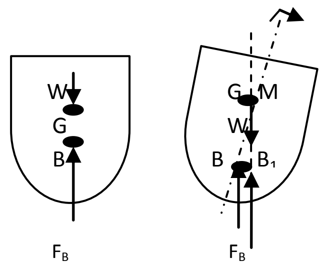
For stable equilibrium, M is above G i.e. MG is positive. If disturbing couple is applied in clockwise direction, the couple due to W and F_B produces moment in anticlockwise direction (restoring moment), thus bringing the body in stable condition.

Unstable equilibrium



For unstable equilibrium, M is below G i.e. MG is negative. If disturbing couple is applied in clockwise direction, the couple constituting W and F_B produces moment in clockwise direction (overturning moment), thus moving the body further away.

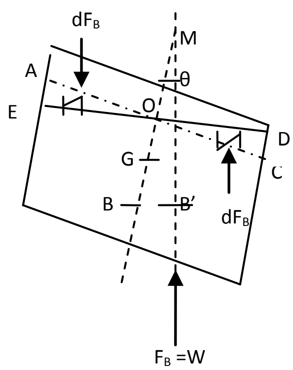
Neutral equilibrium



For neutral equilibrium, M and G are at the same point. If a small angular displacement is given, no any moment is produced in such equilibrium.

3.2.8 Determination of metacentric height

a. Analytical method



Consider a vessel with water line AC, B as the CB and G as the CG in original position. When the vessel is tilted through a small angle θ, the CB changes from B to B', the position of water line changes to ED and two wedges AOE and COD are formed. M is the metacenter, W is the weight of vessel and F_B is the buoyant force.

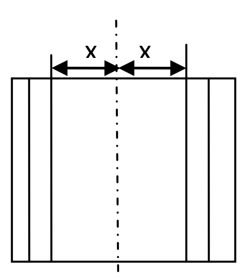
Consider an element of area dA at a distance x from the CG of plan of vessel on either side.

Volume of element (dV) = dA.xθ

(xθ = height for small angle)

Buoyant force on the element of left side (dF_B) = Weight of element

$= \gamma\ dV = \gamma dAx\theta$



where γ is the sp wt of water.

Similarly,

Buoyant force on the element of right side (dF_B) = $\gamma dAx\theta$
Buoyant force on the two elements produces a couple.
Moment of couple = $\gamma dAx\theta(x + x)$ = $\gamma\theta(2x^2dA)$
Total moment due to altered displacement (M) = $\int \gamma\theta(2x^2dA)$
=$\gamma\theta \int(2x^2dA)$
$\int(2x^2dA)$ = I = Second moment of plan of the vessel about an axis through CG (Moment of inertia, I_{YY})
$M = \gamma\theta I$ (a)
Moment due to the movement of CB from B to B' (M') = F_B BB' = W BB'
$M' = \gamma\ V\ BM\ \theta$ (b)
Where V is the volume of water displaced.
For equilibrium, M = M'

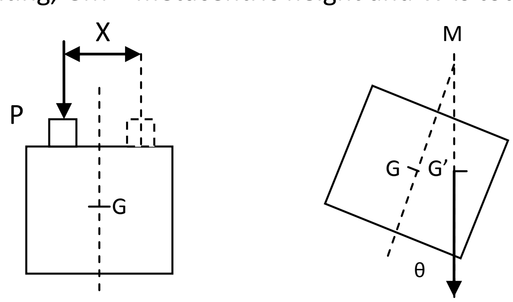
$$\gamma\theta I = \gamma\ V\ BM\ \theta$$

$$BM = \frac{I}{V}$$

GM = BM-BG $= \frac{I}{V} - BG$
If G is below B, then GM = BM+BG $= \frac{I}{V} + BG$
If GM is positive, floating body is stable. If GM is negative, floating body is unstable.
Pitching: oscillatory motion about longitudinal axis, rolling: oscillatory motion about transverse axis
Metacentric height for pitching = $\frac{I_{xx}}{V} - BG$, Metacentric height for rolling = $\frac{I_{yy}}{V} - BG$

b. Experimental method

For experimental determination of metacentric height, a load of value P is moved through a distance x across the vessel and the angle of tilt θ caused by the moving the load is measured. G= original CG, G' = CG after tilting, GM = metacentric height and W is total weight of vessel including load P.

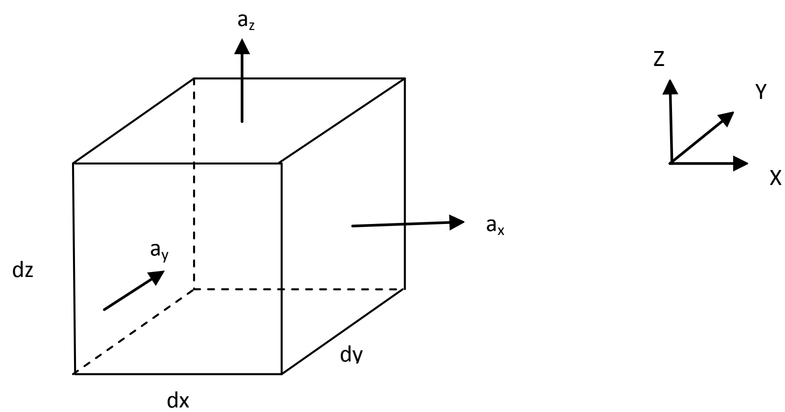


Overturning moment due to movement of load P = Px
Righting moment due to displacement of CG from G to G' = W GM tanθ
For equilibrium,
Px = W GM tanθ
GM = Px/W tanθ
For small θ, tan θ is approximated as θ.
GM = Px/Wθ

3.3 Relative equilibrium

In case of static equilibrium, the fluid is at rest and there exist only pressure force on it and there are no shear forces. If the container containing fluid is given a continuous acceleration, this will be transmitted to the fluid and affect the pressure distribution in it. The fluid and the container in this situation are in motion together as a solid body. Since the fluid remains at rest relative to the container, there is no relative motion of the particles of the fluid, and therefore, no shear stresses. The fluid pressure acts normal to the surface at each point. Under these conditions, the fluid is said to be in relative equilibrium. In such case, hydrostatic law is applicable for the evaluation of pressure.

3.3.1 General expression for pressure in relative equilibrium



Let $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}$ and $\frac{\partial P}{\partial z}$ are the rate of change of pressure P in X,Y, and Z directions and a_x, a_y and a_z the accelerations. ρ = density of fluid

In X-direction

$\sum F_x = m\,a_x$

$P\,dydz - \left(P + \frac{\partial P}{\partial x} dx\right) dydz = \rho dxdydz\, a_x$

$-\frac{\partial P}{\partial x} = \rho a_x$

In Y-direction

$\sum F_y = m\,a_y$

$P\,dxdz - \left(P + \frac{\partial P}{\partial y} dy\right) dxdz = \rho dxdydz\, a_y$

$-\frac{\partial P}{\partial y} = \rho a_y$

In Z-direction

$\sum F_z = m\,a_z$

$P\,dxdy - \left(P + \frac{\partial P}{\partial z}dz\right)dxdy - weight = \rho dxdydz\,a_z$

$P\,dxdy - \left(P + \frac{\partial P}{\partial z}dz\right)dxdy - \rho g dxdydz = \rho dxdydz\,a_z$

$-\frac{\partial P}{\partial z} = \rho(g + a_z)$

If we only consider x-z plane

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial z}dz$$

$$dP = -\rho a_x dx - \rho(g + a_z)dz$$

$$P = -\int \rho a_x dx - \int \rho(g + a_z)dz$$

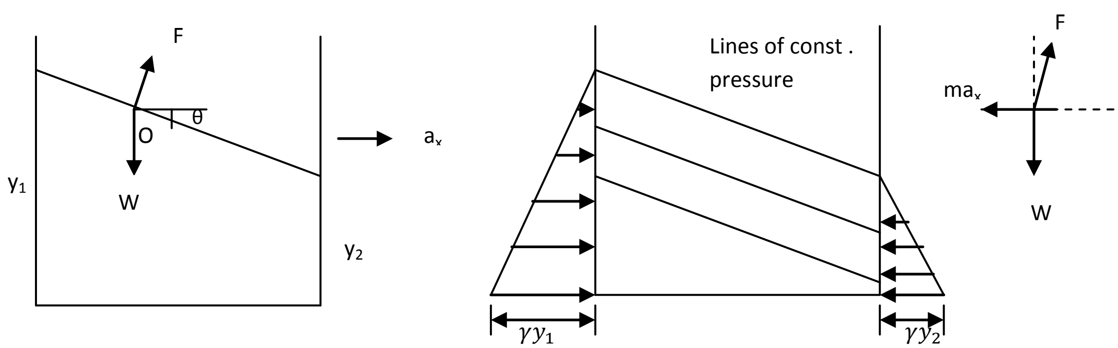
$$P = -\rho a_x x - \rho(g + a_z)z + const$$

At x = 0, z = 0, let p = P_0. Then const = P_0

$$P = P_0 - \rho a_x x - \rho(g + a_z)z$$

This is the equation to compute pressure at any in case of acceleration in x and z direction.

3.3.2 Uniform horizontal acceleration



Pressure diagram

Consider a tank subjected to horizontal acceleration a_x. A particle of mass m on the free surface at O will have same acceleration a_x as the tank. The particle will be in equilibrium under three forces- Weight (W), Pressure force (F) and inertia force. Let θ is the angle of inclination of the water surface to the horizontal.

Resolving horizontally

$FSin\theta = ma_x$ (a)

Resolving vertically

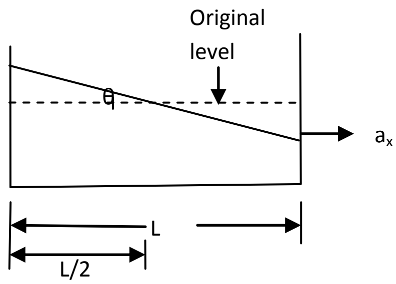
$FCos\theta = W = mg$ (b)

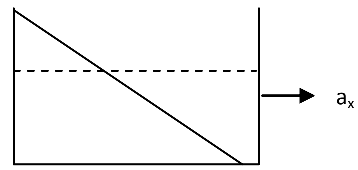
Dividing a by b

$tan\theta = \frac{a_x}{g}$

The slope is constant at all points on the free surface, which is the line of constant pressure.

The water surface inclines at the midpoint of original level until there is no spilling of water. After the water spills, water level to the left of midpoint of original level.



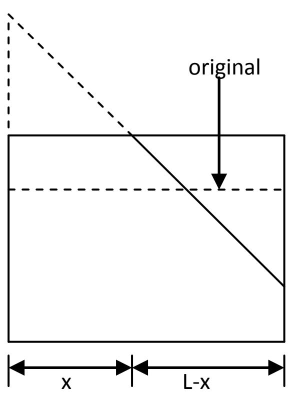


Water surface: spilling case

Water surface: no spilling case

As there is no vertical acceleration, pressure at any depth y below the surface = γy.
Pressure force on front and rear edge can be calculated by using
Pressure force = Area of pressure diagram x width or by using $F = \gamma A\bar{y}$.
Force on side is computed by taking mean depth using $F = \gamma A\bar{y}$.
Force on bottom = weight of fluid

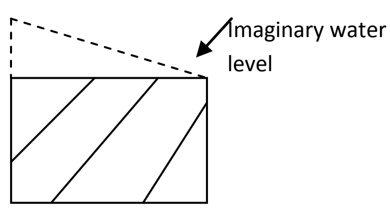
Closed tank



Partially full

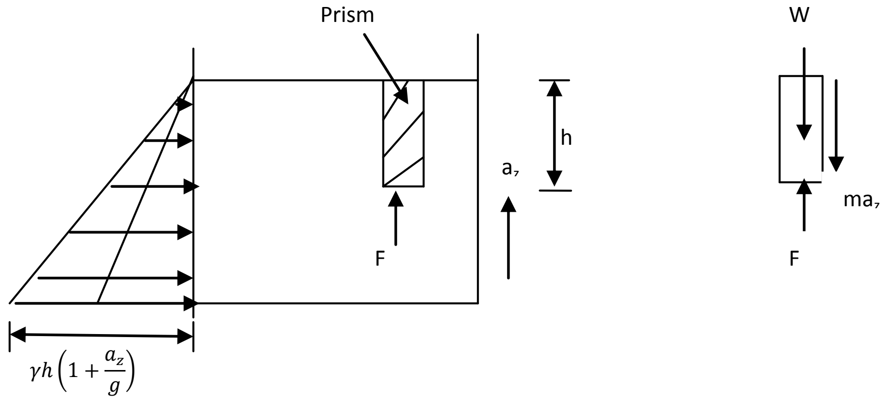
Equate volume of water before acceleration and after acceleration, find position of x and free surface.

In case of closed tank completely filled with water, imaginary water level should be drawn above the tank.



Completely full

3.3.3 Uniform vertical acceleration

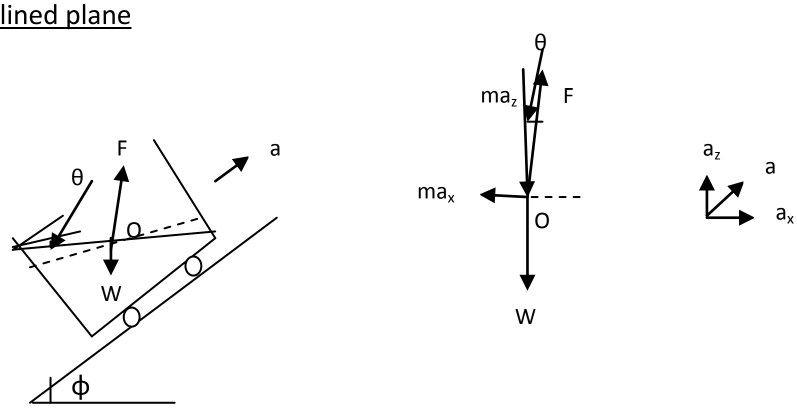


Consider a tank subjected to vertical acceleration a_z. The water surface in this case will be horizontal. Consider a vertical prism of cross-sectional area A subjected to an upward acceleration a_z. At depth h, pressure is P.

Force due to Pressure (F) = PA
Weight of prism (W) = ρgAh
Resolving vertically
$F - W = ma_z$
$PA - \rho gAh = \rho Aha_z$
$P = \rho gh\left(1+\frac{a_z}{g}\right)$
Or,
$P = \gamma h\left(1+\frac{a_z}{g}\right)$
Pressure force on each side can be calculated by using
Pressure force on longer and shorter side = Area of pressure diagram x width.
Force on bottom = PA_b where A_b = Bottom area

In case of tank moving vertically downward, take negative sign of a_z. In this case,
$P = \gamma h\left(1-\frac{a_z}{g}\right)$

3.3.4 Acceleration in an inclined plane



Consider a container with a body of fluid is accelerated with a_s in a plane inclined an at angle ϕ with the horizontal. A particle of mass m on the free surface at O will have same acceleration a as the container. The particle will be in equilibrium under three forces- Weight (W), pressure force (F) and inertia force. Let θ is the angle of inclination of the water surface to the horizontal.

$$a_x = a\ cos\varphi, a_z = a\ sin\varphi$$

Resolving horizontally

$FSin\theta = ma_x$ (a)

Resolving vertically

$FCos\theta = mg + ma_z = m(g + a_z)$

$FCos\theta = m(g + a_z)$ (b)

Dividing a by b

$tan\theta = \frac{a_x}{g+a_z}$ (slope w.r.t. horizontal)

As there is vertical acceleration, the pressure distribution will be same as that for the case of uniform vertical acceleration.

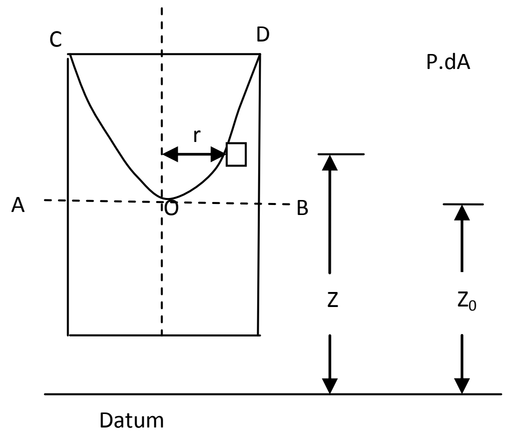
$P = \gamma h\left(1 + \frac{a_z}{g}\right)$

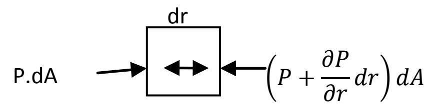
Force on front and rear edge = area of pressure diagramxwidth

Force on side is computed by taking mean depth.

Force on bottom = weight of fluid

3.3.5 Radial acceleration (Forced vortex)





Consider a vessel containing a body of fluid of density ρ and specific weight γ, which is rotating about a vertical axis with angular velocity ω. After some time, the fluid will rotate with the same angular velocity ω as the vessel and there is no relative motion.

Consider an elementary fluid mass of area dA and length dr at a radius r from the axis of rotation. Let P is the pressure on left side and $\left(P+\frac{\partial P}{\partial r}dr\right)dA$ is the pressure on right side of the element.
Centrifugal force acting away from the axis of rotation = mass x radial acceleration = $\rho dAdr\, r\omega^2$

Net pressure force on element = Inertia force

$$PdA-\left(P+\frac{\partial P}{\partial r}dr\right)dA = -\rho dAdr\, r\omega^2$$

$$\frac{\partial P}{\partial r}=\rho r\omega^2$$

Pressure varies with r as well as elevation (Z) from datum

$$dP=\frac{\partial P}{\partial r}dr+\frac{\partial P}{\partial z}dz$$

$$dP=\rho r\omega^2 dr+(-\gamma)dz$$

Integrating

$$P=\frac{1}{2}\rho w^2r^2-\gamma z+c$$

If bottom of the considered as reference level,
Then at O, r=0, Z=Z_0 and P = Pa = Atmospheric pressure

$$c=Patm+\gamma Z_0$$

$$P=Pa+\frac{1}{2}\rho w^2r^2-\gamma(z-Z_0)$$

This is the equation to compute pressure at any point for vortex motion.

For any point at the free surface P = Patm

$$\frac{1}{2}\rho w^2r^2-\gamma(z-Z_0)=0$$

$$(Z - Z_0) = \frac{\rho r^2 \omega^2}{2\gamma} = \frac{r^2 \omega^2}{2g}$$

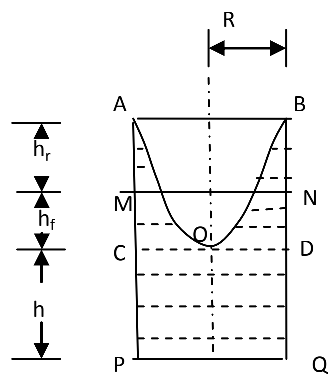
If the datum is assumed to pass through the vertex, then $Z_0 = 0$. Thus for a point on free liquid surface at radius r and with elevation Z,

$$Z = \frac{r^2 \omega^2}{2g}$$

This equation shows that the water surface is paraboloid.
Z is known as centrifugal head.

Open cylinder case

When the water surface is below or just touches the rim (water on the verge of spilling)
Rise of liquid levels at the end = fall of liquid level at the axis of rotation



R = Radius of cylinder
MN = Water level at absolute equilibrium (original water level)
After rotation, AOB is the profile of the liquid surface.
h_r = Rise of liquid at end
h_f = Fall of liquid at the end

Volume of liquid before rotation = $\pi R^2 (h + h_f)$
Volume of liquid rotation = Volume of cylinder ABQP- Volume of paraboloid AOB

$$= \pi R^2 (h + h_f + h_r) - \frac{1}{2} \pi R^2 (h_f + h_r)$$

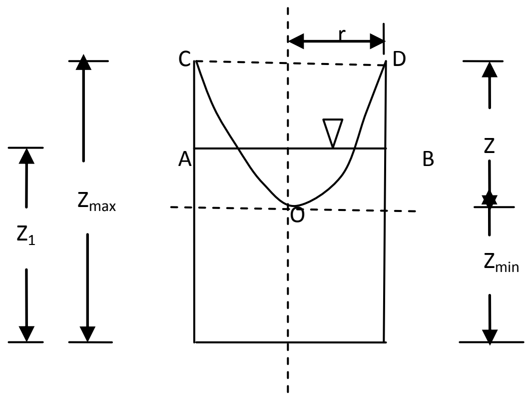
Volume of liquid before rotation = Volume of liquid after rotation

$$\pi R^2 (h + h_f) = \pi R^2 (h + h_f + h_r) - \frac{1}{2} \pi R^2 (h_f + h_r)$$

$$h_r = h_f$$

Hence, rise of liquid level at the end = fall of liquid level at the axis of rotation

Relationships



$Z = \frac{r^2\omega^2}{2g}$

$Z_{max} = Z_{min} + Z$

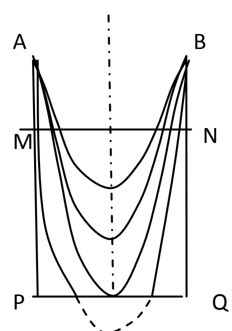
$Z_{max} - Z_1 = Z/2$

$Z_{max} = Z_1 + \frac{r^2\omega^2}{4g}$

Volume of parabola = ½(volume of circumscribing cylinder)

Shape of parabola with increase of rotational speed for open cylinder

When water just begins to spill or when the tank is full of water, the two ends of parabola touches the rim. After further increasing the speed of rotation, water spills out and the depth of parabola increases further.

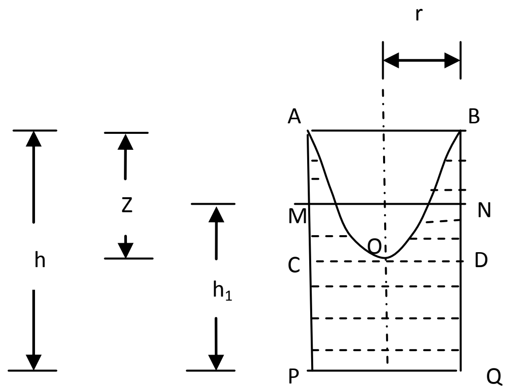


Volume of water spilled = volume of parabola

(If the bottom is exposed, Volume of water spilled =part of parabola within the cylinder)

Closed cylinder case

No spilling in closed cylinder



MN = Original liquid level (before rotation)

AOB = liquid surface after rotation

a. Centrifugal head $(Z) = \frac{r^2\omega^2}{2g}$

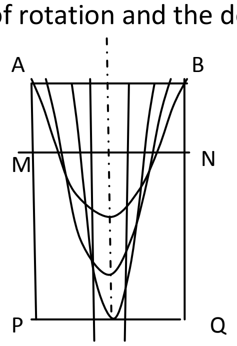
b. Volume of cylinder above MN (before rotation) = Volume of parabola AOB (after rotation)

$$\pi r^2(h - h_1) = \frac{1}{2}\pi r^2 Z$$

If the closed cylinder is completely filled, the imaginary parabola will be formed on the top of vessel.

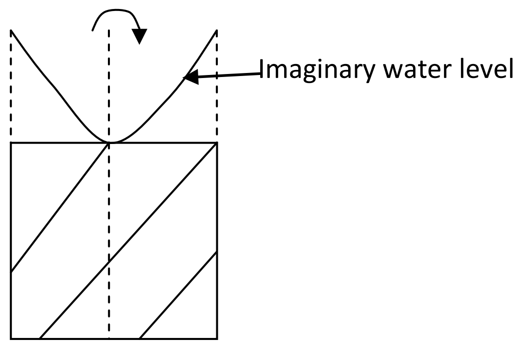
Shape of parabola with increase of rotational speed for closed cylinder

At lower speed, the two ends of parabola touches the rim. After further increasing the speed of rotation, the ends of parabola shifts towards axis of rotation and the depth of parabola increases.



Closed cylinder completely filled with water

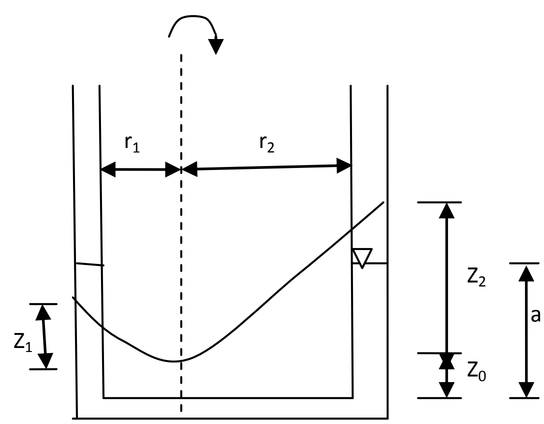
Imaginary parabola above the cylinder is considered.

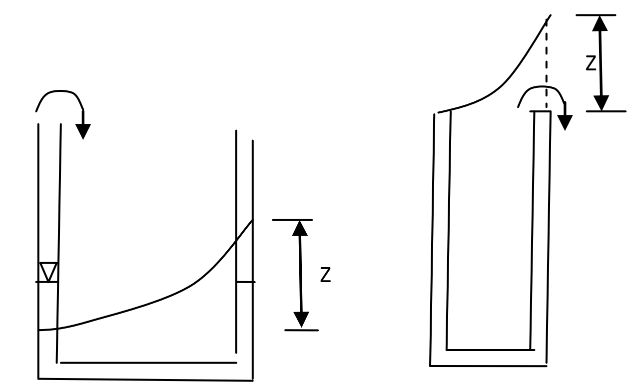


U-tube rotation

Vortex of parabola (lowest point) is at the axis of rotation.

For symmetric axis of rotation, the parabola is also symmetric. For unsymmetric axis of rotation, the end of parabola at lesser radial distance from the axis of rotation falls below original water level, while the other end moves up the original level.





both open   full and one end closed

a. Axis of rotation in between two legs   b. One leg as axis of rotation

For case (a)

$Z_1 = \frac{r_1^2\omega^2}{2g}, Z_2 = \frac{r_2^2\omega^2}{2g}$

Until no spilling

Total height of fluid before and after rotation is same

a+a= $(Z_0+Z_1)+(Z_0+Z_2)$

(If one of the leg is closed and the tube is rotated about the leg with open end, imaginary parabola will be considered above the closed end)

CHAPTER 4. FLUID KINEMATICS

4.1 Introduction

Fluid kinematics is the study of fluid motion without considering the forces causing motion. It is mainly concerned with the velocity of the fluid. Once velocity is known, pressure and force can be determined. In kinematics, motion of fluid is described in terms of space-time relationship. Flow field is the region of interest in flow.

Velocity

Velocity is the rate of change of displacement of fluid particle. If ds is the displacement in time dt, then velocity V is $V = \lim_{dt \to 0} \frac{dS}{dt}$

Velocity is a vector quantity, which acts tangential to the path of the particle.

V = f(position, time)

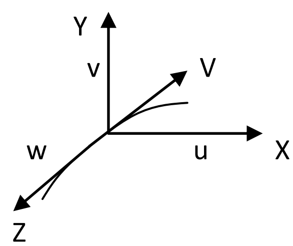
V can be resolved into three mutually perpendicular direction x, y and z respectively.

V= f(x,y,z,t)

u= f1(x,y,z,t)

v= f2(x,y,z,t)

w = f3(x,y,z,t)



$u = \frac{\partial x}{\partial t}, v = \frac{\partial y}{\partial t}, w = \frac{\partial z}{\partial t}$

Change of velocity is

$dv = \frac{\partial v}{\partial s} ds + \frac{\partial v}{\partial t} dt$

Acceleration

Acceleration is the rate of change of velocity. It is also a vector quantity. It acts in the direction of velocity vector as well as in the normal direction. (Since acceleration depends on the rate of change of magnitude of velocity and rate of change of direction of velocity)

Acceleration (a) = $\frac{dV}{dt} = \frac{\partial v}{\partial s}\frac{dS}{dt} + \frac{\partial v}{\partial t} = v\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}$

The first term represents the change of velocity in space, which is known as convective acceleration. The second term represents the change of velocity with time, which is known as local acceleration.

Tangential acceleration: rate of change of velocity in the direction of motion, e.g. flow through straight converging boundary
Normal acceleration: rate of change of velocity perpendicular to the direction of motion, e.g. flow through converging curved boundary

4.2 Method of describing fluid motion

a. Lagrangian method: This method deals with the motion of single fluid particle. This approach is more complex and the equations of motion are difficult to solve.

If a fluid particle is at point (a, b, c) at time t=0 and at point (x, y, z) at time t, then
$x = f_1(a, b, c, t)$, $y = f_2(a, b, c, t)$, $z = f_3(a, b, c, t)$
Velocities u, v and w in x, y and z direction are
$u = \frac{\partial x}{\partial t}, v = \frac{\partial y}{\partial t}, w = \frac{\partial z}{\partial t}$
Accelerations in x, y and z direction are
$a_x = \frac{\partial^2 x}{\partial t^2}, a_y = \frac{\partial^2 y}{\partial t^2}, a_z = \frac{\partial^2 z}{\partial t^2}$
Resultant velocity (V) = $\sqrt{u^2 + v^2 + w^2}$

b. Eulerian approach: This method deals with the motion of fluid particles at a point in space. This method is simple and widely used as the resulting equations can be solved easily.

Velocities at any point (x, y, z): Number of particles pass through point (x,y,z), but we are concerned with the velocity at that point.
$u = f_1(x, y, z, t)$, $v = f_2(x, y, z, t)$, $w = f_3(x, y, z, t)$
Resultant velocity (V) = $\sqrt{u^2 + v^2 + w^2}$ which is also a function of (x, y, z, t)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt$$

$$a_x = \frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly,

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Main fluid properties: velocity, pressure, depth

4.3 General types of fluid flow

a. Based on time criteria

Steady flow: If the fluid properties at a point do not change with time, the flow is known as steady flow.

$$\frac{\partial(fluid\ property)}{\partial t} = 0$$

e.g. flow through a pipe at constant rate

Unsteady flow: If the fluid properties at a point change with time, the flow is known as unsteady flow.

$$\frac{\partial(fluid\ property)}{\partial t} \neq 0$$

e.g. flow in pipe whose valve is closed or opened gradually

b. Based on space

Uniform flow: If the fluid properties at any given time do not vary with distance, the flow is known as uniform flow.

$$\frac{\partial(fluid\ property)}{\partial s} = 0$$

e.g. flow through a straight pipe of constant diameter

Non-uniform flow: If the fluid properties at any given time vary with distance, the flow is known as non-uniform flow.

$$\frac{\partial(fluid\ property)}{\partial s} \neq 0$$

e.g. flow through a tapering pipe, flow through canal bend

c. Based on both time and space criteria

Steady uniform flow: In this type of flow, fluid properties do not change with space or time. e.g. flow of water in a pipe of constant diameter at a constant rate

Steady non-uniform flow: In this type of flow, fluid properties change from point to point, but not with time. e.g. flow of water in a tapering pipe at a constant rate

Unsteady uniform flow: In this type of flow, fluid properties remain same from point to point, but change with time. e.g. flow of water through pipe of constant diameter at increasing or decreasing rate

Unsteady non-uniform flow: In this type of flow, fluid properties change with both time and space. e.g. flow of water through tapering pipe at increasing or decreasing rate, wave travelling along a channel.

d. Laminar and turbulent flow

If the fluid particles move in layers with one layer of fluid sliding smoothly over adjacent layers, the flow is known as laminar flow. It is also called viscous or streamline flow. This type of flow occurs in fluid of high viscosity. The laminar flow is governed by Newton's law of viscosity. The velocity of flow is low and there is no intermixing of particles. Laminar flow is usually rare. e.g. flow through smooth pipe having low velocity, groundwater flow, flow of blood in veins

If the fluid particles move in random and zigzag way, the flow is known as turbulent flow. This type of flow occurs in fluid of low viscosity. Similar form of equation as Newton's law of viscosity is applicable for turbulent flow. The velocity of flow is high and there is intermixing of particles leading to momentum transfer. Turbulent flow is common type of flow in nature. e.g. flow through river, high velocity flow in a conduit of large size

Reynold number (Re), which is a ratio of inertia force to viscous force, is used as a measure to distinguish between laminar and turbulent flow.
$Re = \frac{\rho VL}{\mu}\ or \frac{VL}{\upsilon}$ where ρ =density of fluid, V = mean velocity of flow, L = Characteristic length, μ = dynamic viscosity, υ = kinematic viscosity
For pipe flow:
If Re<2000, the flow is laminar
If Re>4000, the flow is turbulent

e. Compressible and incompressible flow

If the density of fluid changes during flow, the flow is said to be compressible flow. e.g. flow of gas through nozzles. The change in density is due to the variation of pressure and temperature.

If the density of fluid is constant, the flow is said to be incompressible flow. e.g. flow of water through channel

f. Rotational and irrotational flow

If the fluid particles within a region rotate about their own axes, the flow is called rotational flow. e.g. motion of liquid in a rotating tank

If the fluid particles within a region do not rotate about their own axes, the flow is called rotational flow. e.g. flow above wash basin

g. One, two and three dimensional flow

In one dimensional (1D) flow, fluid properties are a function of time and one space co-ordinate. In other words, conditions vary only in the direction of flow not across the cross-section.
v = f(x, t)
The streamlines in 1D flow are straight and parallel.
e.g. flow in pipe

In two dimensional (2D) flow, fluid properties are a function of time and two space co-ordinates. In other words, conditions vary in the direction of flow and in one direction at right angle to this.
v = f(x, y, t)
The streamlines in 2D flow are curves (plane curves).
e.g. flow in main stream of wide river, flow between parallel plates

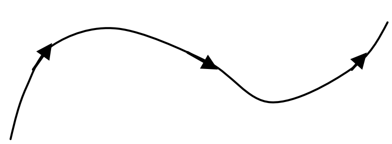
In three dimensional (3D) flow, fluid properties are a function of time and three space co-ordinates. In other words, conditions vary in the direction of flow, across the cross section and across the depth of flow.
v = f(x, y, z, t)
The streamline in 3D flow are space curves (curve in 3D space).
e.g. flood flow in river, flow in converging or diverging pipe.

In general fluid flow is three-dimensional. In many cases the greatest changes only occur in two directions or even only in one. The changes in the other direction can be effectively ignored making analysis much more simpler.

4.4 Different types of flow lines

a. Path line

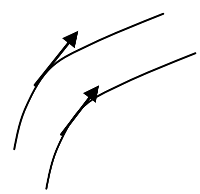
The path traced by a single fluid particle in motion over a period of time in a flow field is known as path line. It indicates the direction of velocity of same particle at successive interval of time. Path lines can intersect themselves.



Equation for drawing pathline
$u = \frac{dx}{dt}, v = \frac{dy}{dt}, w = \frac{dz}{dt}$
Substitute the values of x, y and z. Integrate by taking lower and upper limits of space and time.

b. Stream line

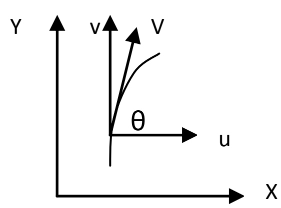
Stream line is an imaginary curve drawn through the flow field in such a way that the tangent to it at any point indicates the direction of velocity at that point. As stream lines join points of equal velocity, these are velocity contours. It is useful to visualize the flow pattern.



Properties

- Close to a solid boundary, streamlines are parallel to that boundary.
- Fluid cannot cross a streamline. (as the velocity component perpendicular to it is zero)
- Streamlines cannot cross each other.
- Any particles starting on one streamline will stay on that same streamline.
- Streamline spacing varies inversely with velocity.
- Series of streamlines represent flow pattern at an instant.
- In unsteady flow, position of streamlines can change with time.
- In steady flow, the position of streamlines does not change.

Differential equation of streamline



$$tan\theta = \frac{dy}{dx} = \frac{v}{u}$$

$$\frac{dx}{u} = \frac{dy}{v}$$

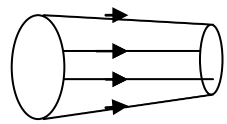
In 3D:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Obtain equation of streamline by integrating.

c. Stream tube

An imaginary tube formed by a group of streamlines through a small closed curve is called stream tube. e.g. pipes, nozzles. This concept is useful for the analysis of flow of large body of fluid.

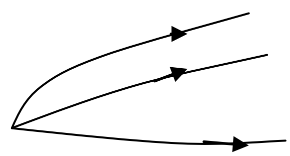


Properties

- The walls of a streamtube are streamlines.
- It has finite dimensions.
- Fluid cannot flow across a streamline, so fluid cannot cross a streamtube wall.
- A streamtube is not like a pipe. Its walls move with the fluid.
- In unsteady flow streamtubes can change position with time.
- In steady flow, the position of streamtubes does not change.

d. Streak line

Streak line is a locus of fluid particles passing through a specified point in the flow field. e.g. path taken by smoke coming out of chimney, movement of particles after dye is injected.



Equation for drawing pathline

$u = \frac{dx}{dt}, v = \frac{dy}{dt}, w = \frac{dz}{dt}$

Substitute the values of x, y and z. Integrate by taking lower and upper limits of space and time (t_0 and t).

4.5 Discharge and mean velocity

The total quantity of fluid flowing per unit time through a particular cross-section is called discharge or flow rate. If discharge is measured in terms of mass, it is called mass flow rate. If it is measured in terms of volume, it is called volumetric flow rate. It is represented by Q.

Unit: m^3/s or litres/s (volumetric flow rate), kg/s (mass flow rate)

In many cases, the variation of velocity over the cross-section can be neglected. The velocity is assumed to be constant and is equal to the mean velocity.

If V is mean velocity and A is cross-sectional area, then discharge (Q) is

Q = AV

In terms of mass flow rate

Q = ρAV

Mass flow rate = ρ x Volumetric flow rate

Computation of average velocity given velocity profile v

$V = \frac{Q}{A}$.

For an elementary strip of area dA and velocity v, discharge (dQ) = vdA

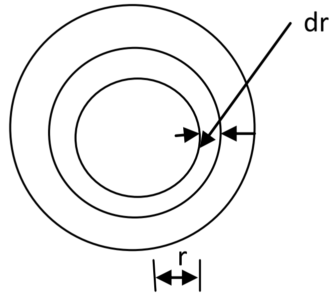
$Q = \int vdA$

$V = \frac{\int vdA}{A}$

Example: pipe (Radius R)

dA= 2πrdr

$V = \frac{\int_0^R vx2\pi rdr}{\pi R^2}$



4.6 Basic principle of fluid flows

I. Principle of conservation of mass

It states that mass can neither be created nor destroyed, i.e. total mass of a system remains constant. Continuity equation is derived from this principle.

II. Principle of conservation of energy

It states that energy can neither be created nor destroyed, i.e. total energy of a system remains constant. Energy equation is derived from this principle.

III. Principle of conservation of momentum

It states that the change in momentum of a body is equal to the product of force and time increment during which it acts. Momentum equation is derived from this principle.

4.7 Continuity equation for 1D steady flow

Mass of fluid entering = Mass of fluid leaving + Change in mass

Change in mass is zero for steady flow. In such case, Mass of fluid entering = Mass of fluid leaving

Consider cross-section of pipe at 1-1 and 2-2. Let ρ_1, V_1 and A_1 be the density, velocity and cross-sectional area at 1-1 and ρ_2, V_2 and A_2 be the density, velocity and cross-sectional area at 2-2.

Mass of fluid entering section 1-1 per unit time = $\rho_1 A_1 V_1$

Mass of fluid leaving section 2-2 per unit time = $\rho_2 A_2 V_2$

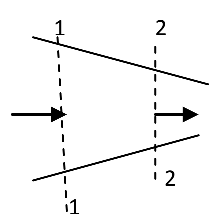
According to principle of conservation of mass,

$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$

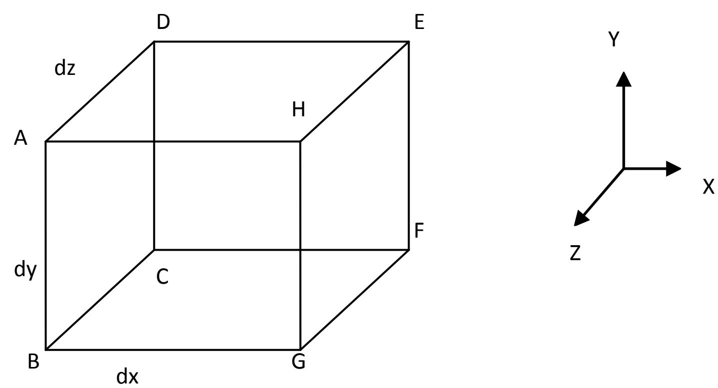
For constant density,

$A_1 V_1 = A_2 V_2$

$Q_1 = Q_2$



4.8 Continuity equation in 3D Cartesian co-ordinate



Consider a fluid element of length dx, dy and dz in X, Y and Z direction respectively. Let u, v and w be the velocity in X, Y and Z direction respectively, and ρ be the density of fluid.

Mass of fluid entering face ABCD per unit time = $\rho udydz$
Mass of fluid leaving face EFGH per unit time = $\rho udydz + \frac{\partial(\rho udydz)}{\partial x}dx$
Gain of mass in X-direction = $\rho udydz - \left[\rho udydz + \frac{\partial(\rho udydz)}{\partial x}dx\right] = -\frac{\partial(\rho u)}{\partial x}dxdydz$ (a)

Similarly,
Gain of mass in Y-direction = $-\frac{\partial(\rho v)}{\partial y}dxdydz$ (b)
Gain of mass in Z-direction = $-\frac{\partial(\rho w)}{\partial z}dxdydz$ (c)
Net gain of mass = $-\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]dxdydz$ (d)
Rate of increase of mass of fluid in the element = $\frac{\partial(\rho dxdydz)}{\partial t} = \frac{\partial \rho}{\partial t}dxdydz$ (e)
Equating d and e

$$-\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]dxdydz = \frac{\partial \rho}{\partial t}dxdydz$$

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = 0$$

This is the continuity equation in 3D Cartesian co-ordinate.
For steady flow, $\frac{\partial \rho}{\partial t} = 0$. The continuity equation is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

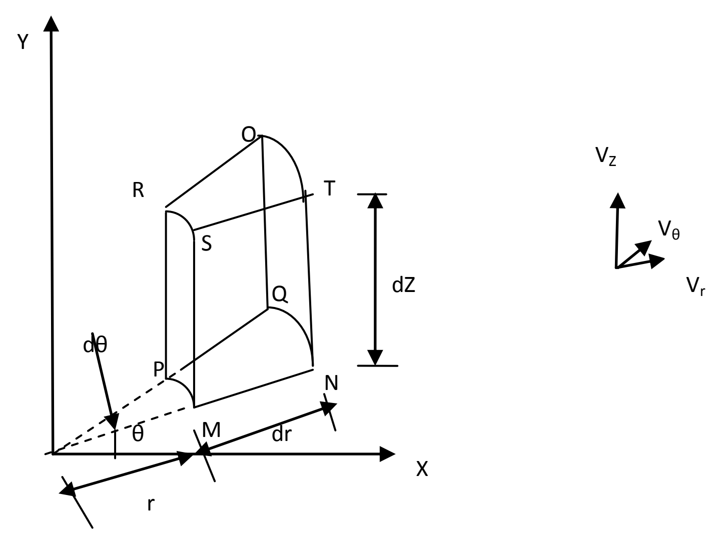
For incompressible flow, $\rho = const$. The continuity equation is

$$\frac{\rho\partial(u)}{\partial x}+\frac{\rho\partial(v)}{\partial y}+\frac{\rho\partial(w)}{\partial z}=0$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$
This is the 3D continuity equation for steady incompressible flow.

For 2D flow, w = 0. The continuity equation is
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

For 1D flow, v = 0, w = 0. The continuity equation is
$$\frac{\partial u}{\partial x}=0$$
u =const
If A is cross-sectional area, Au = Q = Const.

4.9 Continuity equation in 3D cylindrical polar co-ordinate



Consider any point M(r, θ, Z) in space. Let dr, dθ and dZ be small increments in r, θ and Z direction respectively. Let V_r, V_θ and V_z be the velocity in r, θ and Z direction respectively, and ρ be the density of fluid.

Mass of fluid entering face PQNM per unit time = $\rho V_z dr\, rd\theta$
Mass of fluid leaving face RSTO per unit time = $\rho V_z dr\, rd\theta+\frac{\partial(\rho V_z dr\, rd\theta)}{\partial z}dz$
Gain of mass in z-direction = $\rho V_z dr\, rd\theta-\left[\rho V_z dr\, rd\theta+\frac{\partial(\rho V_z dr\, rd\theta)}{\partial z}dz\right]=-\frac{\partial(\rho V_z dr\, rd\theta)}{\partial z}dz$ (a)

Similarly,

Gain of mass in r-direction $= -\frac{\partial(\rho V_r dz\, rd\theta)}{\partial r} dr$ (b)

Gain of mass in θ-direction $= -\frac{\partial(\rho V_\theta drdz\,)}{\partial \theta} d\theta$ (c)

Net gain of mass $= -\frac{\partial(\rho V_z dr\, rd\theta)}{\partial z} dz - \frac{\partial(\rho V_r dz\, rd\theta)}{\partial r} dr - \frac{\partial(\rho V_\theta drdz\,)}{\partial \theta} d\theta$

$= -\left[\frac{\partial(\rho V_z)}{\partial z} + \frac{\partial(\rho V_r r)}{r\partial r} + \frac{\partial(\rho V_\theta\,)}{r\partial \theta}\right] dr\, rd\theta\, dz$ (d)

Rate of increase of mass per unit time $= \frac{\partial(\rho drdz\, rd\theta)}{\partial t}$ (e)

Equating d and e

$$-\left[\frac{\partial(\rho V_z)}{\partial z} + \frac{\partial(\rho V_r r)}{r\partial r} + \frac{\partial(\rho V_\theta\,)}{r\partial \theta}\right] dr\, rd\theta\, dz = \frac{\partial(\rho drdz\, rd\theta)}{\partial t}$$

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho V_r r)}{r\partial r} + \frac{\partial(\rho V_\theta\,)}{r\partial \theta} + \frac{\partial(\rho V_z)}{\partial z}\right] = 0$$

This is the continuity equation in 3D cylindrical polar co-ordinate.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$. The continuity equation is

$$\frac{\partial(\rho V_r r)}{r\partial r} + \frac{\partial(\rho V_\theta\,)}{r\partial \theta} + \frac{\partial(\rho V_z)}{\partial z} = 0$$

For incompressible flow, $\rho = const$. The continuity equation is

$$\frac{\partial(V_r r)}{r\partial r} + \frac{\partial V_\theta}{r\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

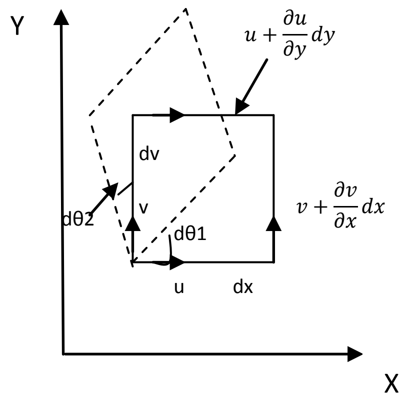
For 2D flow, $V_Z = 0$. The continuity equation is

$$\frac{\partial(V_r r)}{r\partial r} + \frac{\partial V_\theta}{r\partial \theta} = 0$$

4.10 Rotation and vorticity

Rotation of a fluid particle at a point is the average angular velocity of two differential linear elements of fluid particle originally perpendicular to each other.

2D



Consider the rotation of rectangular fluid element in 2D as shown in the figure. During time interval dt, the position of fluid is shown by dotted rectangle.

Angular velocity 1= $\lim_{\delta t\to 0}\frac{\delta\theta 1}{\delta t}=\frac{\left(v+\frac{\partial v}{\partial x}dx-v\right)\delta t}{\delta x\delta t}=\frac{\partial v}{\partial x}$

Angular velocity 2= $\lim_{\delta t\to 0}\frac{\delta\theta 2}{\delta t}=-\frac{\left(u+\frac{\partial u}{\partial y}dy-u\right)\delta t}{\delta y\delta t}=-\frac{\partial u}{\partial y}$

(clockwise: +, anticlockwise: - with respect to origin)

$$\omega_z = average\ of\ two\ velocity = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

Similarly,

$$\omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

Rotational components: (Curl of V)/2

i $\quad j$ $\quad k$

$\frac{\partial}{\partial x}$ $\quad\frac{\partial}{\partial y}$ $\quad\frac{\partial}{\partial z}$

u $\quad v$ $\quad w$

$\omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\ \omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

Vorticity is twice the rotation.

4.11 Velocity potential

Velocity potential, represented by φ, is a scalar function of space and time, such that its negative derivative with respect to any direction gives the fluid velocity in that direction.

$\varphi=f(x,y,z)$ for steady flow

Velocity component u, v and w is

$$u = -\frac{\partial\varphi}{\partial x}, v = -\frac{\partial\varphi}{\partial y}, w = -\frac{\partial\varphi}{\partial z}$$

(-ve sign indicates that φ decreases with the increase in x, y, z or flow is in the direction of decreasing φ)
Substituting u, v and w in continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial\varphi}{\partial z}\right) = 0$$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

This is known as Laplace equation.
The rotational components are

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For irrotational flow, the rotational component should be zero.

Properties of velocity potential

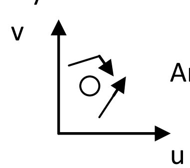
- If velocity potential exists, the flow should be irrotational. (The flow should be irrotational for φ to exist)
- If velocity potential satisfies Laplace equation, it represents the possible steady incompressible irrotational flow.

In polar co-ordinate

$$v_r = -\frac{\partial\varphi}{\partial r}, v_\theta = -\frac{\partial\varphi}{r\partial\theta}, v_z = -\frac{\partial\varphi}{\partial z}$$

4.12 Stream function

Stream function, represented by ψ, is a scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It cannot be expressed mathematically in 3D. Velocity component at right angles to the direction means it is for 2D.



Anticlockwise: +, clockwise: -

For 2D steady flow, ψ = f(x,y)

$$\frac{\partial\psi}{\partial x} = v, \qquad \frac{\partial\psi}{\partial y} = -u$$

The continuity equation for 2D is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting values of u and v

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0$$

$$-\frac{\partial^2 \psi}{\partial x \partial y}+\frac{\partial^2 \psi}{\partial x \partial y}=0$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component

$$\omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]=\frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z=0$, $\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0$ which is Laplace equation for ψ.

In polar coordinate

$v_\theta=\frac{\partial \psi}{\partial r}$, $v_r=-\frac{\partial \psi}{r\partial \theta}$

Properties

- If ψ exists, it is possible case of fluid flow which may be rotational or irrotational. (The continuity equation should be satisfied for ψ to exist.)
- If ψ satisfies Laplace equation, it is a possible case of an irrotational flow.

Relationship between φ and ψ

$u=-\frac{\partial \varphi}{\partial x}=-\frac{\partial \psi}{\partial y}$

$v=-\frac{\partial \varphi}{\partial y}=\frac{\partial \psi}{\partial x}$

The relationship is $\frac{\partial \varphi}{\partial x}=\frac{\partial \psi}{\partial y}$ and $-\frac{\partial \varphi}{\partial y}=\frac{\partial \psi}{\partial x}$. This relationship can be used to derive φ given ψ or to derive ψ given φ.

Alternatively, φ and ψ can be obtained from velocity components.

$$d\varphi=\frac{\partial \varphi}{\partial x}dx+\frac{\partial \varphi}{\partial y}dy$$

$$d\varphi=-udx-vdy$$

Integrate this expression to get φ.

$$d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy$$

$$d\psi=vdx-udy$$

Integrate this expression to get ψ.

After obtaining stream function ψ, streamline can be plotted by taking different values of ψ. ψ is constant along a streamline.

4.13 Equipotential line

An equipotential line is a line along which velocity potential is constant.

ϕ= const., dϕ = o

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy$$

$$0 = -udx - vdy$$

$\frac{dy}{dx} = -\frac{u}{v}$ = slope of equipotential line

4.14 Line of constant stream function

ψ = const., $d\psi$ =0

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$0 = vdx - udy$$

$\frac{dy}{dx} = \frac{v}{u}$ = slope of stream line

The product of slope of equipotential line and stream line is -1. That means lines equipotential lines are orthogonal to streamlines at all points of intersection.

4.15 Flow net

A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net.

For drawing flow net

- The flow should be steady
- The flow should be irrotational (no or negligible viscosity)
- The flow should not be governed by gravity. If there is gravity force, the correct shape of the free water boundary should be fixed first.

Plotting of flow net

Analytical method: Solving Laplace equation to obtain ϕ and ψ, and plotting

Graphical method: Streamlines are plotted graphically and a set of equipotential lines are drawn so as to intersect the streamlines perpendicularly.

CHAPTER 5. HYDRODYNAMICs

5.1 Introduction

Dynamics of fluid is governed by Newton's second law of motion.
Resultant force =mass x acceleration
$\sum F = ma$
where $\sum F$ = resultant external force acting on a fluid element, m = mass of fluid element, a = total acceleration. The acceleration vector has the direction of resultant force vector. Therefore, forces and acceleration can be resolved into three reference direction x, y and z.
$\sum F_x = ma_x, \sum F_y = ma_y, \sum F_z = ma_z$

Forces acting on a fluid in motion
a. Gravity force (F_g): due to weight of fluid
b. Pressure force (F_p): due to pressure of fluid
c. Viscous force (F_v): due to viscosity of flowing fluid
d. Turbulent force (F_t): due to turbulence of flow
e. Surface tension force (F_s): due to cohesive property of fluid
f. Compressibility force (F_c): due to elastic property of fluid

If a certain mass of fluid is influenced by all six forces, then the equation of motion is

$$F_g + F_p + F_v + F_t + F_s + F_c = ma$$

Resolving forces and acceleration in 3 directions,

$$F_{gx} + F_{px} + F_{vx} + F_{tx} + F_{sx} + F_{cx} = ma_x$$
$$F_{gy} + F_{py} + F_{vy} + F_{ty} + F_{sy} + F_{cy} = ma_y$$
$$F_{gz} + F_{pz} + F_{vz} + F_{tz} + F_{sz} + F_{cz} = ma_z$$

Simplification of complete equation of fluid motion

a. F_s and F_c are not significant in many cases. So they can be neglected.

$$F_g + F_p + F_v + F_t = ma$$

This is known as Reynolds' equation of motion, which is useful for the analysis of turbulent flows.
b. For laminar flow, F_t, F_s and F_c are insignificant. Neglecting these forces,

$$F_g + F_p + F_v = ma$$

This is known as Navier-Stoke's equation of motion, which is useful for the analysis of laminar flows.
c. For fluid having low viscosity or for ideal fluid, F_v, F_t, F_s and F_c are insignificant. Neglecting these forces,

$$F_g + F_p = ma$$

This is known as Euler's equation of motion.

Euler's equation of motion in 3D

$X_B - \frac{1}{\rho}\frac{\partial P}{\partial x} = a_x$

$Y_B - \frac{1}{\rho}\frac{\partial P}{\partial y} = a_y$

$Z_B - \frac{1}{\rho}\frac{\partial P}{\partial z} = a_z$

Here,

$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$

$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$

$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$

Navier-Stoke's equation in 3D

a. Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

b. Momentum equation

$X - \frac{1}{\rho}\frac{\partial P}{\partial X} + \frac{\mu}{\rho}\nabla^2 u = a_x$

$Y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\nabla^2 v = a_y$

$Z - \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\mu}{\rho}\nabla^2 w = a_z$

X_B, Y_B, Z_B: body force (gravity force) per unit mass in X, Y and Z direction

$\frac{\partial P}{\partial X}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}$ = pressure gradient in X, Y and Z direction

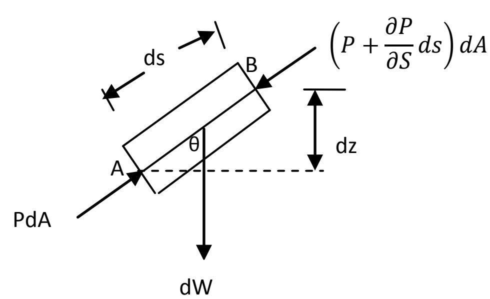
a_x, a_y, a_z: acceleration in X, Y and Z direction

μ = dynamic viscosity, ρ = density of fluid

u, v, w: velocity in X, Y and Z direction

∇^2 = Laplace operator = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

5.2 Euler's equation of motion along a streamline (1D Euler's equation)



Consider a streamline AB in which flow is taking place in S direction. Consider a cylindrical element of length ds and area dA. Let P be the pressure on left face and $\left(P + \frac{\partial P}{\partial s}ds\right)$ on the right face. Forces acting: pressure force and gravity force

Resultant force in S- direction = m a_s where a_s = acceleration in s direction

$$PdA - \left(P + \frac{\partial P}{\partial S}ds\right)dA - dW = ma_s$$

$$-\frac{\partial P}{\partial S}dsdA \; - \rho gdAdsCos\theta = \rho dAdsa_s$$

$$-\frac{\partial P}{\partial S} - \rho g\frac{dz}{ds} = \rho\frac{dV}{dt}$$

$$-\frac{\partial P}{\partial S} - \rho g\frac{dz}{ds} = \rho\left(\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial S}\right)$$

For steady flow $\frac{\partial V}{\partial t} = 0$

$$-\frac{\partial P}{\partial S} - \rho g\frac{dz}{ds} = \rho V\frac{\partial V}{\partial S}$$

Dividing by ρ

$$\frac{1}{\rho}\frac{\partial P}{\partial S} + g\frac{dz}{ds} + V\frac{\partial V}{\partial S} = 0$$

As all are function of S,

$$\frac{1}{\rho}\frac{dP}{dS} + g\frac{dz}{ds} + V\frac{dV}{dS} = 0$$

Or,

$$\frac{dP}{\rho} + gdZ + \; VdV = 0$$

This is 1D Euler's equation in differential form.

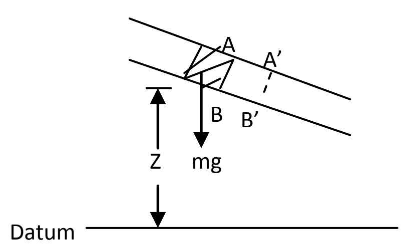
5.3 Different types of energies (heads) of liquid in motion

Energy is the capacity of doing work. Head is the energy per unit weight, having unit of length. Fluid possesses three types of energy (head):

a. Potential energy or elevation energy (potential head): Fluid possesses potential energy due to position or elevation above some datum line.

b. Kinetic energy (velocity head): Fluid possesses kinetic energy due to its velocity.

c. Pressure energy (pressure head): A steadily flowing stream of fluid can also do work because of its pressure. At any given cross-section, the pressure generates a force and, as the fluid flows, this cross-section will move forward and so work will be done.



Consider an element of fluid, at elevation Z above a datum, which is flowing with velocity V. Let m be the mass of fluid element, P is the pressure and A is the cross-sectional area.

Weight of element (W) =mg

Potential energy = mgZ

potential head = $\frac{mgZ}{mg}$ = Z

Kinetic energy = $\frac{1}{2}mV^2$

Velocity head = $\frac{1}{2}\frac{mV^2}{mg} = \frac{V^2}{2g}$

Force exerted on AB = PA

Work done = Force x distance AA' = $PdAds$= Pressure energy

Pressure head = $\frac{PdAds}{mg} = \frac{PdAds}{\rho dAdsg} = \frac{P}{\rho g}$

5.4 Bernoulli's equation (energy equation)

The Bernoulli's equation is a statement of the principle of conservation of energy along a streamline. It states that in a steady, ideal flow of an incompressible fluid, the total energy at any point of the fluid is constant.

Mathematically,

Pressure head + velocity head + elevation head = constant

$\frac{P}{\gamma} + \frac{V^2}{2g} + Z = const$

where $\frac{P}{\gamma}$ = pressure head, $\frac{V^2}{2g}$ = velocity head and Z = elevation head

Assumptions

- The fluid is ideal.
- The fluid is incompressible.
- The flow is steady and continuous.
- The flow is irrotational.
- Velocity is uniform over the section.

It relates the states at two points along a single streamline (not conditions on two different streamlines) All these conditions are impossible to satisfy at any instant in time. Fortunately, for many real situations where the conditions are approximately satisfied, the equation gives very good results.

5.5 Derivation of Bernoulli's equation from 1D Euler's equation

Euler's equation in 1D is

$\frac{dP}{\rho} + gdZ + VdV = 0$

Integrating

$\frac{1}{\rho}\int dP + \int gdZ + \int VdV = 0$

$\frac{P}{\rho} + gZ + \frac{V^2}{2} = const$

Dividing by g

$\frac{P}{\rho g}+\frac{V^2}{2g}+Z=const$

or, $\frac{P}{\gamma}+\frac{V^2}{2g}+Z=const$

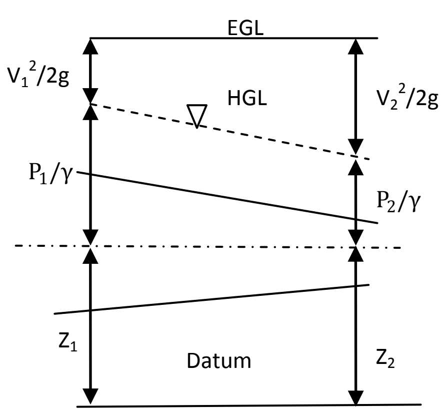
5.6 Hydraulic gradient line (HGL) and Energy gradient line (EGL)

Hydraulic gradient line (HGL) is the graphical representation of sum of potential head and pressure head of a flowing fluid with respect to some reference line. Energy gradient line (EGL) or total energy line (TEL) is the graphical representation of sum of total head, i.e. sum of potential head, pressure head and velocity head of a flowing fluid with respect to some reference line.

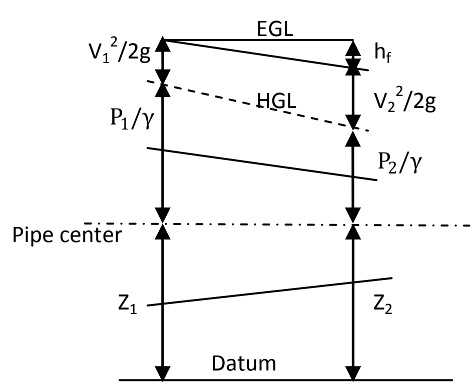
Features

- The HGL is the height to which the liquid would rise in a piezometer tube. So it is also called piezometric head.
- HGL may rise or fall depending on the pressure change.
- If the HGL drops below pipe elevation this means negative gauge pressures (i.e. less than atmospheric).
- HGL is always below the EGL and the vertical intercept between EGL and HGL is equal to $V^2/2g$.
- For a pipe of uniform cross section the slope of the HGL is equal to the slope of EGL.
- For open-channel flows, pressure is atmospheric (i.e. p = 0) at the surface. So, the HGL is the height of the free surface.
- For real fluid, EGL always drops in the direction of flow due to friction.

Bernoulli's equation at two sections



Ideal fluid



Real fluid

For ideal fluid (no friction loss), the Bernoulli's equation at section 1 and 2 is

$$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2$$

For real fluid, loss of head should be considered. The Bernoulli's equation at section 1 and 2 is

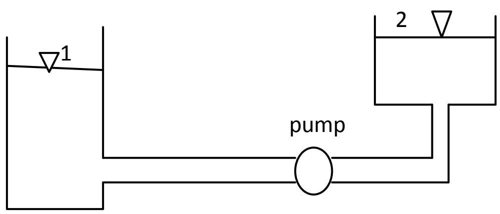
$$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2+h_f$$

where h_f = loss of head

The loss of head is mainly due to friction.

5.7 Bernoulli's equation with pump and turbine

Pump



If the pump supplies head (h_p), then energy equation for two end points is

E1+h_p = E2 + Loss of head between 1 and 2 (h_L)

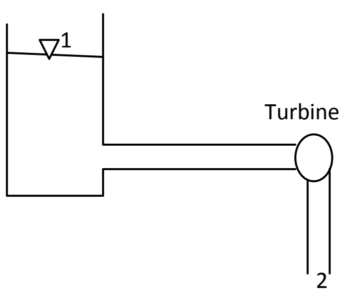
$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1+h_p=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2+h_L$

Energy supplied by the pump = $\gamma Q h_p$ (theoretical)

If η = efficiency

Actual energy supplied by the pump = $\gamma Q h_p/\eta$

Turbine



If the turbine extracts head (h_t), then energy equation for two end points is

E1-h_t = E2 + Loss of head between 1 and 2 (h_L)

$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1-h_t=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2+h_L$

Energy output = $\gamma Q h_t$

If η = efficiency

Actual energy output = $\eta\gamma Q h_t$

CHAPTER 6. FLOW MEASUREMENT

6.1 Orifice

An orifice is an opening on the side or at the bottom of a tank through which fluid flows. A mouthpiece is a short length of pipe attached to the orifice. The cross-section of opening may be rectangular, square, circular or triangular. The purpose of orifice is to measure discharge.

Classification of orifice

a. According to size

- Small: small dimension compared to head, head from center of orifice>5xdepth of orifice
- Large: large dimension compared to head, head from center of orifice<5xdepth of orifice

b. According to shape

- Rectangular
- Triangular
- Circular
- Square

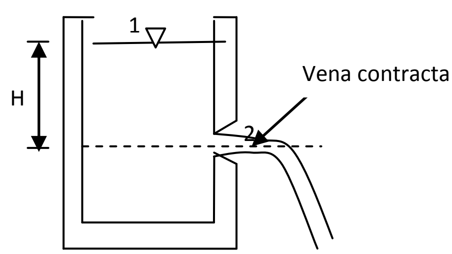
c. According to shape of upstream edge

- Sharp-edged
- Bell mouthed

d. According to discharge conditions

- Free discharging: discharge into atmosphere
- Submerged: discharge into another liquid

6.1.1 Flow through small orifice



The liquid flows through the orifice in the form of jet. The area of the jet of fluid goes on decreasing and becomes minimum. This contracted section is called vena-contracta. This section is approximately at a distance of half of diameter of the orifice. At this section, streamlines are parallel to each other. Beyond vena-contracta, the jet diverges and is attracted in the downward direction by gravity.

Consider point 1 on the surface of the liquid and point 2 at the center of the orifice. Let the flow is steady and at a constant head H.

Applying Bernoulli's equation at 1 and 2,

$$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2$$

Taking datum line through orifice, Z_1 = H, Z_2 =0

V_1 = 0 (negligible at surface), P_1 = P_2 = 0 (atmospheric pressure)

With these values, above equation becomes

$$0+0+H=0+\frac{V_2^2}{2g}+0$$

$$V_2=\sqrt{2gH}$$

This is the theoretical velocity and the actual velocity is less than this due to friction loss. Above equation is known as Toricelli's equation.

Hydraulic coefficients

a. Coefficient of velocity (C_v)

$$C_v=\frac{Actual\ Velocity\ of\ jet\ at\ vena-contracta}{Theoretical\ velocity}=\frac{V}{\sqrt{2gH}}$$

$$V=C_V\sqrt{2gH}$$

Normal value of C_v = 0.95-0.99

b. Coefficient of contraction (C_c)

$$C_c=\frac{Area\ of\ jet\ at\ vena-contracta}{Area\ of\ orifice}=\frac{a_c}{a}$$

Normal value of C_c = 0.61-0.69

c. Coefficient of discharge (C_d)

$$C_d=\frac{Actual\ discharge}{Theoretical\ discharge}$$

Normal value of C_d = 0.61-0.65

Relationship between C_v, C_c and C_d

$$C_d=\frac{Actual\ discharge}{Theoretical\ discharge}=\frac{Actual\ areaxactual\ velocity}{Theoretical\ areaxtheoretical\ velocity}$$

$$C_d=C_c\,C_v$$

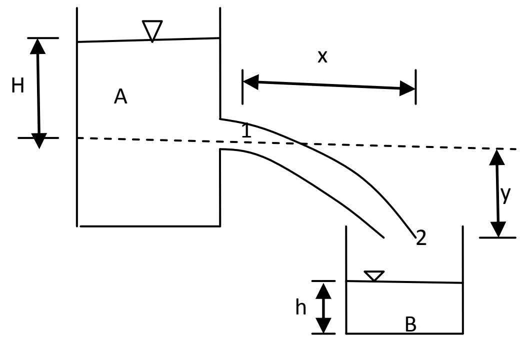
Discharge through orifice

Actual discharge (Q) = Actual area of jet x Actual velocity

$$=C_c a\ C_V\sqrt{2gH}$$

$$Q=C_d a\sqrt{2gH}$$

6.1.2 Determination of Hydraulic coefficients



Water from tank A is made to flow through an orifice under a constant head H and is collected in measuring tank B. Rise in water level (h) in tank B for a known time t is noted down. Let A = area of measuring tank B and a = area of orifice

Coefficient of discharge (C_d)

Actual discharge $(Q_a) = \frac{Ah}{t}$

Theoretical discharge $(Q_{th}) = a\sqrt{2gH}$

$C_d = \frac{Q_a}{Q_{th}} = \frac{Q_a}{a\sqrt{2gH}}$

Coefficient of velocity (C_v)

Let V =actual velocity at vena-contracta, x = horizontal distance between 1 and 2, y = vertical distance between 1 and 2.

x = V t

$t = \frac{x}{V}$ (a)

$y = \frac{1}{2}gt^2$ (b)

From a and b

$y = \frac{1}{2}g\left(\frac{x}{V}\right)^2$

$V = x\sqrt{\frac{g}{2y}}$

$C_v = \frac{Actual\ Velocity}{Theoretical\ velocity} = \frac{x\sqrt{\frac{g}{2y}}}{\sqrt{2gH}} = \frac{x}{\sqrt{4yH}}$

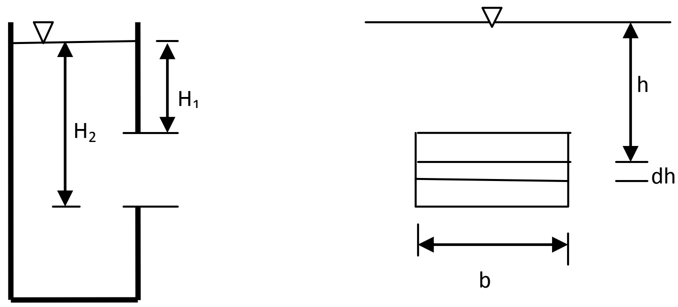
Coefficient of contraction (C_c)

$C_c = \frac{C_d}{C_v}$

For identical orifice, C_v is same.

6.1.3 Flow through large rectangular orifice

In large orifice, velocity of jet is not constant over the entire cross=section of the jet.



Consider a tank having large rectangular orifice discharging freely under a constant head. Let b = width of orifice, H_1 = Height of liquid above the upper edge of orifice, H_2 = Height of liquid above the lower edge of orifice.

Consider an elementary strip of thickness dh at depth h from free surface.
Area of strip =bdh
Velocity of flow through the strip = $\sqrt{2gh}$
Discharge through strip (dQ) = Area of strip x velocity = $bdh\sqrt{2gh}$
For the whole opening

$$Q = b\sqrt{2g}\int_{H_1}^{H_2} h^{1/2}\,dh$$

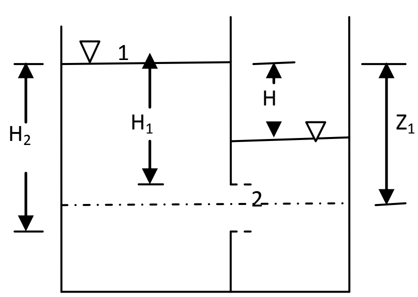
$$= \frac{2}{3}b\sqrt{2g}\left(H_2^{3/2} - H_1^{3/2}\right)$$

This is theoretical discharge.

$$Q_{actual} = \frac{2}{3}C_d\, b\sqrt{2g}\left(H_2^{3/2} - H_1^{3/2}\right)$$

6.1.4 Discharge through fully sub-merged orifice

In fully sub-merged (drowned) orifice, the whole of the outlet side of the orifice is submerged under liquid of same kind.



Let H_1 = height of liquid above the upper edge of orifice, H_2 = height of liquid above the lower edge of orifice, H= Difference in liquid level between two tanks, b = width of orifice, and C_d = coefficient of discharge

Consider point 1 on the liquid surface of tank 1 and point 2 at the center of orifice. Applying Bernoulli's equation at 1 and 2

$$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2$$

Taking datum through the center of orifice, $Z_2 = 0$

$V_1 = 0$, $P_1 = 0$ (atmospheric), $\frac{P_2}{\gamma}=Z_1-H$

Substituting above values in Bernoulli's equation

$$0+0+Z_1=Z_1-H+\frac{V_2^2}{2g}+0$$

$$V_2=\sqrt{2gH}$$

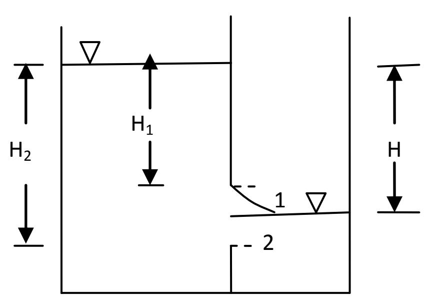
Discharge through orifice = C_d x Area of orifice x velocity

$$Q=C_d b(H_2-H_1)\sqrt{2gH}$$

or, $Q=C_d bd\sqrt{2gH}$ where d = depth of orifice

6.1.5 Discharge through partially submerged orifice

In partially sub-merged orifice, the outlet side of the orifice is partially submerged under liquid. This type of orifice has two portions: free discharging orifice at the upper part (1) and submerged orifice at the lower part (2).



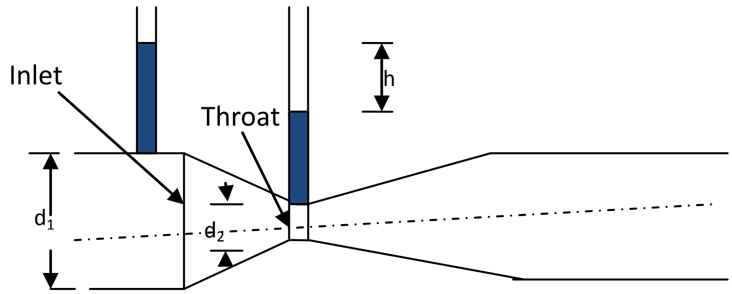
Let H_1 = height of liquid above the upper edge of orifice, H_2 = height of liquid above the lower edge of orifice, H= Difference in liquid level between two tanks, b = width of orifice, and C_d = coefficient of discharge

Discharge through orifice = Discharge through free portion (treated as large rectangular orifice) + Discharge through submerged portion

$$Q=\frac{2}{3}C_d\, b\sqrt{2g}\left(H^{3/2}-H_1^{3/2}\right)+C_d b(H_2-H)\sqrt{2gH}$$

6.2 Venturimeter

A venturimeter is a device to measure discharge through a pipe. In this instrument, the cross-sectional area of the flow passage is reduced to create pressure difference. Based on Bernoulli's equation together with continuity equation, the formula to compute discharge through venturimeter is derived.



The instrument consists of three parts.

- Inlet section followed by convergent cone: short pipe, angle: $21^0 \pm 1^0$, Length =2.7(d_1-d_2)
- Cylindrical throat: d_2 = 1/3 to 3/4 pipe diameter, length of throat = diameter of throat,
- Gradually divergent cone: angle: 5^0 -15^0 (preferably 6^0)

In the convergent part, the velocity increases rapidly in a short length without resulting in appreciable loss of energy. In the divergent part, deceleration takes place and if it is made shorter, the fluid particles will not remain in contact with the boundary i.e. the flow separates and eddies are formed resulting in excessive loss of energy. Hence, to avoid separation loss and consequent energy loss, it is made longer with gradual divergence. Due to separation problem, the pressure is measured at inlet and throat section. So, the divergent portion is not used in discharge measurement.

Let d_1, A_1, V_1, P_1 = Diameter, cross-sectional area, velocity and pressure at inlet
d_2, A_2, V_2, P_2 = Diameter, cross-sectional area, velocity and pressure at throat

Applying Bernoulli's equation at inlet and throat

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

Taking datum through the center of pipe, $Z_1 = Z_2 = 0$

$$\frac{P_1}{\gamma} - \frac{P_2}{\gamma} = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$\frac{P_1}{\gamma} - \frac{P_2}{\gamma} = h$ = piezometric head

$h = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$ (a)

From continuity equation

$$A_1V_1 = A_2V_2$$

$V_1 = \frac{A_2V_2}{A_1}$ (b)

From a and b

$$h = \frac{V_2^2}{2g} - \frac{\left(\frac{A_2V_2}{A_1}\right)^2}{2g} = \frac{V_2^2}{2g}\left(\frac{A_1^2 - A_2^2}{A_1^2}\right)$$

$$V_2 = \frac{A_1\sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}}$$

Theoretical Discharge $(Qth) = A_2V_2 = \frac{A_1A_2\sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}}$

Actual discharge $(Q) = C_d \frac{A_1A_2\sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}}$

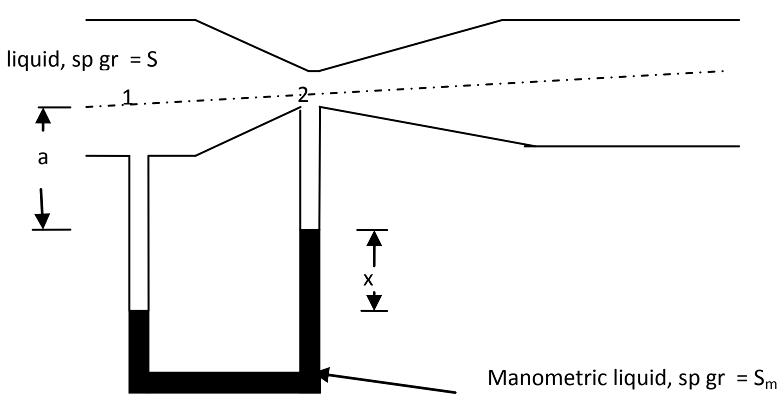
where C_d =Discharge coefficient of venturimeter

C_d = 0.96-0.98

For horizontal venturimeter, piezometric head: $h = \frac{P_1}{\gamma} - \frac{P_2}{\gamma}$

For vertical or inclined venturimeter, piezometric head: $h = \left(\frac{P_1}{\gamma} + Z_1\right) - \left(\frac{P_2}{\gamma} + Z_2\right)$

Relationship between piezometric head and deflection in U-tube manometer



Writing equation of pressure

$P_1 + \gamma(a + x) - \gamma_m\, x - \gamma\, a = P_2$

where γ = sp wt of liquid flowing through pipe, γ_m= sp wt of manometric liquid (heavier than fluid flowing through pipe), x = deflection of mercury

$$P_1 - P_2 = x(\gamma_m - \gamma)$$

$$P_1 - P_2 = \gamma x\left(\frac{\gamma_m}{\gamma} - 1\right)$$

$\frac{P_1 - P_2}{\gamma} = x\left(\frac{S_m}{S} - 1\right)$

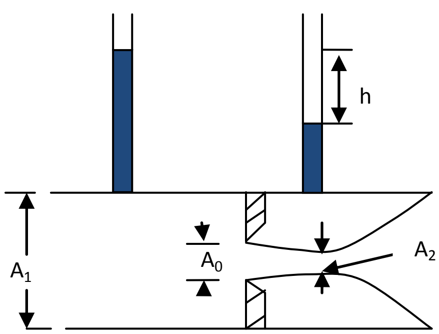
$h = x\left(\frac{S_m}{S} - 1\right)$

$h = x\left(\frac{S_m}{S} - 1\right)$ for $S_m > S$

$h = x\left(1 - \frac{S_m}{S}\right)$ for $S_m < S$

6.3 Orifice meter or orifice plate

An orifice meter is a simple device for measuring rate of flow through pipe. It is cheaper than venturimeter. It also works on the same principle as that of venturimeter. It consists of a flat circular plate which has a circular sharp edged hole called orifice, which is concentric with the pipe. The diameter of orifice is generally 0.5 times the diameter of pipe.



Similar to venturimeter, discharge through orificemeter (Q) is given by

$$Q = \frac{C A_0 A_1 \sqrt{2gh}}{\sqrt{A_1^2 - A_0^2}}$$

C = coefficient of discharge of orifice meter, A_1 = cross-sectional area at section 1, A_0 = Area of orifice, h = difference in piezometric head between 1 and 2

Merits and demerits of venturimeter and orificemeter

Merits of Venturimeter

- High coefficient of discharge due to low loss
- Useful in measuring flow through large size pipes
- Useful for higher rates of flow

Limitationof Venturimeter

- Not suitable in case of space limitation as the divergence part is long
- Higher cost of installation

Merits of orificemeter

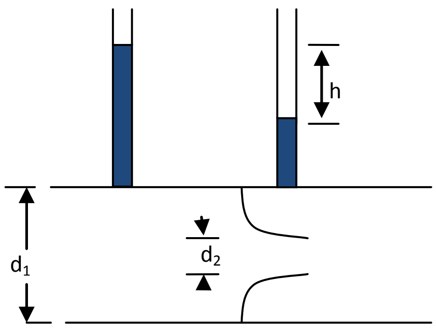
- Cheaper than venturimeter
- Requires less space

Limitations of orificemeter

- Low coefficient of discharge due to high loss

6.4 Nozzle meter or flow nozzle

Nozzlemeter is also used for measuring discharge through pipe. It consists of streamlined convergent nozzle through which the fluid is gradually accelerated. Therefore, a nozzle meter is essentially a venturimeter with the divergent part omitted, and hence the basic equation to compute discharge is derived in a similar manner as that of venturimeter.

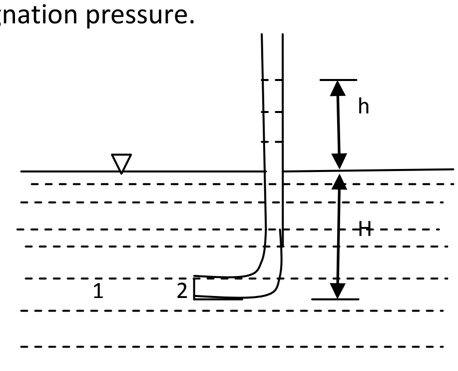


$$Q = C_d \frac{A_1 A_2 \sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}}$$

6.5 Pitot tube

Pitot tube is a device used to measure velocity of flow through a pipe or a channel. It is based on the principle that if the velocity of flow at a point becomes zero, the pressure there is increased due to the conversion of kinetic energy into pressure energy.

It consists of a glass tube bent at right angles. The lower end of the tube is placed in the direction of flow and the other end is open to the atmosphere. The pressure far away from the tube is known as static pressure and the pressure at the tip of the lower end of the tube, where the fluid velocity is reduced to zero, is called stagnation pressure.



Let P_1, V_1 = pressure and velocity at point 1

P_2, V_2 = Pressure and velocity at point 2 at the tip of pitot tube

h = rise of liquid above the free surface, H = depth of stem in the liquid

Applying Bernoulli's equation at 1 and 2

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$Z_1 = Z_2$, $V_2 = 0$, $\frac{P_1}{\gamma} = H, \frac{P_2}{\gamma} = H + h$

Substituting above values in Bernoulli's equation

$$H + \frac{V_1^2}{2g} = H + h$$

$H + \frac{V_1^2}{2g} = stagnation\ pressure$ = static + dynamic pressure

$$V_1 = \sqrt{2gh}$$

Actual velocity = $= C_v\sqrt{2gh}$

where C_v = Coefficient of pitot tube (usually 0.95)

6.6 Flow over notches or weirs

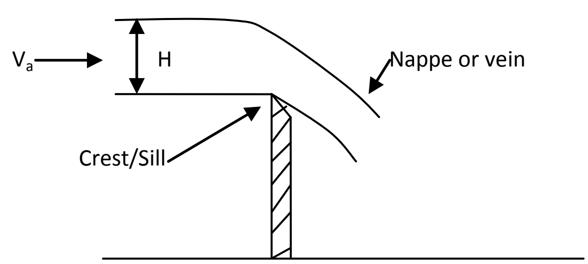
A weir or notch is an opening in the side of a tank or a small channel in such a way that the liquid surface in the tank or channel is below the top edge of opening. It is regarded as a large orifice.

A weir is an obstruction constructed across the channel through which flow occurs. It is used to raise the water level on the upstream side and to allow the excess water to flow over its entire length on the downstream side.

Difference between weir and notch

Weir	Notch
Bigger in size Used to measure large flow from a river or open channel Made of masonry or concrete	Smaller in size Used to measure small flow from a river or open channel Made of metallic plate

Parts



Crest/Sill: top of weir/notch

Nappe/vein: sheet of water flowing through a notch or over a weir

Head (H): Depth of water measure vertically above the crest level

Velocity of approach (V_a): velocity with which the water reaches the weir/notch before it flows over it

Classification of notches/weirs

a. According to shape of opening

Rectangular

Triangular

Trapezoidal

Circular

Compound

b. According to the effect of the sides on the nappe

Suppressed (without end contraction): crest length = width of channel

Contracted weir (With end contraction): crest length < width of channel

c. According to shape of crest

Sharp crested: crest is sharp, width of weir<Head/2

Narrow crested: crest is not sharp, width of weir<Head/2

Broad crested: Crest is wide, width of weir>Head/2

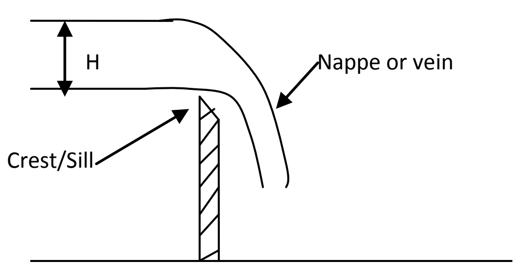
Ogee shaped: portion of dam over which water spills (spillway)

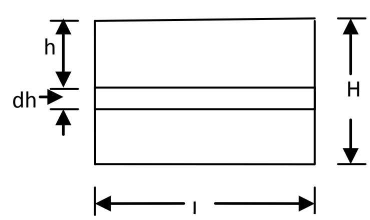
d. According to discharge conditions

Ordinary/freely discharging: downstream water level<crest level

Submerged/drowned: downstream water level>crest level

6.6.1 Discharge over a rectangular weir/notch





Section at crest

Consider a rectangular weir/notch over which water flows. Let H = Head of water over crest, L = Length of weir

Consider an elementary strip of thickness dh and length L at a depth h from the free surface.

Discharge (dQ) through the strip = C_d x Area of strip x Theoretical velocity

$$dQ = C_d\, Ldh\, \sqrt{2gh}$$

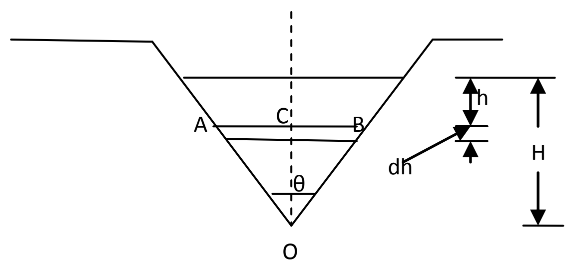
where C_d = coefficient of discharge

Total discharge over whole weir/notch is

$$Q = \int_0^H C_d\, L\, \sqrt{2g}\, \sqrt{h} dh$$

$$Q = \frac{2}{3} C_d\, L\, \sqrt{2g} H^{3/2}$$

6.6.2 Discharge over a triangular notch/weir (V-notch)



Consider a triangular notch/weir over which water flows. Let H = head of water above the notch/weir, θ = angle of notch. Consider and elementary strip of thickness dh at a depth h from the free surface of water.
In triangle AOC

$$Tan\frac{\theta}{2} = \frac{AC}{H-h}$$

$$AC\ = (H-h)Tan\frac{\theta}{2}$$

AB = 2 AC = $2(H-h)Tan\theta/2$
Discharge (dQ) through the strip = C_d x Area of strip x Theoretical velocity

$$dQ = C_d x\ 2(H-h)Tan\frac{\theta}{2}dh\ \sqrt{2gh}$$

where C_d = coefficient of discharge
Total discharge over whole weir/notch is

$$Q = \int_0^H 2C_d Tan\frac{\theta}{2}\sqrt{2gh}\ (H-h)dh$$

$$= 2C_d Tan\frac{\theta}{2}\sqrt{2g}\int_0^H \left(H\sqrt{h} - h^{3/2}\right)dh$$

$$= 2C_d Tan\frac{\theta}{2}\sqrt{2g}\left|\frac{2}{3}Hh^{3/2} - \frac{2}{5}h^{5/2}\right|_0^H$$

$$Q = \frac{8}{15}C_d\sqrt{2g}Tan\frac{\theta}{2}H^{5/2}$$

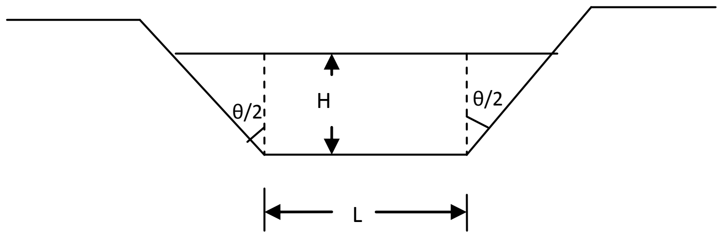
For right angled weir, θ = 90^0 and C_d =0.6

$$Q = 1.417H^{5/2}$$

Advantaged of V-notch over rectangular weir/notch

- Simple expression of Q for right angled weir
- More accurate for measuring low flow
- Only H is required for discharge computation
- No need of ventilation

6.6.3 Discharge over a trapezoidal notch/weir



Let H = Height of water over the notch, L = Length of the crest of the weir

Discharge over trapezoidal weir = Discharge over rectangular portion+ Discharge over two triangular portion

= Discharge over rectangular weir+ Discharge over single triangular weir

$$Q = \frac{2}{3} C_{d1}\, L \sqrt{2g} H^{3/2} + \frac{8}{15} C_{d2} \sqrt{2g}\, Tan \frac{\theta}{2} H^{5/2}$$

where C_{d1} = coefficient of discharge for rectangular portion and C_{d2} = coefficient of discharge for triangular portion

Effect on discharge over a notch/weir due to error in the measurement of H

Compute dQ/dH. Express dQ in terms of dH.

Find dQ/Q substituting the value of error dH.

<u>Consideration of Velocity of approach</u>

The velocity with which water reaches the weir/notch before it flows over it is called velocity of approach (V_a). Due to V_a, an additional head h_a acts on water flowing over the weir/notch.

Velocity head $(h_a) = \frac{V_a^2}{2g}$

This additional head should be considered while finding Q by integrating.

Initial head = h_a, final head = $H+h_a$

Discharge over rectangular weir considering velocity approach

$$Q = \frac{2}{3} C_d\, L \sqrt{2g} \left[(H + h_a)^{3/2} - h_a^{3/2} \right]$$

Discharge over triangular weir considering velocity approach

$$Q = \frac{8}{15} C_d \sqrt{2g}\, Tan \frac{\theta}{2} \left[(H + h_a)^{5/2} - h_a^{5/2} \right]$$

Discharge computation procedure considering velocity of approach

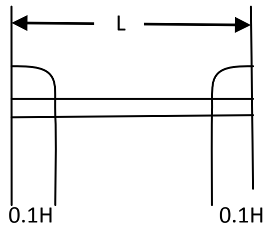
I. Compute Q without considering velocity of approach

II. V_a = Q/A where Q = preciously computed discharge, A = c/s area

III. Compute Velocity head $(h_a) = \frac{V_a^2}{2g}$ and compute new discharge Q by considering h_a. If newly computed Q is almost equal to previously computed Q, final Q = newly computed Q. Else repeat steps II-III until the two values of Q does not change significantly.

Effect of end contraction: Francis formula

If the crest length is not equal to the width of channel, the effect of end contraction should be considered.



Reduction in effective length and reduction in discharge due to end contraction

Effective length = L-0.2H for two end contractions

For n end contractions, discharge is

$$Q = \frac{2}{3} C_d \,(L - 0.1nH)\,\sqrt{2g} H^{3/2}$$

Considering velocity of approach: $Q = \frac{2}{3} C_d \,\sqrt{2g}\big(L - 0.1n(H + h_a)\big)\left[(H + h_a)^{3/2} - h_a^{3/2}\right]$

(If a weir is divided into bays by vertical post, 1 bay = 2 end contractions, L= total width-n x width of vertical post)

In general C_d = 0.623

Discharge without considering velocity of approach:

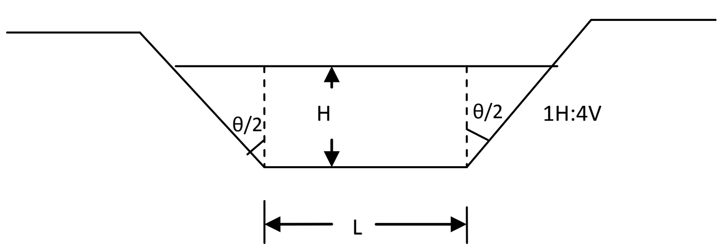
$$Q = \frac{2}{3} x0.623\,(L - 0.1nH)\,\sqrt{2x9.81} H^{3/2} = 1.84(L - 0.1nH)H^{3/2}$$

Discharge by considering velocity of approach:

$$Q = 1.84\big(L - 0.1n(H + h_a)\big)\left[(H + h_a)^{3/2} - h_a^{3/2}\right]$$

Experimental value of C_d (Chow): $C_d = 0.61 + 0.08\frac{H}{P}$ where H = head over crest and P = Height of weir/notch

6.6.4 Cippoleti weir/notch



Cippoleti weir is a trapezoidal weir having side slopes of 1H to 4V. Discharge through this weir is

$$Q = \frac{2}{3} C_d \,L\,\sqrt{2g} H^{3/2}$$

Proof: For a rectangular weir with two end contractions, the discharge is given by

$$Q = \frac{2}{3} C_d \,(L - 0.2H)\, \sqrt{2g} H^{3/2}$$
$$= \frac{2}{3} C_d \, L \sqrt{2g} H^{3/2} - \frac{2}{15} C_d \, \sqrt{2g} H^{5/2}$$

From the above expression, it is clear that discharge of a rectangular weir is reduced by $\frac{2}{15} C_d \, \sqrt{2g} H^{5/2}$ due to end contractions. This decrease in discharge is compensated by providing side slopes making the weir trapezoidal. The side slope should be such that the increase in discharge due to two triangular portions is equal to the loss of discharge due to end contraction.

Discharge through triangular = loss of discharge

$$\frac{8}{15} C_d \sqrt{2g} Tan \frac{\theta}{2} H^{5/2} = \frac{2}{15} C_d \, \sqrt{2g} H^{5/2}$$
$$Tan \frac{\theta}{2} = \frac{1}{4}$$
$$\frac{\theta}{2} = 14^0$$

6.6.5 Discharge over sharp crested weir

Rectangular weir: $Q = \frac{2}{3} C_d \, L \sqrt{2g} H^{3/2}$

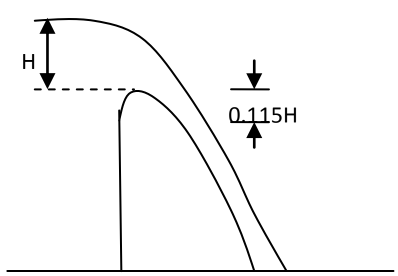
Triangular weir: $Q = \frac{8}{15} C_d \sqrt{2g} Tan \frac{\theta}{2} H^{5/2}$

6.6.6 Discharge over narrow crested weir

Discharge over narrow crested weir is same as discharge over rectangular weir

$$Q = \frac{2}{3} C_d \, L \sqrt{2g} H^{3/2}$$

6.6.7 Discharge over an ogee weir

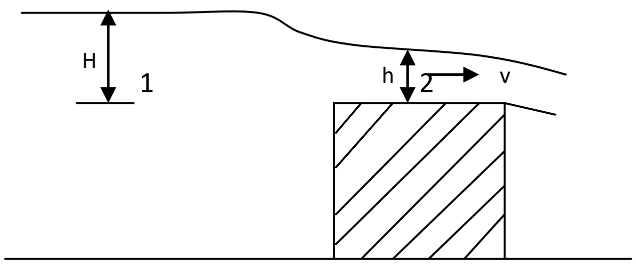


Crest of weir rises up to a maximum of 0.115H.

Discharge over an ogee weir is same as discharge over rectangular weir

$$Q = \frac{2}{3} C_d \, L \sqrt{2g} H^{3/2}$$

6.6.8 Discharge over broad crested weir



Let H = height of water above crest, L = Length of crest, h =height of water at the middle of weir which is constant, v = velocity of flow over weir

Applying Bernoulli's equation at 1 and 2

$$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2$$

$Z_1 = Z_2$, $V_1 = 0$, $\frac{P_1}{\gamma}=H$, $\frac{P_2}{\gamma}=h$, $V_2 = v$

Substituting above values

$$H=\frac{v^2}{2g}+h$$

$$v=\sqrt{2g(H-h)}$$

Discharge over weir = C_d x Area of flow x velocity

$$Q=C_d\,Lh\sqrt{2g(H-h)}$$

For maximum discharge, dQ/dh = 0

$$\frac{d\left(C_d\,Lh\sqrt{2g(H-h)}\right)}{dh}=0$$

$$\frac{d\left(\sqrt{(h^2H-h^3)}\right)}{dh}=0$$

$$\frac{1}{\sqrt{(h^2H-h^3)}}(2hH-3h^2)=0$$

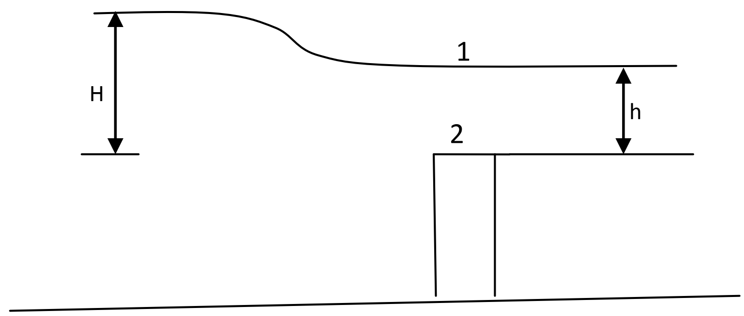
$$h=\frac{2}{3}H$$

Substituting the value of h

$$Q_{max}=C_d\,L\left(\frac{2}{3}H\right)\sqrt{2g\left(H-\left(\frac{2}{3}H\right)\right)}$$

$$Q_{max}=1.7C_d\,LH^{3/2}$$

6.6.9 Discharge over submerged (drowned weir)



The total discharge of submerged weir is computed by dividing into two portions. The portion between upstream and downstream weir surface is treated as free weir (1) and the portion between downstream water surface and crest of weir is drowned portion (2).

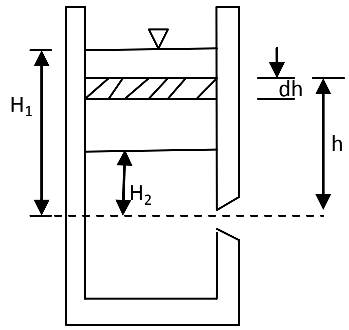
Discharge through submerged weir = Discharge through free portion + Discharge through drowned portion

$$Q = \frac{2}{3} C_{d1} L \sqrt{2g} (H - h)^{3/2} + C_{d2} L h \sqrt{2g(H - h)}$$

(Free portion: rectangular, drowned portion: broad-crested)

6.7 Emptying and filling of tank (Unsteady flow, variable head flow)

6.7.1 Without inflow



Consider a tank with orifice at the side, where there is no inflow. Let A = cross-sectional area of tank, a = cross-sectional area of orifice, H_1 = Initial height of liquid, H_2 = Final height of liquid, T = time to fall from H_1 to H_2. Let at any instant of time, the liquid surface is at height h above the orifice and let dh is the decrease of liquid surface in an interval of time dt.

According to continuity,
Volume of water leaving the tank = Volume of liquid flowing through the orifice

$$-Adh = Qdt$$

(-ve sign means head decreases with increase in t)

$$T = -\int_{H_1}^{H_2} \frac{Adh}{Q}$$

where Q= Discharge through orifice = $C_d a\sqrt{2gh}$ and C_d = Coefficient of discharge

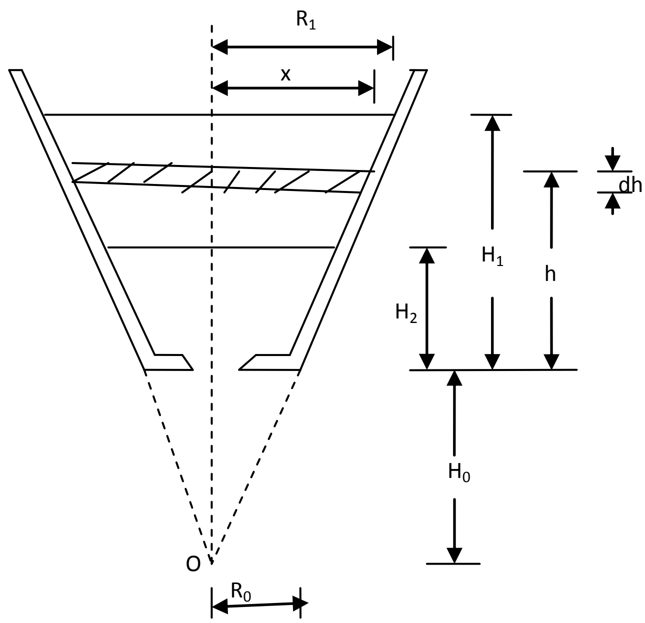
$$T = -\int_{H_1}^{H_2} \frac{Adh}{C_d a\sqrt{2gh}}$$

I. Time of emptying cylindrical tank without inflow

$$T = -\frac{A}{C_d a\sqrt{2g}}\int_{H_1}^{H_2} h^{-1/2}dh$$

$$T = -\frac{2A}{C_d a\sqrt{2g}}\left(\sqrt{H_2} - \sqrt{H_1}\right)$$

II. Time of emptying conical tank without inflow



Let A = cross-sectional area of conical tank, a = cross-sectional area of orifice, H_1 = Initial level of liquid, H_2 = Final level of liquid, R = radius of tank at H_1, R_0 = radius of tank at the bottom, T = time to fall from H_1 to H_2. Let at any instant of time, the liquid surface is at height h above the orifice and let dh is the decrease of liquid surface in an interval of time dt.

From similar triangles

$$\frac{R_1}{H_1 + H_0} = \frac{x}{h + H_0}$$

$$x = \frac{R_1(h + H_0)}{H_1 + H_0}$$

Time of emptying tank from H_1 to H_2

$$T = -\int_{H_1}^{H_2} \frac{Adh}{C_d a\sqrt{2gh}}$$

$$T = -\frac{1}{C_d a\sqrt{2g}} \int_{H_1}^{H_2} A\, h^{-1/2} dh \text{ (A also varies with h)}$$

$$= -\frac{1}{C_d a\sqrt{2g}} \int_{H_1}^{H_2} \pi x^2\, h^{-1/2} dh$$

$$= -\frac{1}{C_d a\sqrt{2g}} \int_{H_1}^{H_2} \pi \left[\frac{R_1(h + H_0)}{H_1 + H_0}\right]^2\, h^{-1/2} dh$$

$$= -\frac{1}{C_d a\sqrt{2g}}\frac{\pi R_1^2}{(H_1+H_0)^2}\int_{H_1}^{H_2}\left(h^{3/2}+2h^{1/2}H_0+h^{-1/2}H_0^2\right)dh$$

$$= K\left[\left|\frac{2}{5}h^{5/2}\right|_{H_1}^{H_2}+\frac{4}{3}H_0\left|h^{3/2}\right|_{H_1}^{H_2}+2H_0^2\left|h^{1/2}\right|_{H_1}^{H_2}\right]$$

where $K = \frac{1}{C_d a\sqrt{2g}}\frac{\pi R_1^2}{(H_1+H_0)^2}$

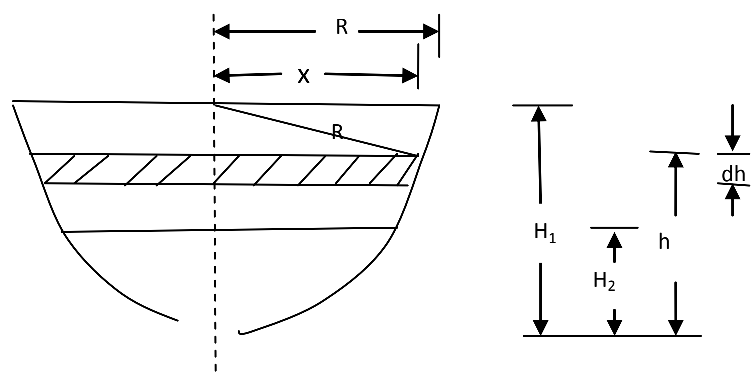
$$T = K\left[\frac{2}{5}\left(H_1^{5/2}-H_2^{5/2}\right)+\frac{4}{3}H_0\left(H_1^{3/2}-H_2^{3/2}\right)+2H_0^2\left(H_1^{1/2}-H_2^{1/2}\right)\right]$$

This is the expression to compute T. H_0 is found from similar triangles.

$$\frac{R_1}{H_1+H_0}=\frac{R_0}{H_0}$$

$$H_0 = \frac{R_0 H_1}{R_1-R_0}$$

III. Time of emptying hemispherical tank without inflow



Let A = cross-sectional area of hemispherical tank, a = cross-sectional area of orifice, H_1 = Initial level of liquid, H_2 = Final level of liquid, R = radius of tank at H_1, T = time to fall from H_1 to H_2. Let at any instant of time, the liquid surface is at height h above the orifice and let dh is the decrease of liquid surface in an interval of time dt.

$$R^2 = x^2+(R-h)^2$$

$$x^2 = 2Rh-h^2$$

Time of emptying tank from H_1 to H_2

$$T = -\int_{H_1}^{H_2}\frac{Adh}{C_d a\sqrt{2gh}}$$

$T = -\frac{1}{C_d a\sqrt{2g}}\int_{H_1}^{H_2} A\, h^{-1/2}dh$ (A also varies with h)

$$= -\frac{1}{C_d a\sqrt{2g}} \int_{H_1}^{H_2} \pi x^2\, h^{-1/2} dh$$

$$= -\frac{1}{C_d a\sqrt{2g}} \int_{H_1}^{H_2} \pi\,(2Rh - h^2) h^{-1/2} dh$$

$$= -\frac{\pi}{C_d a\sqrt{2g}} \int_{H_1}^{H_2} \left(2Rh^{1/2} - h^{3/2}\right) dh$$

$$= -\frac{\pi}{C_d a\sqrt{2g}} \left[2R\left|\frac{2}{3}h^{3/2}\right|_{H_1}^{H_2} - \left|\frac{2}{5}h^{5/2}\right|_{H_1}^{H_2}\right]$$

$$= \frac{\pi}{C_d a\sqrt{2g}} \left[\frac{4R}{3}\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right)\right]$$

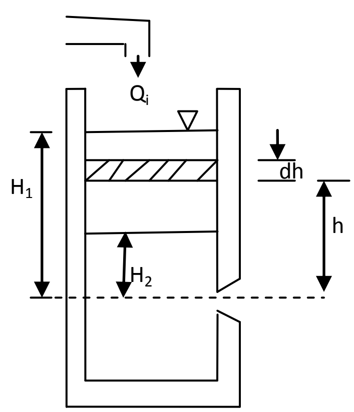
For completely emptying the tank, $H_2 = 0$

$$t = \frac{\pi}{C_d a\sqrt{2g}} \left[\frac{4R}{3} H_1^{3/2} - \frac{2}{5} H_1^{5/2}\right]$$

In this condition $H_1 = R$

$t = \frac{\pi}{C_d a\sqrt{2g}} \left[\frac{4R}{3} R^{3/2} - \frac{2}{5} R^{5/2}\right] = \frac{14}{15} \frac{\pi}{C_d a\sqrt{2g}} R^{5/2}$

6.7.2 Emptying and filling of tank with inflow



Consider a tank with orifice at the side, which is supplied with constant rate of inflow Q_i. Let A = cross-sectional area of tank, a = cross-sectional area of orifice, H_1 = Initial height of liquid, H_2 = Final height of liquid, T = time to fall from H_1 to H_2, Q_0 = outflow. Let at any instant of time, the liquid surface is at height h above the orifice and let dh is the increase of liquid surface in an interval of time dt.

According to continuity,
Volume of inflow- volume of outflow = Volume added to the tank

$$Q_i dt - Q_0 dt = A dh$$

$$dt = \frac{Adh}{Q_i - Q_0}$$
$$dt = \frac{Adh}{Q_i - C_d a\sqrt{2gh}}$$
Let $K = C_d a\sqrt{2g}$
$$dt = \frac{Adh}{Q_i - K\sqrt{h}}$$
$$T = \int_{H_1}^{H_2} \frac{Adh}{Q_i - K\sqrt{h}} \qquad \text{(a)}$$
Let $Q_i - K\sqrt{h} = Z$
$$h = \frac{(Q_i - Z)^2}{K^2}$$
$dh = -\frac{2(Q_i - Z)}{K^2} dZ$ (b)

From a and b
$$T = \int_{H_1}^{H_2} -\frac{2A(Q_i - Z)}{K^2}\frac{1}{Z}dZ$$
For h = H_1, $Z = Q_i - K\sqrt{H_1}$

For h = H_2, $Z = Q_i - K\sqrt{H_2}$
$$T = \int_{Q_i - K\sqrt{H_1}}^{Q_i - K\sqrt{H_2}} -\frac{2A(Q_i - Z)}{K^2}\frac{1}{Z}dZ$$
$$T = -\frac{2A}{K^2}|(Q_i lnZ - Z)|_{Q_i - K\sqrt{H_1}}^{Q_i - K\sqrt{H_2}}$$
$$T = -\frac{2A}{K^2}\left\{Q_i\left(ln\left(Q_i - K\sqrt{H_2}\right) - ln\left(Q_i - K\sqrt{H_1}\right)\right) - \left[Q_i - K\sqrt{H_2} - Q_i + K\sqrt{H_1}\right]\right\}$$
$$T = \frac{2A}{K^2}\left[Q_i ln\left(\frac{Q_i - K\sqrt{H_1}}{Q_i - K\sqrt{H_2}}\right) + K\left(\sqrt{H_1} - \sqrt{H_2}\right)\right]$$

CHAPTER 7. MOMENTUM PRINCIPLE

7.1 Introduction

Momentum principle (Conservation of momentum) is a modified form of Newton's second law of motion. According to Newton's second law of motion,
Force acting on fluid mass = rate of change of momentum

$$F = \frac{d(mv)}{dt}$$

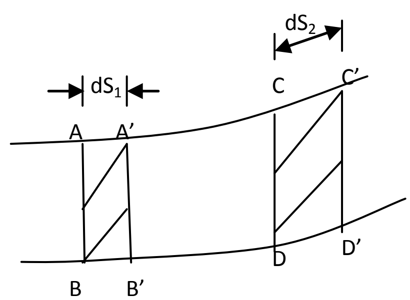
where F = force acting on fluid mass, m = mass of fluid, v = velocity of fluid and mv = momentum
Above equation may also be written as

$$F\,dt = d(mv)$$

In this equation, F dt = impulse of applied force and d(mv) = resulting change of momentum. This equation is known as impulse-momentum equation, which states that impulse of a force F acting on a fluid mass of m in a short interval of time dt is equal to the change of momentum d(mv) in the direction of flow.

7.2 Expression of force based on Impulse momentum equation for fluid flow

The above impulse momentum equation is applicable to finite or discrete bodies, for which the action of any force may take place and be completed in a finite period of time. For continuous motion of fluid, the equation to compute force is derived in the following way.



Consider a fluid flowing through a closed conduit under steady condition. Consider fluid mass enclosed between section AB and CD. Let ρ_1, V_1, A_1 = density, mean velocity and cross-sectional area at AB, ρ_2, V_2, A_2 = density, mean velocity and cross-sectional area at CD. Under the effect of external force, let the fluid mass at AB and CD shifts to new position A'B' and C'D' after short interval of time dt. dS_1 and dS_2 = width of region AA'BB' and CC'DD'.

Momentum of fluid entering section AB in time dt = mass of fluid at AB x V_1

$$= \rho_1 A_1 dS_1 V_1 = \rho_1 A_1 \frac{dS_1}{dt} dt\, V_1 = \rho_1 A_1\, V_1 dt\, V_1 = \rho_1 A_1 V_1^2 dt$$

Similarly, Momentum of fluid leaving section CD in time dt = $\rho_2 A_2 V_2^2 dt$

Change in momentum $d(mv) = (\rho_2 A_2 V_2^2 - \rho_1 A_1 V_1^2)dt$

From impulse-momentum equation,

$$F\,dt = d(mv)$$
$$F\,dt = (\rho_2 A_2 V_2^2 - \rho_1 A_1 V_1^2)dt$$
$$F = (\rho_2 A_2 V_2^2 - \rho_1 A_1 V_1^2)$$

For incompressible flow, $\rho_1 = \rho_2 = \rho = const$

$A_1 V_1 = A_2 V_2 = Q$

With these simplifications,

$$F = (\rho Q V_2 - \rho Q V_1) = \rho Q (V_2 - V_1)$$

where F is the sum of forces acting in any direction.

This is the force exerted by the boundary of conduit on flowing fluid. According to Newton's third law of motion, fluid also exerts equal force in opposite direction.

Practical applications of momentum equations

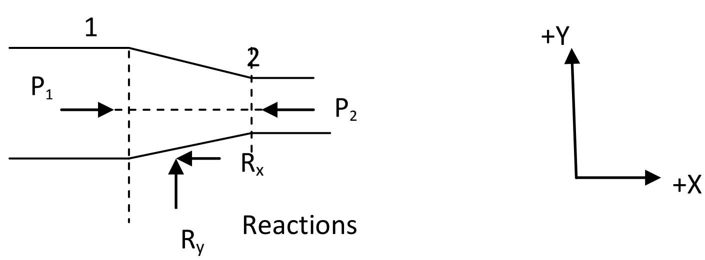
Pipe bends and reducers

Jet propulsion

Moving vane

7.3 Example of force computation

Force on reducer



$\sum Forces\ in\ X\ direction = Rate\ of\ change\ of\ momentum\ in\ X\ direction$

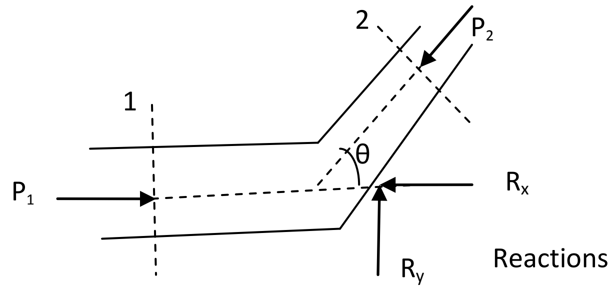
$(P_1\,A_1 - P_2\,A_2) - R_x = \rho Q (V_{2x} - V_{1x})$

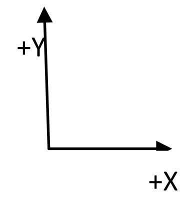
$(P_1\,A_1 - P_2 A_2) - R_x = \rho Q (V_2 - V_1)$

$R_x = (P_1\,A_1 - P_2 A_2) + \rho Q (V_1 - V_2)$

This is the force exerted by the reducer on the fluid. The force exerted by the fluid on the reducer is equal in magnitude and opposite in direction.

Example: Force on pipe bend





$\sum Forces\ in\ X\ direction = Rate\ of\ change\ of\ momentum\ \ in\ X\ direction$
$(P_1\ A_1 - P_2 Cos\theta\ A_2) - R_x = \rho Q(V_{2x} - V_{1x})$
$(P_1\ A_1 - P_2 A_2 Cos\theta) - R_x = \rho Q(V_2 Cos\theta - V_1)$
$F_x = (P_1\ A_1 - P_2 A_2 Cos\theta) + \rho Q(V_1 - V_2 Cos\theta)$

$\sum Forces\ in\ Y\ direction = Rate\ of\ change\ of\ momentum\ \ in\ Y\ direction$
$R_y - P_2 Sin\theta\ A_2 = \rho Q\left(V_{2y} - V_{1y}\right)$
$R_y - P_2\ A_2 Sin\theta = \rho Q(V_2 Sin\theta - 0)$
$R_y = P_2\ A_2 Sin\theta + \rho Q V_2 Sin\theta$

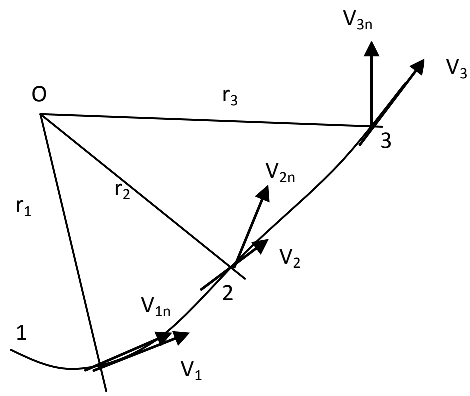
If weight of fluid contained in the bend (W) is considered
$F_y - P_2 Sin\theta\ A_2 - W = \rho Q\left(V_{2y} - V_{1y}\right)$
$F_y - P_2\ A_2 Sin\theta - W = \rho Q(V_2 Sin\theta - 0)$
$F_y = P_2\ A_2 Sin\theta + \rho Q V_2 Sin\theta + W$
For computing unknowns, application of continuity and Bernoulli's equation may also be required depending on the given data.

Resultant force $(F_R) = \sqrt{F_x^2 + F_y^2}$
Resultant force exerted by the water on the bend is equal in magnitude and opposite in direction.
Direction of resultant force = $Tan^{-1}\frac{F_y}{F_x}$

7.4 Moment of momentum

Linear momentum deals with the relationship between linear momentum and force, whereas angular momentum deals with the relationship between angular momentum and torque.

According to moment of momentum principle,
Resulting torque acting on a rotating fluid = rate of change of moment of momentum



Consider a fluid particle of density ρ moving along a curved path. Consider a fixed point O on a surface. Let r_1, r_2, r_3 = radius of curvature at 1, 2 and 3; V_1, V_2, V_3 = tangential velocity at 1, 2 and 3; V_{1n}, V_{2n}, V_{3n} = normal velocity with respect to point O at 1, 2 and 3.

Moment of momentum = Momentum x moment arm
Moment of momentum at 1 = $mV_{1n}r_1$
Moment of momentum at 2 = $mV_{2n}r_2$

Change of angular momentum = $m(V_{2n}r_2 - V_{1n}r_1)$
Rate of change angular momentum = $\frac{m}{t}(V_{2n}r_2 - V_{1n}r_1)$

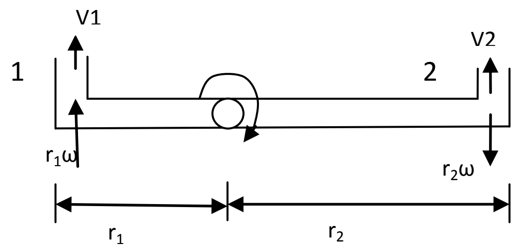
$$= \frac{\rho x Volume}{t}(V_{2n}r_2 - V_{1n}r_1)$$

$$Torque = \rho Q(V_{2n}r_2 - V_{1n}r_1)$$

This is the moment of momentum equation.

Application of moment of momentum equation
Flow in turbines and centrifugal pumps
Torque exerted by water on sprinkler

Example of sprinkler
Assume no friction at the pivot.



Discharge through each nozzle = Q
Relative velocity at outlet of each nozzle= V_1, V_2

a. For torque (T) = 0, Angular speed of rotation (ω)=?
Initial moment of momentum of fluid entering the sprinkler is zero. So torque exerted is equal to the final moment of momentum. As no external torque acts (no friction), final moment of momentum should also be zero.
Jet exerts force in opposite direction at nozzle 1 and 2 (downward direction).
Torque at 1: anticlockwise, torque at 2: clockwise.
As the torque arm for 2 is greater, the sprinkler will rotate clockwise if free to rotate.

Absolute velocity at 1 (V_{1a}) = V_1 + r_1 ω (tangential velocity and relative velocity in the same direction)
Absolute velocity at 1 (V_{2a}) = V_2- r_2 ω (tangential velocity and relative velocity in opposite direction)
Final moment of momentum = $\rho Q V_{2a}\, r_2 - \rho Q V_{1a}\, r_1 = 0$
(Two torques in opposite direction, net torque = greater torque-smaller torque)

b. For ω =0, velocities are V_1 and V_2.
Torque exerted by the water on sprinkler = $\rho Q V_2 r_2 - \rho Q V_1 r_1$

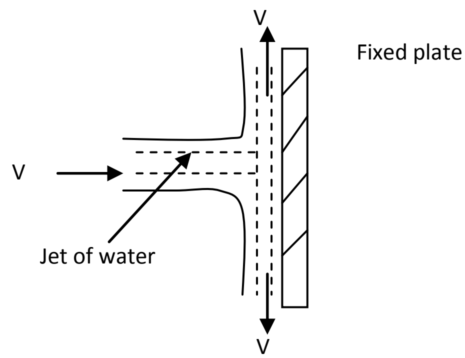
7.5 Impact of jets

The force exerted by the jet of flowing liquid on a plate is computed by using the impulse-momentum equation.

Assumptions
No friction between jet and plate
No energy loss due to impact of jet
Same velocity of jet before and after striking

7.5.1 Force exerted by jet on stationary plate and vane (curved plate)

I) Stationary vertical plate



Consider a jet of water moving with velocity V strikes a stationary vertical plate. After striking the jet gets deflected by 90^0. Let A = c/s area of jet.

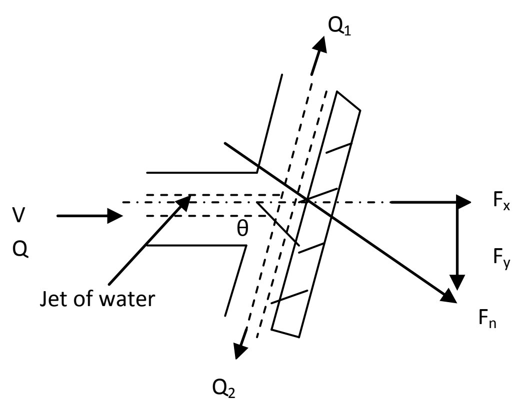
Velocity of jet before striking in X-direction (V_{1x}) = V

Velocity of jet after striking in X-direction (V_{2x}) = 0

Force exerted by the jet on the plate in X-direction $(F_x) = \rho Q(V_{1x} - V_{2x}) = \rho AV(V - 0)$

$$= \rho AV^2$$

II) Stationary inclined plate



Consider a jet of water moving with velocity V strikes an inclined stationary plate. Let θ = angle between jet and plate, F_n = force exerted by the jet on the plate in normal direction. Let A = c/s area of jet.

Velocity of jet before striking in normal direction (V_{1n}) = Vcos(90-θ)=Vsinθ

Velocity of jet after striking in normal direction (V_{2n}) = 0

Force exerted by the jet on the plate in normal direction $(F_n) = \rho Q(V_{1n} - V_{2n}) = \rho AV(Vsin\theta - 0)$

$$= \rho AV^2 sin\theta$$

Force in X-direction $(F_x) = F_n sin\theta = \rho AV^2 sin^2\theta$

Force in X-direction $(F_y) = F_n cos\theta = \rho AV^2 sin\theta cos\theta$

Division of flow

Momentum equation along inclined plate in the direction of Q_1 (resultant force = 0)

$$(\rho Q_1 V_1 - \rho Q_2 V_2) - \rho Q V cos\theta = 0$$

$V_1 = V_2 = V$. Above equation reduces to

$Q_1 - Q_2 = Qcos\theta$ (a)

From continuity equation,

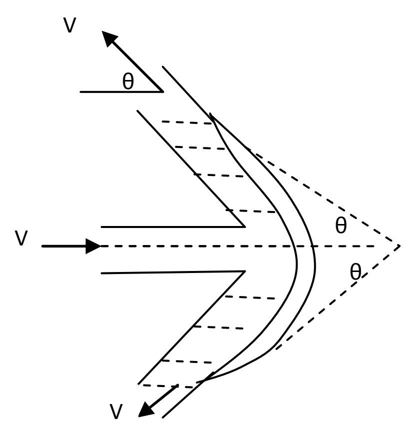
$Q_1 + Q_2 = Q$ (b)

Solving a and b

$Q_1 = \frac{Q}{2}(1 + cos\theta)$, $Q_2 = \frac{Q}{2}(1 - cos\theta)$

III) Stationary curved vane

a. Jet striking curved plate at center



Consider a jet moving with velocity V strikes a curved plate at its center. Let θ = angle made by jet with tangent to the plate.

Force exerted by the jet on the plate in X-direction $(F_x) = \rho Q(V_{1x} - V_{2x}) = \rho AV(V - (-Vcos\theta))$

$$= \rho AV^2(1 + cos\theta)$$

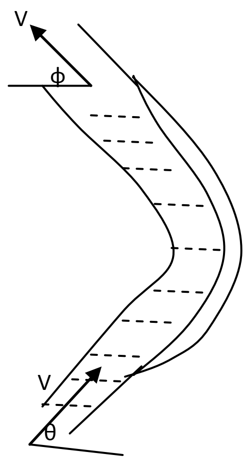
(another way: $F_x = \rho QV - \left(\rho Q_1(-Vcos\theta) + \rho Q_2(-Vcos\theta)\right) = \rho QV - \rho(Q_1 + Q_2)(-Vcos\theta)$

$\rho QV + \rho QVcos\theta = \rho AV^2(1 + cos\theta)$

Force exerted by the jet on the plate in Y-direction $\left(F_y\right) = \rho Qx0 - (\rho Q_1 Vsin\theta + \rho Q_2(-Vsin\theta))$=0

θ = 90^0: flat plate, θ = 0^0: semicircular

b. Jet striking unsymmetrical curved plate at one end tangentially



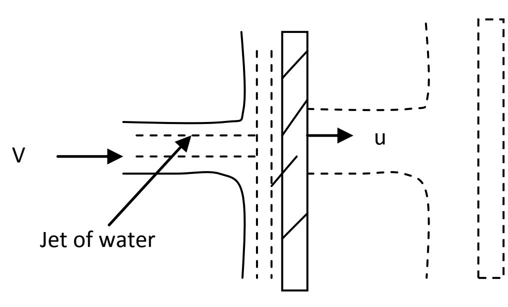
Consider a jet moving with velocity V strikes an unsymmetrical curved plate at one end tangentially. Let θ = angle made by jet with X-axis at the inlet, ϕ = angle made by jet with X-axis at the outlet.

Force exerted by the jet on the plate in X-direction $(F_x) = \rho Q(V_{1x} - V_{2x}) = \rho AV(Vcos\theta - (-Vcos\varphi))$
$= \rho AV^2(cos\theta + cos\varphi)$
Force exerted by the jet on the plate in Y-direction $(F_y) = \rho Q(V_{1y} - V_{2y}) = \rho AV(Vsin\theta - Vsin\varphi)$
$= \rho AV^2(sin\theta - sin\varphi)$

In case of symmetrical vane, θ = ϕ
$F_x = 2\rho AV^2 cos\theta, F_y = 0$

7.5.2 Force exerted by jet on moving plate

I) Moving vertical plate



Consider a jet moving with velocity V strikes a flat vertical plate, which is moving with velocity u. The velocity with which the jet strikes the plate will be the relative velocity V-u.

Force exerted by the jet in X-direction (F_x) = mass of water striking/sec x ($V_{1x} - V_{2x}$)

$$= \rho A(V-u)[(V-u)-0] = \rho A(V-u)^2$$

Work done per sec by the jet on the plate = $Force\ \frac{Distance\ in\ the\ direction\ of\ flow}{time}$

$$= \rho A(V-u)^2 u$$

Efficiency (η) = $\frac{Work\ done\ per\ sec\ by\ jet}{Energy\ supplied\ by\ jet\ per\ sec} = \frac{\rho A(V-u)^2 u}{\frac{1}{2}\rho AVV^2} = \frac{2u(V-u)^2}{V^3}$

For maximum efficiency, $\frac{d\eta}{du} = 0$

$\frac{d}{du}\left(\frac{2(uV^2-2u^2V+u^3)}{V^3}\right) = 0$

$V^2 - 4uV + 3u^2 = 0$

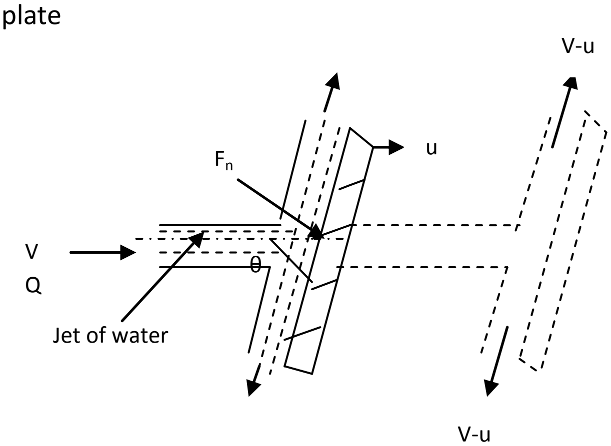
(V-u)(v-3u) = 0

V=u (work done =0)

V=3u

$\eta = \frac{2u(V-u)^2}{V^3} = \frac{2u(3u-u)^2}{(3u)^3} = \frac{8}{27}$

II) Moving inclined plate



V= Velocity of jet, u = velocity of plate

Force exerted by the jet on the plate in normal direction (F_n) = Mass of water striking/sec x ($V_{1n} - V_{2n}$)

$$= \rho A(V-u)[(V-u)sin\theta - 0] = \rho A(V-u)^2 sin\theta$$

Force in X-direction $(F_x) = F_n sin\theta = \rho A(V-u)^2 sin^2\theta$

Force in X-direction $(F_y) = F_n cos\theta = \rho A(V-u)^2 sin\theta cos\theta$

Work done per sec by the jet on the plate = $F_x u$

$$= \rho A(V-u)^2 sin^2\theta\ u$$

Efficiency (η) = $\frac{Work\ done\ per\ sec\ by\ jet}{Energy\ supplied\ by\ jet\ per\ sec} = \frac{\rho A(V-u)^2 sin\theta\ u}{\frac{1}{2}\rho AVV^2} = \frac{2u(V-u)^2 sin^2\theta}{V^3}$

For maximum efficiency, $\frac{d\eta}{du} = 0$

$\frac{d}{du}\left(\frac{2(uV^2-2u^2V+u^3)sin^2\theta}{V^3}\right) = 0$

$V^2 - 4uV + 3u^2 = 0$

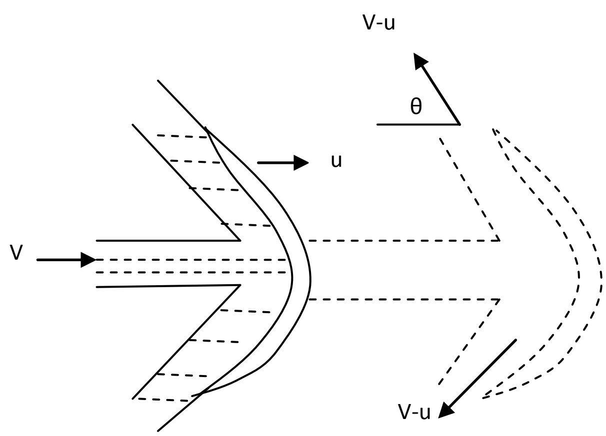
(V-u)(v-3u) = 0

V=u (work done =0)

V=3u

$\eta = \frac{2u(V-u)^2}{V^3} = \frac{2u(3u-u)^2 sin^2\theta}{(3u)^3} = \frac{8}{27} sin^2\theta$

III) Moving symmetrical curved vane, jet striking at center



Force exerted by the jet on the plate in X-direction = Mass of water striking/sec x ($V_{1x} - V_{2x}$)

$$F_x = \rho A(V-u)(V_{1x} - V_{2x}) = \rho A(V-u)[(V-u) — (V-u)cos\theta)$$
$$= \rho A(V-u)^2(1 + cos\theta)$$

Force exerted by the jet on the plate in Y-direction = Mass of water striking/sec x ($V_{1y} - V_{2y}$)

$$F_y = \rho A(V-u)(V_{1y} - V_{2y}) = \rho A(V-u)(0 - (V-u)sin\theta) == -\rho A(V-u)^2 sin\theta$$

Work done per sec by the jet on the plate = $F_x u$

$$= \rho A(V-u)^2(1 + cos\theta)\, u$$

Efficiency (η) $= \frac{Work\ done\ per\ sec\ by\ jet}{Energy\ supplied\ by\ jet\ per\ sec} = \frac{\rho A(V-u)^2(1+cos\theta)\, u}{\frac{1}{2}\rho A V V^2} = \frac{2u(V-u)^2(1+cos\theta)}{V^3}$

For maximum efficiency, $\frac{d\eta}{du} = 0$

$\frac{d}{du}\left(\frac{2(uV^2 - 2u^2V + u^3)(1+cos\theta)}{V^3}\right) = 0$

$V^2 - 4uV + 3u^2 = 0$

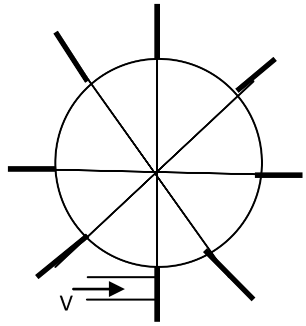
(V-u)(v-3u) = 0

V=u (work done =0)

V=3u

$\eta = \frac{2u(V-u)^2}{V^3} = \frac{2u(3u-u)^2(1+cos\theta)}{(3u)^3} = \frac{8}{27}(1 + cos\theta)$

(IV) Jet striking series of flat plates mounted on a wheel



V = absolute velocity of jet, u = absolute velocity of plate

Force exerted by the jet in X-direction (F_x) = mass of water striking/sec x ($V_{1x} - V_{2x}$)

$$= \rho AV[(V-u)-0] = \rho AV(V-u)$$

(Since entire fluid mass issuing from the jet strikes the plate, mass of water striking/sec = ρAV)

Work done per sec by the jet on the plate = F_xu = $\rho AV(V-u)u$

Efficiency (η) = $\frac{Work\ done\ per\ sec\ by\ jet}{Energy\ supplied\ by\ jet\ per\ sec} = \frac{\rho AV(V-u)u}{\frac{1}{2}\rho AVV^2} = \frac{2u(V-u)}{V^2}$

For maximum efficiency, $\frac{d\eta}{du} = 0$

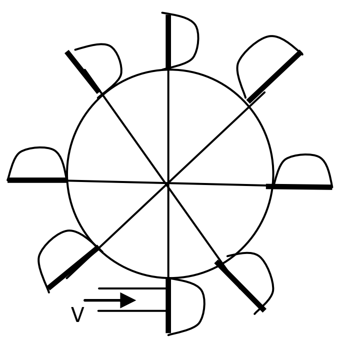
$\frac{d}{du}\left(\frac{2u(V-u)}{V^2}\right) = 0$

$V - 2u = 0$

V=2u

$\eta = \frac{2u(V-u)}{V^2} = \frac{2u(2u-u)}{(2u)} = \frac{1}{2}$

V) Jet striking series of curved vane mounted on a wheel



V = absolute velocity of jet, u = absolute velocity of plate, Let θ = angle made by tangent to the vane at its outlet

Force exerted by the jet in X-direction (F_x) = mass of water striking/sec x ($V_{1x} - V_{2x}$)

$$= \rho AV\left[(V-u) - \left(-(V-u)\right)cos\theta\right] = \rho AV(V-u)(1+cos\theta)$$

(Since entire fluid mass issuing from the jet strikes the plate, mass of water striking/sec = ρAV)

Work done per sec by the jet on the plate = F_xu = $\rho AV(V-u)(1+cos\theta)u$

Efficiency (η) = $\frac{Work\ done\ per\ \sec by\ jet}{Energy\ supplied\ by\ jet\ per\ sec} = \frac{\rho AV(V-u)(1+cos\theta)u}{\frac{1}{2}\rho AVV^2} = \frac{2u(V-u)(1+cos\theta)}{V^2}$

For maximum efficiency, $\frac{d\eta}{du} = 0$

$$\frac{d}{du}\left(\frac{2u(V-u)(1+cos\theta)}{V^2}\right) = 0$$

$$V - 2u = 0$$

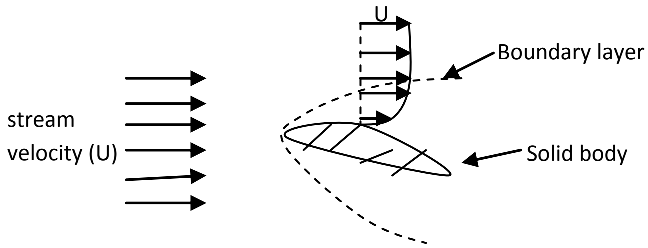
V=2u

$$\eta = \frac{2u(V-u)(1+cos\theta)}{V^2} = \frac{2u(2u-u)(1+cos\theta)}{(2u)} = \frac{1}{2}(1 + cos\theta)$$

CHAPTER 8. BOUNDARY LAYER THEORY

8.1 Introduction

When a real fluid flows past a solid boundary, the fluid particle on the surface will have the same velocity as that of the surface because of viscosity. This is called no-slip condition. If the boundary is stationary, the velocity of the fluid at the boundary is zero. Further away from the boundary, the velocity gradually increases.

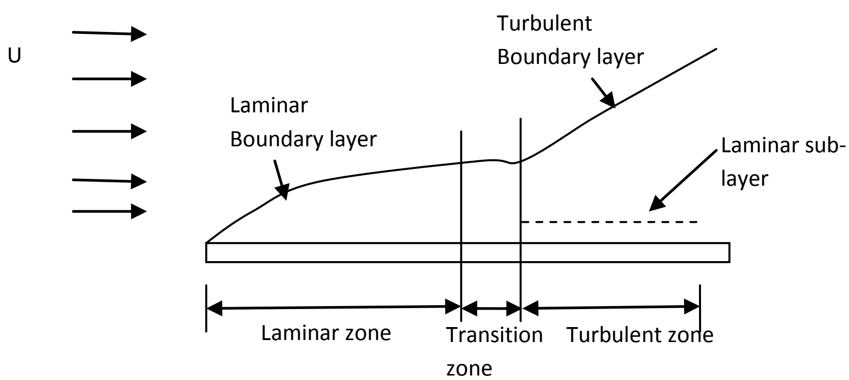


The term boundary layer is defined as the thin layer of the flow on the boundary within which the velocity varies from zero at the solid boundary to the free stream velocity in the direction normal to the boundary. In the boundary layer velocity gradient (du/dy) is large and the shear stress exerted by the fluid is given by Newton's law of viscosity as $\tau = \mu \frac{du}{dy}$. Outside the boundary, velocity is constant and velocity gradient is zero and hence shear stress is zero. Hence, there are two regions of flows: one is the boundary layer zone close to the boundary where the effect of viscosity is mostly confined and another region outside the boundary layer zone where the flow is inviscid (ideal). Thus, the boundary layer theory helps in simplifying the complex motion of fluid by considering the region where the velocity gradient exists.

Boundary layer thickness (δ)

It is defined as the distance from the boundary of the solid body measured in the Y-direction to the point, where the velocity of fluid is approximately 0.95 times the free stream velocity.

8.2 Boundary layer along a thin flat plate



Consider the flow of fluid, having free stream velocity U, over a smooth thin plate which is flat and parallel to the direction of free stream of fluid.

At the leading edge of the fluid, the thickness of the boundary layer is zero. On the downstream, for the fluid in contact with the boundary, the velocity of flow is zero and at some distance from the boundary the velocity is u. Hence a velocity gradient is set up which retards the motion of fluid due to the shear resistance. Near the leading edge, the fluid is retarded in thin layer. At subsequent point downstream the leading edge, the boundary layer region increases because the retarded fluid is further retarded.

Laminar boundary layer

As the boundary layer develops, up to some distance from the leading edge, the flow in the boundary layer is laminar irrespective of whether the incoming flow is laminar or turbulent. This is known as laminar boundary layer.

When the boundary layer is thin, the velocity gradient is large. As the boundary layer thickens, velocity gradient reduces and shear stress decreases. Eventually it is too small to drag the slow fluid along. Up to this point the flow is laminar. Viscous force is dominant in this case.

Transition zone

As the boundary layer grows to a certain thickness, instability occurs, leading to transition from laminar to turbulent boundary layer. This length is known as transition zone.

Turbulent boundary layer

After the transition zone, the boundary layer is turbulent and continues to grow in thickness. This layer is known as turbulent boundary layer.

If the viscous forces were the only action, the fluid would come to a rest. Viscous shear stresses have held the fluid particles in a constant motion within layers. Eventually they become too small to hold the flow in layers and the fluid starts to rotate. The fluid motion rapidly becomes turbulent. Momentum transfer occurs between fast moving main flow and slow moving near wall flow. Thus the fluid by the wall is kept in motion. The net effect is an increase in momentum in the boundary layer.

Laminar sub-layer

If the plate is very smooth, even in the region of turbulent boundary layer, there is a very thin layer just adjacent to the boundary, in which the flow is laminar. This layer is known as laminar sub-layer.

Characteristics of boundary layer

- The boundary layer consists of laminar, transition zone and turbulent zone.
- The flow in the boundary layer is rotational and strongly influenced by viscosity.
- δ (thickness of boundary layer) increases as the distance (x) from the leading edge increases.
- δ decreases as velocity (U) increases (no retardation).
- δ increases as kinematic viscosity increases.
- Shear stress $(\tau_0) \approx \mu \frac{U}{\delta}$

  τ_0 decreases as x increases. However, when the boundary layer becomes turbulent, it shows a sudden increase and then decreases with increasing x.
- If U decreases in the downward direction, flow near the boundary is further retarded, boundary layer growth is faster and is susceptible to separation.
- The various properties of boundary layer on a flat plate, e.g. δ, τ_0 are governed by inertia and viscous forces. Hence, they are functions of Reynold no. (Re) ($Re = Ux/\upsilon$ where U = velocity, x = distance from leading edge and υ = kinematic viscosity)
- If Re < $3x10^5$, the boundary layer is laminar and the velocity distribution is parabolic

  If Re > $5x10^5$, the boundary layer is turbulent and the velocity distribution follows logarithmic law or power law. (Re from $3x10^5$-5x105: transition)
- Critical value of Re at which boundary layer changes from laminar to turbulent depends on turbulence in ambient flow, surface roughness (faster transition), pressure gradient, plate curvature, and temperature difference between fluid and boundary. (Lower for +ve dp/dx and higher for –ve dp/dx)
- In laminar sub-layer, the velocity distribution can be assumed to be linear.

Laminar boundary layer

Thickness of laminar boundary layer, $\delta = \frac{5x}{\sqrt{Re_x}}$

Drag coefficient $C_f = \frac{0.664}{\sqrt{Re_x}}$

Shear stress $(\tau) = \frac{1}{2} C_f \rho U^2$

Boundary conditions for laminar flow: (a) At y = 0, u = 0, (b) At $y = \delta, u = U$, (c) At $y = \delta, \frac{du}{dy} = 0$ and in between y = 0 to δ, velocity gradient exists.

The general equation of velocity profile for laminar boundary layer is u = $ay+by^2+cy^3+dy^4$ or $\frac{u}{U} = f\left(\frac{y}{\delta}\right)$.

Common form of equations for laminar boundary layer

a. u = ay+b

Use boundary condition: u = 0 at y = 0 , $u = U$ at $y = \delta$ to determine coefficients

b. u = ay^2+by+c

Use boundary condition: u = 0 at y = 0 , $u = U$ at $y = \delta$ and $\frac{du}{dy} = 0$ at $y = \delta$ to determine coefficients

c. u = ay^2+by+c

Use boundary condition: u = 0 at y = 0 , $u = U$ at $y = \delta$, $\frac{du}{dy} = 0$ at $y = \delta$ and $\frac{d^2u}{dy^2} = 0$ at y= 0 to determine coefficients

d. u = $ay^4+by^3+cy^2+dy+e$

Use boundary condition: u = 0 at y = 0 , $u = U$ at $y = \delta$, $\frac{du}{dy} = 0$ at $y = \delta$, $\frac{d^2u}{dy^2} = 0$ at y= 0 and $\frac{d^2u}{dy^2} = 0$ at $y = \delta$ to determine coefficients

Turbulent boundary layer

For Re_x between $5x10^5$ to $2x10^7$

Thickness of turbulent boundary layer, $\delta = \frac{0.377x}{{Re_x}^{0.2}}$

Drag coefficient $C_f = \frac{0.059}{{Re_x}^{0.2}}$

For Re_x between $>2x10^7$

$\delta = \frac{0.22x}{{Re_x}^{1/6}}$

Drag coefficient $C_f = \frac{0.37}{(LogRe_x)^{.2.58}}$

In case of turbulent boundary layer, the general equation for velocity distribution is $u{\sim}logy\ or\ u{\sim}y^{1/n}$.

8.3 Hydrodynamically smooth and rough boundary

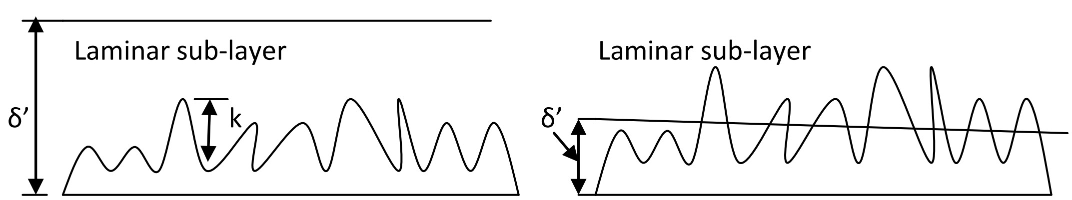
The velocity distribution within laminar sub-layer is parabolic. As its thickness is very small, parabolic velocity profile is replaced by straight line. Experimentally, the thickness of laminar sub-layer (δ') is given by

$\delta' = \frac{11.6\nu}{V^*}$

where ν = kinematic viscosity of fluid, V* =shear velocity = $\sqrt{\tau_0/\rho}$, τ_0 = wall shear stress, ρ = density of fluid.

$\tau_0 = \frac{f\rho V^2}{8}$ where f = friction factor

Turbulent boundary layer Turbulent boundary layer



Smooth boundary Rough boundary

Smooth and rough boundary

If k is much less than δ', the boundary is called hydrodynamically smooth boundary. In this case eddies cannot reach the surface irregularities. If δ' is much less than k, the boundary is called hydrodynamically rough boundary. In this case eddies come in contact with the surface irregularities and lot of energy will be lost.

From Nikuradse's experiment:

$\frac{k}{\delta'}$<0.25: Smooth boundary

$\frac{k}{\delta'}$>6: rough boundary

0.25<$\frac{k}{\delta'}$<6: transition

In terms of roughness, Reynold number (Re) = $\frac{V_* K}{\upsilon}$ where V_* = shear velocity, k = roughness height

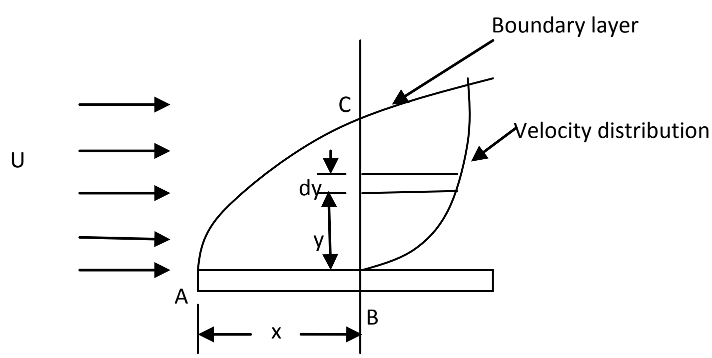
Re<4: smooth

Re>70: rough

4<Re<70: transition

8.4 Displacement thickness

It is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in flow rate on account of boundary layer formation. It is denoted by δ^*.



Let the fluid of density ρ with velocity U flows past a stationary plate of width b. Consider a section at a distance x from leading edge. Consider an elementary strip of thickness dy at a distance y from the plate. Let u be the velocity of fluid at this strip. Distance BC = δ is the boundary layer thickness.

Mass of fluid flowing per sec through elemental strip = $\rho ubdy$

Mass of fluid flowing per sec through elemental strip in absence of plate = $\rho Ubdy$

Reduction in mass of fluid flowing per sec through strip = $\rho Ubdy - \rho ubdy = \rho bdy(U-u)$

Total reduction in mass of fluid flowing through BC due to plate = $\int_0^\delta \rho bdy(U-u)$

$= \rho b \int_0^\delta (U-u)dy$ (a)

Let the plate is displaced by a distance δ^*, which is the displacement thickness, and the velocity of the flow for the distance δ^* is equal to U.

Loss of mass of fluid per sec flowing through distance δ^* = $\rho U b \delta^*$ (b)
Equating a and b

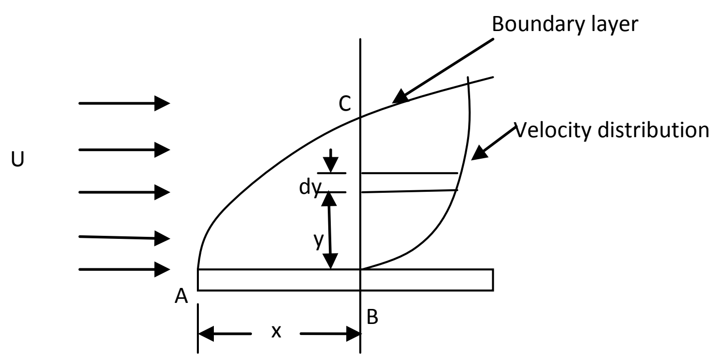
$$\rho b \int_0^{\delta} (U-u)dy = \rho U b \delta^*$$

$$\delta^* = \int_0^{\delta} \frac{1}{U}(U-u)dy$$

$$\delta^* = \int_0^{\delta} \left(1-\frac{u}{U}\right) dy$$

8.5 Momentum thickness

It is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in momentum of the flowing fluid on account of boundary layer formation. It is denoted by θ.



Let the fluid of density ρ with velocity U flows past a stationary plate of width b. Consider a section at a distance x from leading edge. Consider an elementary strip of thickness dy at a distance y from the plate. Let u be the velocity of fluid at this strip. Distance BC = δ is the boundary layer thickness.

Momentum of fluid per sec through elemental strip = $(\rho u b dy)u$
Momentum of fluid per sec through elemental strip in absence of plate = $(\rho u b dy)U$
Loss of momentum per sec through strip = $(\rho u b dy)U - (\rho u b dy)u = \rho b u dy(U-u)$
Total Loss of momentum per sec through BC due to plate = $\int_0^{\delta} \rho b u dy(U-u)$
$= \rho b \int_0^{\delta} u(U-u)dy$ (a)
Let the plate is displaced by a distance θ, which is the momentum thickness, and the velocity of the flow for this distance is equal to U.
Loss of momentum per sec of fluid flowing through distance θ = $(\rho \theta b U)U$ (b)

Equating a and b

$$\rho b\int_0^\delta u(U-u)dy = (\rho\theta bU)U$$

$$\theta = \int_0^\delta \frac{u}{U^2}(U-u)dy$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

Separation of Boundary layer

- du/dy>0 and dp/dx<0 attached (pressure gradient is negative i.e. decreasing pressure in the direction of flow)
- du/dy =0 and dp/dx=0 verge of separation
- du/dy<0 and dp/dx>0 separation (positive or adverse pressure gradient together with shear causing fluid at boundary to come to rest, and consequent separation of boundary layer)

Methods of controlling the formation and separation of Boundary layer

- Acceleration of the fluid in the boundary layer
- Motion of solid boundary
- Suction of the fluid from the boundary layer
- Streamlining of body shapes

CHAPTER 9. FLOW PAST THROUGH SUBMERGED BODIES

9.1 Introduction

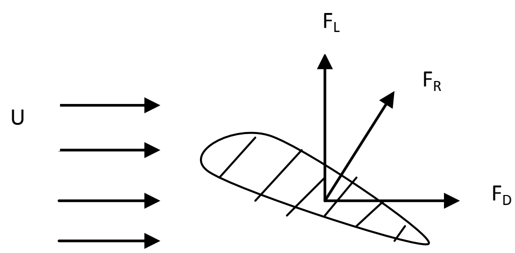
It is often necessary to find the force on objects moving through a stationary fluid, or force on the stationary bodies over which fluid flows, or both the object and fluid in motion. When the body immersed in a fluid, moves through the fluid, then a force is exerted on it by the fluid. The body, in turn, exerts a force on the fluid, which is equal in magnitude, but opposite to the direction. The force exerted on the fluid or the body results from relative motion. So the magnitude of force is same whether the body moves through the static fluid or the fluid moves over static body.

Example of fluids flowing over stationary bodies or bodies moving in a fluid

- Force on bodies like aeroplane, submarine, automobile, ship moving through a static fluid
- Force on chimney, cable subjected to wind
- Force on buildings submerged in air
- Force on bridges submerged in water

9.2 Drag and Lift

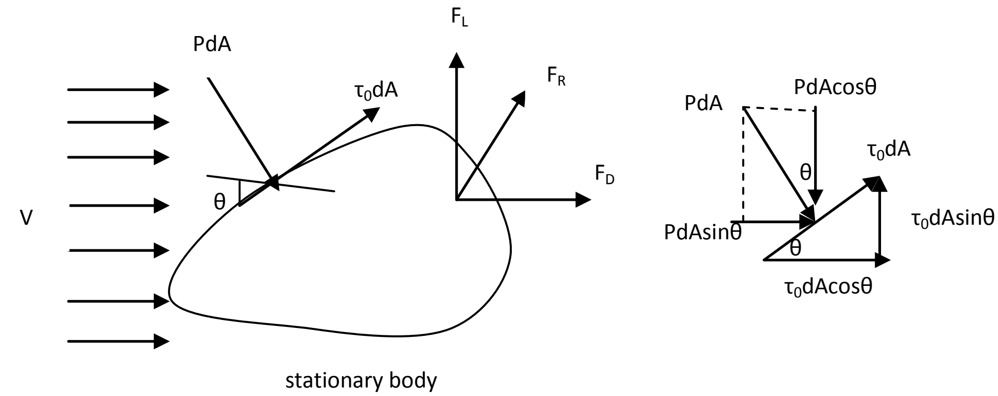
When a fluid flows over a stationary body, it will exert a force on the body.



The total force exerted by the fluid on the body is perpendicular to the surface of the body. This force is inclined to the direction of motion. This force has two components. The component of total force in the direction of motion is called drag force (F_D), and the component of the force perpendicular to the direction of motion is called lift force (F_L). Lift force occurs only when the axis of the body is inclined to the direction of flow. If the axis of the body is parallel to the direction of flow, lift force is zero.

Expression for drag and lift

Consider an arbitrary shaped solid body placed in a real fluid, which is flowing with a uniform velocity V in a horizontal direction. Consider a small elemental area dA on the surface of the body. The forces acting on surface dA are: pressure force acting perpendicular surface, shear force acting tangential to the surface. Let θ be the angle made by the pressure force with horizontal direction.



Drag force

Drag force on elemental area = Total force in the direction of motion

$$dF_D = PdAsin\theta + \tau_0 dAcos\theta$$

Total drag is

$$F_D = \int PdAsin\theta + \int \tau_0 dAcos\theta$$

Types of drag
First term: pressure drag/form drag
Second term: friction drag/skin drag/shear drag

The existence of viscosity for real fluids is the main cause of drag on the bodies. In the boundary layer zone, due to the velocity gradient, considerable shear stresses are caused. These shear stresses exert a tangential force on the object, which is called shear or friction drag.

If the surface of the immersed object, along which boundary layer forms, is such that it curves away from the flow, there exists a tendency of flowing fluid to leave the boundary. This phenomenon is known as separation of flow. Due to this, low pressure region known as wake is formed. As the pressure in the upstream side is higher, there exists a pressure difference which causes drag on the object. This is known as pressure drag.

Due to viscosity, deformation of fluid particles take place. In order to cause deformation, certain forces are necessarily developed, which offer an additional resistance to the motion. The component of such force in the direction of motion is called deformation drag. The deformation drag mainly exists in the case of very small objects moving at very small velocities through fluids of large viscosity.

Lift force

Lift force on elemental area = Total force in the direction perpendicular to the direction of motion

$$dF_L = -PdAcos\theta + \tau_0 dAsin\theta$$

Total lift is

$$F_L = -\int PdAcos\theta + \int \tau_0 dAsin\,\theta$$

Mathematically, drag and lift for a body moving in a fluid of density ρ, at a uniform velocity V are calculated by the following equation.

$$F_D = \frac{1}{2} C_D \rho A V^2$$

$$F_L = \frac{1}{2} C_L \rho A V^2$$

where C_D = coefficient of drag

C_L = coefficient of lift

A = planform area of immersed body (projected area perpendicular to the direction of flow)

The resultant force $(F_R) = \sqrt{F_D^2 + F_L^2}$

9.3 Drag on a flat plate

Plate held parallel to the direction of flow of fluid: When a thin plate is held parallel to the direction of flow of fluid, then total drag force exerted on the fluid is equal to the friction drag which is due to the formation of boundary layer. The magnitude of drag force depends on whether the boundary layer is laminar or turbulent or transitional.

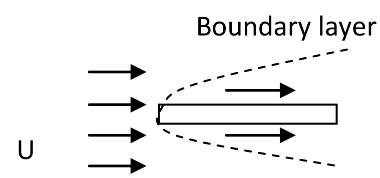


Plate held perpendicular to the direction of flow of fluid: When a thin plate is held perpendicular to the direction of flow of fluid, then the friction drag is negligible compared to the pressure drag. The flow separates at the edge forming a turbulent wake behind the plate. So the effect of inertia force becomes predominant even at lower Reynold number (R_e). Drag coefficient is a function of Re only at low and moderate values of R_e. However, as the value of R_e exceeds 1000, C_D assumes a constant value of about 0.2. A reduction in the value of C_D occurs if the ratio of length of plate to its width is not very large.

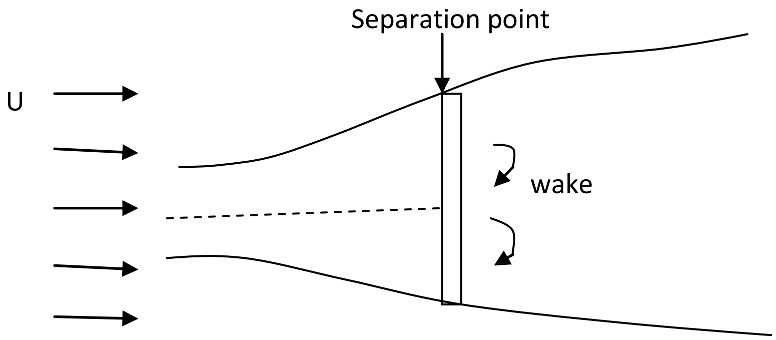
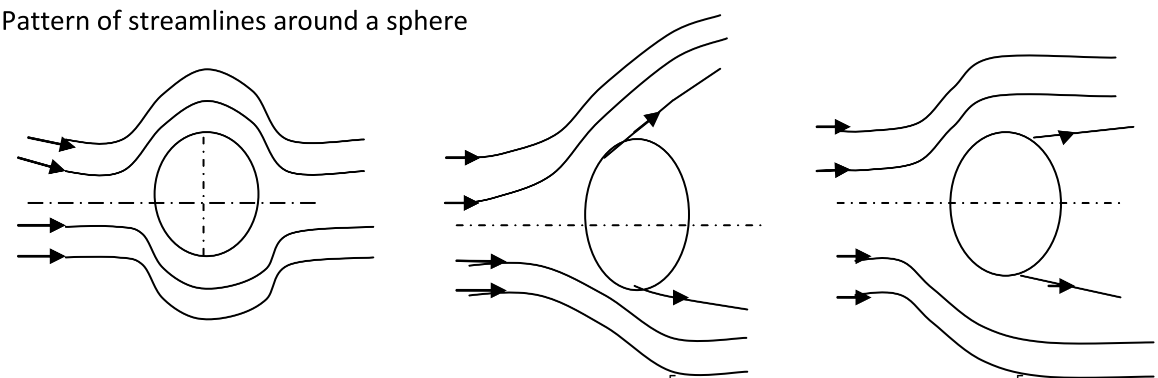


Plate held at an angle with the direct of flow: In this case, total drag = pressure drag + friction drag

9.4 Drag on sphere

Pattern of streamlines around a sphere



Re<0.1 Re: 1000-2.5x10^5 Re: >2.5x10^5

In case of ideal fluid, the flow pattern is symmetrical on the front and rear of the sphere and drag force is zero due to the absence of viscosity and symmetrical pressure distribution. In the laminar boundary layer, the points of separation are located on the upstream half portion (both pressure and friction drag). When the boundary layer becomes turbulent, the points of separation shift farther downstream towards the rear of the cylinder (only pressure drag).

Drag force on sphere depends on the Reynolds number (R_e), which is given by

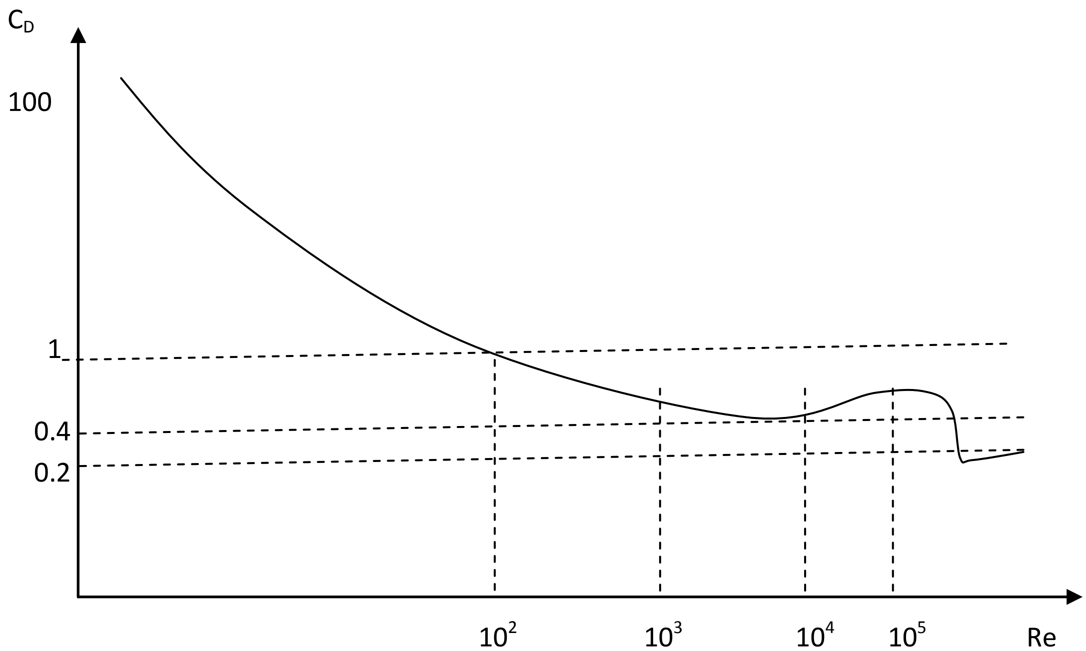
$$R_e = \frac{\rho DV}{\mu}$$

where ρ = density of fluid, D = Diameter of sphere, V = Velocity of flow over the sphere, μ = coefficient of viscosity

According to Stoke, the drag force on sphere for Re<0.2 is given by

$$F_D = 3\pi\mu DV$$

Pressure drag = $\frac{1}{3}F_D$ and friction drag = $\frac{2}{3}F_D$



C_D-Re diagram for sphere

Value of coefficient of drag of sphere for different R_e

a. R_e<0.2 or even up to 1

Equating equation of drag force with Stoke's equation

$$\frac{1}{2}C_D\rho AV^2 = 3\pi\mu DV$$

$$\frac{1}{2}C_D\rho\frac{\pi}{4}D^2V^2 = 3\pi\mu DV$$

$$C_D = \frac{24}{\frac{\rho DV}{\mu}} = \frac{24}{R_e}$$

b. For R_e between 0.2-5

$$C_D = \frac{24}{R_e}\left(1 + \frac{3}{16R_e}\right)$$

c. For R_e between 5-1000, C_D = 0.4

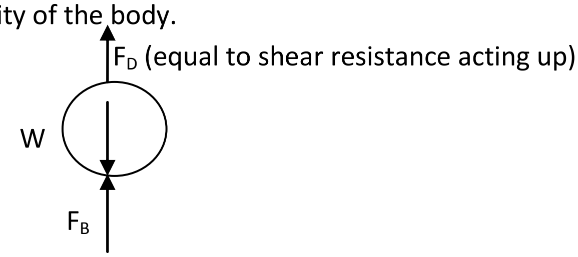
d. For R_e between 1000-100000, C_D close to 0.5

e. For R_e>100000, C_D about 0.2 (for fully developed turbulent flow)

Find R_e and corresponding C_D, then compute F_D.

9.5 Terminal velocity

When a body falls from rest in the atmosphere, its velocity increases and the drag force opposing its motion also increases. A stage is reached when the upward drag force is equal to the weight of the body. The net force acting on the body will be zero and the body will move with constant speed. The constant velocity is called terminal velocity of the body.



If a body, drops in a fluid, then after attaining terminal velocity (V), the forces acting on the body are: drag force (F_D), weight of the body (W) and buoyant force (F_B)

$W = F_D + F_B$

$F_D = 3\pi\mu DV$ if the Stoke's law is valid,

9.6 Drag on cylinder

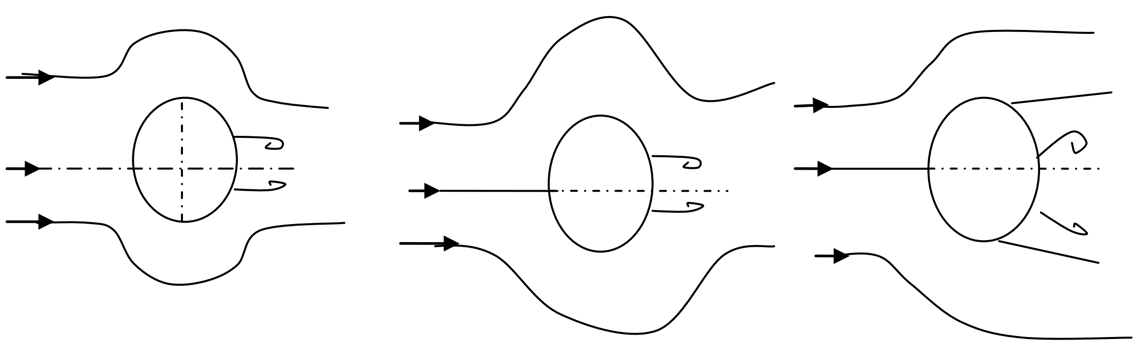
Drag force on cylinder depends on the Reynolds number (R_e), which is given by

$$R_e = \frac{\rho DV}{\mu}$$

where ρ = density of fluid, D = Diameter of cylinder, V = Velocity of flow over the cylinder, μ = coefficient of viscosity. The length of cylinder is perpendicular to the direction of flow.

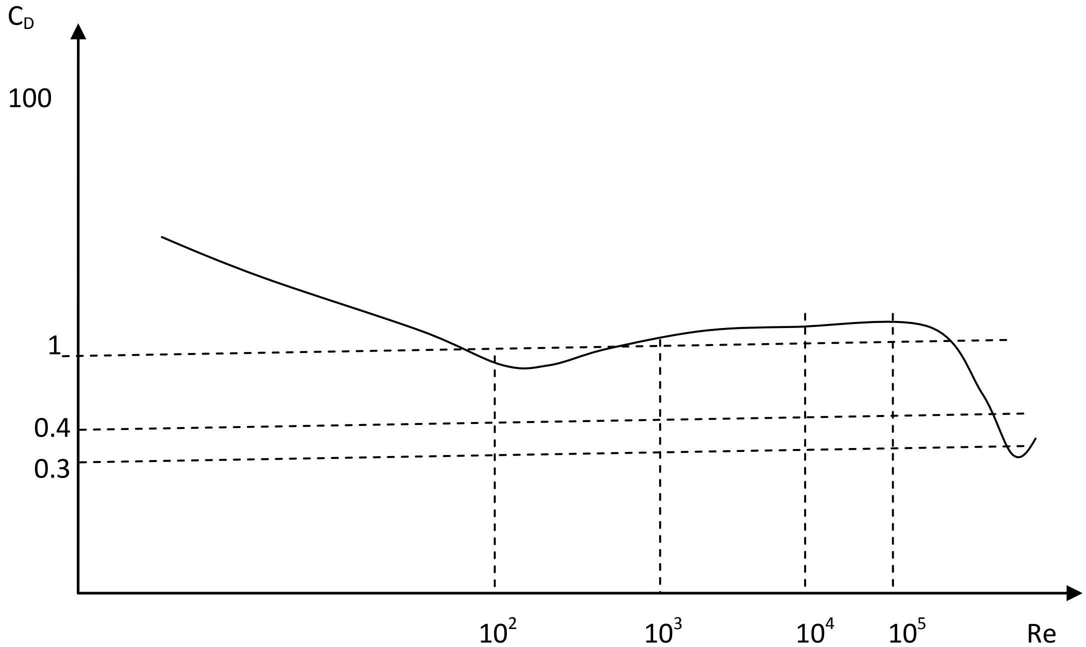
The pattern of streamlines around a cylinder is similar to that of sphere.

Variation of flow around a cylinder with different Re



Re = 2 to 30 Re = 40 to 70 Re = 90

For R_e<0.2, the inertia force is negligible and the flow pattern is symmetrical. With the increase of R_e, the flow pattern becomes unsymmetrical with respect to the axis perpendicular to the direction of flow. At Re from 2 to 30, very weak vortices are formed on the downstream of the cylinder. It is the initial stage for the development of the wake. At Re from 40 to 70, the wake as well as a pair of vortices become quite distinct. With further increase in the value of Re, the vortices become more and more elongated in the direction of flow. At Re = 90, these vortices become cylindrical, they leave the cylinder and slowly move in the downstream direction.



C_D-Re diagram for cylinder

Value of coefficient of drag of cylinder for different R_e

a. For R_e<1, $C_D \propto \frac{1}{R_e}$

b. For R_e between 1-2000: C_D decreases and reaches a minimum value of 0.95 at R_e = 2000.

c. For $R_e = 2x10^3$-$2x10^5$: C_D is close to 1 to of 1.2.

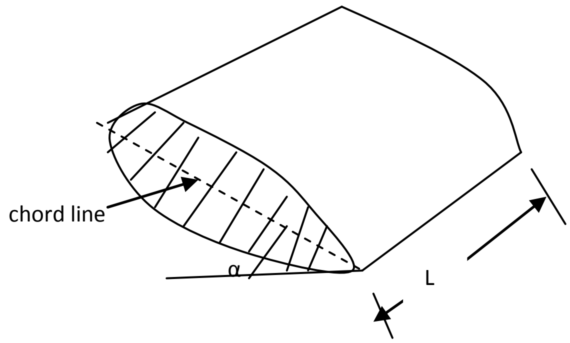
d. For R_e >$3x10^5$: C_D decreases to 0.3

9.7 Lift on airfoil

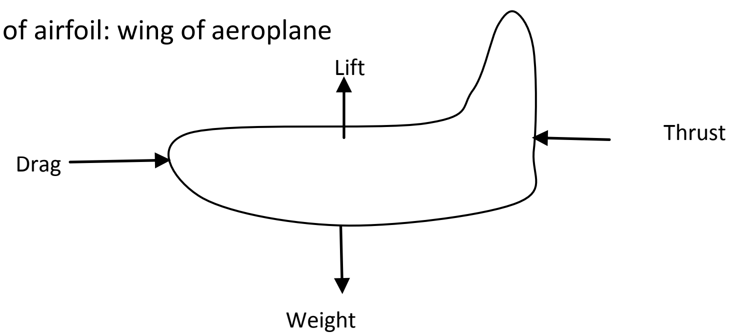
A body whose shape coincides with the streamlines when placed in a flow is called streamlined body. An airfoil is a streamlined body which may be either symmetrical or unsymmetrical. Some of the terminologies used to characterize airfoil are as follows.

Chord line: It is the line joining the front and rear edge.
Angle of attack: It is the angle between the direction of flowing fluid and chord line.
Camber: It is the curvature of an airfoil.
Span: The overall length of airfoil is called its span (L).
Aspect ratio: The ratio of span (L) to mean chord (C) is called aspect ratio.

Stall: An airfoil is said to be in stall condition when the angle of attack of an airfoil is greater than the angle of attack at maximum lift. At stall, the air separates from airfoil and eddies are formed, as a result of which there is considerable increase in the drag coefficient.



Example of airfoil: wing of aeroplane



The wings provide lift by creating a situation where the pressure above the wing is lower than the pressure below the wing. Since the pressure below the wing is higher than the pressure above the wing, there is a net force upwards. To create this pressure difference, the surface of the wing must satisfy one or both of the following conditions. The wing surface must be Cambered (curved) and/or inclined relative to the airflow direction.

Viscosity is essential in generating lift. The effects of viscosity lead to the formation of the starting vortex which, in turn is responsible for producing the proper conditions for lift. To satisfy the conservation of angular momentum, there must be an equivalent motion to oppose the vortex movement. This takes the form of circulation around the wing. The velocity vectors from this counter circulation add to the free flow velocity vectors, thus resulting in a higher velocity above the wing and a lower velocity below the wing.

From the theoretical analysis, the circulation (Γ) developed on the airfoil so that the rear edge of the airfoil is tangential to is, is given by

$$\Gamma = \pi C V sin\alpha$$

where C = chord length, V = free stream velocity of airfoil.

Lift force (F_L) is given by

$F_L = \rho V L \Gamma = \pi \rho C V^2 L sin\alpha$ (a)

$F_L = \frac{1}{2} C_L \rho A V^2$ (b)

Equating a and b

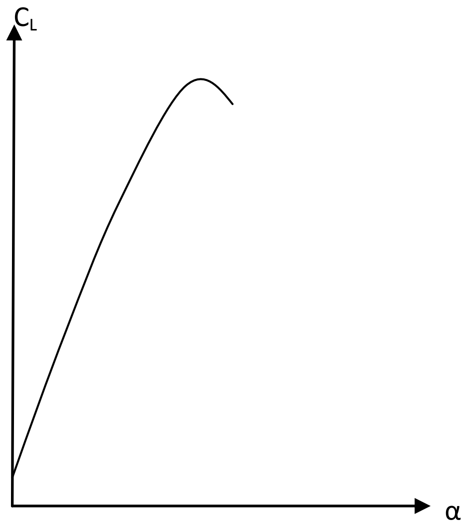
$$\pi \rho C V^2 L sin\alpha \;= \frac{1}{2} C_L \rho A V^2$$

$$\pi \rho C V^2 L sin\alpha \;= \frac{1}{2} C_L \rho C L V^2$$

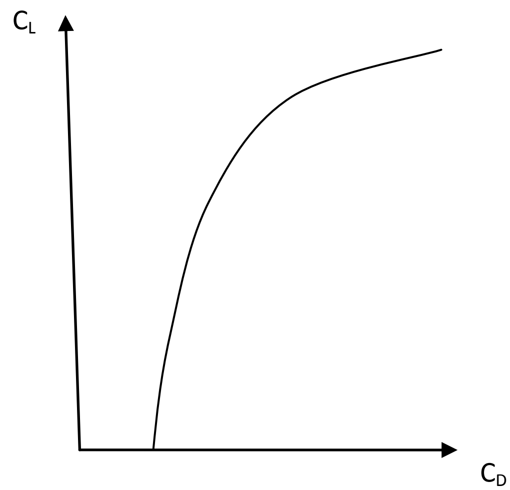
$$C_L = 2\pi sin\alpha$$

This is theoretical value of C_L

C_L depends on angle of attack only.



Lift coefficient for airfoil
($C_L$max = 1.72 at stall)



Relationship of C_L and C_D for airfoil

CHAPTER 10. SIMILITUDE AND PHYSICAL MODELLING

10.1 Introduction

Dimensional analysis is a technique to establish a relationship between variables by using dimensions. Application of fluid mechanics in design makes use of experimental results. The results are often difficult to interpret. Dimensional analysis provides a strategy for choosing relevant data and helps to analyze fluid flow especially when fluid flow is too complex for mathematical analysis. Dimensional analysis gives a single equation, which relates all the physical factors of a problem to each other. The result of this analysis depends on the correct identification of variables and their relationship. It doesn't give the complete answer. So experiments are necessary to complete the solution.

Dimension

Dimension is a property of physical quantity expressed in terms of mass, length and time.
Fundamental dimension: Mass (M), Length (L), time (T)
Derived/Secondary quantity: Quantity containing more than one fundamental dimension, e.g.
velocity = distance/time = LT^{-1}

Dimension of most widely used physical quantities
a. fundamental: Mass = M, Length = L, Time = T
b. Geometric: Area = L^2, Volume = L^3
c. Kinematic quantities
Velocity = LT^{-1}
Angular velocity = T^{-1}
Acceleration = LT^{-2}
Angular acceleration = T^{-2}
Discharge = L^3T^{-1}
Kinematic viscosity = L^2T^{-1}
d. Dynamic quantities
Force, Weight = MLT^{-2}
Density = ML^{-3}
Specific weight = $ML^{-2}T^{-2}$
Dynamic viscosity = $ML^{-1}T^{-1}$
Pressure = $ML^{-1}T^{-2}$
Surface tension = MT^{-2}
Work, energy = ML^2T^{-2}
Shear stress = $ML^{-1}T^{-2}$
Power = ML^2T^{-3}
Torque = ML^2T^{-2}
Momentum = MLT^{-1}

Dimensional homogeneity

Dimensional homogeneity means dimensions of each terms in an equation on both sides are equal. Dimensionally homogeneous equations are independent of the system of units.
e.g. Discharge through rectangular orifice is

$$Q = \frac{2}{3}\sqrt{2g}LH^{3/2}$$

Dimension of Left hand side = L^3T^{-1}

Dimension of right hand side = $(LT^{-2})^{1/2}LL^{3/2} = L^3T^{-1}$

Dimensional homogeneity can be useful for the following purposes.

- Checking units of equations
- Converting between two sets of units
- Defining dimensionless relationships

10.2 Methods of dimensional method

10.2.1 Rayleigh's method

In this method, a functional relationship of variables is expressed in terms of one exponential equation which must be dimensionally homogeneous.

Steps

- Write down the functional relationship between dependent and independent variables.
- Change the functional relationship to the equation of exponential form.
- Write down the dimension of variables.
- Equate the powers of dimensions and find the constants.

X = dependent variable
$X_1, X_2, X_3,\ldots\ldots\ldots,X_n$ = independent variables
Functional relationship: $X = f(X_1, X_2, X_3,\ldots\ldots\ldots,X_n)$
Equation in exponential form: $X = KX_1^aX_2^bX_3^c \ldots \ldots \ldots X_n^Z$
where a, b, c…….Z are constants
After equating the power of dimensions, three equations are formed. If there are more than three constants, then three appropriate constants are expressed in terms of other constants. The variables of the expression to be derived are examined, and the powers of the three variables which are outside of functional form, are usually expressed in terms of other constants.

10.2.2 Buckingham's π theorem

It states that if there are n variables (dependent and independent) in a physical phenomenon and if these variables contain m fundamental dimensions, then the variables are arranged into (n-m) dimensionless terms. Each term is called π term.

Rayleigh method is not practicable for large number of variables due to the difficulty in finding the constants. This is overcome by Buckingham's π theorem, which requires lesser number of dimensional groups of variables.

Steps

- Write down the functional relationship between dependent and independent variables.
  X_1 = dependent variable
  $X_2, X_3, X_4, \ldots\ldots, X_n$ = independent variables
  $X_1 = f(X_2, X_3, X_4, \ldots\ldots, X_n)$
  $f_1(X_1, X_2, X_3, X_4, \ldots\ldots, X_n) = 0$ (I)

- Count total number of variables (n) and fundamental dimensions (m)
  No. of π terms = n-m
- Write down the functional relationship in π terms.
  $\phi\,(\pi_1, \pi_2, \pi_3, \ldots\ldots, \pi_{n-m}) = 0$ (II)
- For each π term, write down equation in exponential form with (m+1) variables, where m=3 is called fundamental dimension and it is also called repeating variables. These variables appear repeatedly in each of π terms.
  Each π group is a function of n repeating variables plus one of the remaining variables. The π term is dimensionless.
  Let X_2, X_3, X_4 be the repeating variables.
  $$\pi_1 = X_2^{a1} X_3^{b1} X_4^{c1} X_1$$
  $$\pi_2 = X_2^{a2} X_3^{b2} X_4^{c2} X_5$$
  ………………………………..
  $$\pi_{n-m} = X_2^{a(n-m)} X_3^{b(n-m)} X_4^{c(n-m)} X_n$$
- Solve each equation by the principle of dimensional homogeneity (writing dimensions and equating the powers of M, L and T to get constants.)
- Obtain $\pi_1, \pi_2, \pi_3, \ldots\ldots, \pi_{n-m}$ and substitute in eqn. (II). Establish functional relationship between π terms.

Criteria for selecting repeating variables

Repeating variables appear in most of the π groups. They have a large influence on the problem. There is great freedom in choosing these.
Some rules which should be followed are

- There are n (= 3) repeating variables.
- In combination they must contain all of dimensions (M, L, T)
- The repeating variables must not form a dimensionless group.
- Dependent variable should not be selected as repeating variable.
- No two repeating variables should have the same dimensions.
- They should be measurable in an experiment.
- They should be of major interest to the designer.

In repeating variables, it is usually possible to take one variable representing fluid property (e.g. density or viscosity), next variable representing flow property (e.g. velocity) and the last variable representing geometry (e.g. diameter or length).

Manipulation of the π groups

Once identified the π groups can be changed to derive the required equation. The number of groups does not change, but their appearance may change drastically.
Taking the defining equation as:
$\phi\ (\pi_1, \pi_2, \pi_3, \ldots\ldots\ldots\ldots, \pi_{n\text{-}m}) = 0$
The following changes are permitted:
I. Combination of existing groups by multiplication or division to form a new group to replaces one of the existing.
E.g. π_1 and π_2 may be combined to form $\pi_{1a} = \pi_1 / \pi_2$. So the defining equation becomes
$\phi(\pi_{1a}, \pi_2, \pi_3, \ldots\ldots\ldots\ldots, \pi_{n\text{-}m}) = 0$
II. Reciprocal of any group is valid.
$\phi\ (1/\pi_1, \pi_2, \pi_3, \ldots\ldots\ldots\ldots, 1/\pi_{n\text{-}m}) = 0$
III. A group may be raised to any power.
$\phi\ (\pi_1^{2}, (\pi_2)^{1/2}, \pi_3, \ldots\ldots\ldots\ldots, \pi_{n\text{-}m}) = 0$
iv. Any groups are multiplied by a constant.
$\phi\ (k\pi_1, \pi_2, \pi_3, \ldots\ldots\ldots\ldots, \pi_{n\text{-}m}) = 0$
v. A group is expressed as a function of other groups
$\pi_2 = f\ (\pi_1, \pi_3, \ldots\ldots\ldots\ldots, \pi_{n\text{-}m}) = 0$

10.3 Model and prototype

The small scale replica of actual structure or the machine is known as its model. The actual structure or the machine is known as prototype.

Before the construction of hydraulic structure such as dam or hydraulic machine such as pump, turbines, it is necessary to know how the structure or machine would behave when it is actually constructed. For this purpose, experimental investigation is needed, which cannot be carried out on the full size of structure or machine. Hence it is necessary to construct a model (replica) on which the tests are performed to obtain the desired information.

10.3.1 Similitude

The similarity between the prototype and model in every aspect is known as similitude. There are in general three types of similarities to be established for complete similarity to exist between the model and its prototype: geometric, kinematic and dynamic

a. Geometric similarity

Geometric similarity is the similarity of shape between the model and the prototype. Geometric similarity exists between the model and the prototype if the ratios of corresponding length dimensions in the model and the prototype are equal. Such a ratio is called scale ratio. e.g.

$$\frac{L_m}{L_p} = \frac{b_m}{b_p} = \frac{d_m}{d_p} = L_r$$

subscript m: model, subscript p: prototype, l= Length, b= Breadth, d = height, L_r = Scale ratio
For area and volume ratio, the relationships are

$$\frac{A_m}{A_p} = \frac{L_m b_m}{L_p b_p} = L_r^2$$

$$\frac{Vol_m}{Vol_p} = L_r^3$$

b. Kinematic similarity

Kinematic similarity is the similarity of motion. Geometric similarity implies that in addition to geometric similarity, the ratios of velocities as well as acceleration of fluid particle at a certain point in the model and at the corresponding point of prototype are equal. e.g.

$$\frac{V_{m1}}{V_{p1}} = \frac{V_{m2}}{V_{p2}}$$

$$\frac{a_{m1}}{a_{p1}} = \frac{a_{m2}}{a_{p2}}$$

where V = velocity, a = acceleration, subscript m = model, subscript p = prototype

c. Dynamic similarity

Dynamic similarity is the similarity of forces. Dynamic similarity implies that in addition to geometric and kinematic similarity, the ratios of all the forces acting at a certain point in the model and at the corresponding point of prototype are equal. e.g.

$$\frac{(F_{inertia})m}{(F_{inertia})p} = \frac{(F_{viscous})m}{(F_{viscous})p} = \frac{(F_{gravity})m}{(gravity)p}$$

Forces acting on fluid particle

- Inertia force: resistance by inert mass to acceleration
- Friction or viscous force
- Gravity force
- Pressure force
- Elastic force
- Surface tension force

10.4 Dimensionless numbers based on force ratios

1. Reynolds number (Re) $= \frac{\text{Inetia force}}{\text{Viscous force}}$

$$\frac{\text{Inetia force}}{\text{Viscous force}} = \frac{mass\ x\ acceleration}{shear\ stress\ x\ area}$$

$$Re = \frac{\rho Vol \frac{V}{t}}{\mu \frac{dV}{dy} A} = \frac{\rho \frac{Vol}{t} V}{\mu \frac{V}{L} A} = \frac{\rho QV}{\mu \frac{V}{L} A} = \frac{\rho AV^2}{\mu \frac{V}{L} A} = \frac{\rho VL}{\mu}$$

or $Re = \frac{VL}{\nu}$

In case of pipe flow, linear dimension diameter (D) is taken as linear dimension L.

$$Re = \frac{VD}{\nu}$$

2. Froude number (Fr) = $\frac{\text{Inetia force}}{\text{Gravity force}}$

$$\frac{\text{Inetia force}}{\text{Gravity force}} = \frac{\rho AV^2}{massxg} = \frac{\rho AV^2}{\rho Volg} = \frac{\rho AV^2}{\rho ALg} = \frac{V^2}{Lg}$$

Taking square root

$$Fr = \frac{V}{\sqrt{Lg}}$$

3. Euler number (Eu) = $\frac{\text{Inetia force}}{\text{pressure force}} = \frac{V}{\sqrt{P/\rho}}$

$$\frac{\text{Inetia force}}{\text{pressure force}} = \frac{\rho AV^2}{PA} = \frac{V^2}{P/\rho}$$

Taking square root

$$Eu = \frac{V}{\sqrt{P/\rho}}$$

4. Mach number (Ma) = $\frac{\text{Inetia force}}{\text{Elastic force}} = \frac{V}{\sqrt{K/\rho}}$

$$\frac{\text{Inetia force}}{\text{Elastic force}} = \frac{\rho AV^2}{Elastic\ stress\ x\ A} = \frac{\rho AV^2}{K\ A} = \frac{V^2}{K/\rho}$$

Taking square root

$$Ma = \frac{V}{\sqrt{K/\rho}}$$

5. Weber number (Wb) = $\frac{\text{Inetia force}}{\text{Surface tension force}} = \frac{V}{\sqrt{\sigma/\rho L}}$

$$\frac{\text{Inetia force}}{\text{Surface tension force}} = \frac{\rho AV^2}{\sigma L} == \frac{\rho L^2 V^2}{\sigma L} = \frac{V^2}{\sigma/\rho L}$$

Taking square root

$$Wb = \frac{V}{\sqrt{\sigma/\rho L}}$$

V = Velocity, L = Length, P = Pressure, ρ = Density, K = Bulk modulus, μ = Dynamic viscosity, υ = Kinematic viscosity, σ = Surface tension

10.5 Similarity laws or model laws

a. Reynolds model law

In addition to inertia force, if viscous force is the only predominant force, then the similarity of flow in the model and its prototype can be established if the Re is same for both systems. This is known as Reynolds model law.

(Re) model = (Re) prototype

$$\frac{V_m\, L_m}{\upsilon_m} = \frac{V_p\, L_p}{\upsilon_p}$$

Application of Reynolds model law: pipe flow, motion of air planes, motion of submarine completely under water, flow around structure under moving fluid

Scale ratios using Reynold's law

$\frac{V_r\, L_r}{\upsilon_r} = 1$ where subscript r represents respective scale ratio.

$V_r\ = \frac{\upsilon_r}{L_r}$

Time scale ratio $(T_r) = \frac{L_r}{V_r}$

Acceleration scale ratio $(a_r) = \frac{V_r}{T_r}$

Force scale ratio $(F_r) = m_r a_r = \rho_r A_r V_r a_r = \rho_r L_r^2 V_r a_r$

Discharge scale ratio $(Q_r) = A_r V_r = L_r^2 V_r$

Work scale ratio $(W_r) = F_r L_r$

Power scale ratio $(P_r) = \frac{F_r L_r}{T_r}$

b. Froude model law

In addition to inertia force, if gravity force is the only predominant force, then the similarity of flow in the model and its prototype can be established if the Fr is same for both systems. This is known as Froude model law.

(Fr) model = (Fr) prototype

$$\frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}}$$

Application of Froude model law: free surface flow such as flow over spillways, sluices, flow jet from an orifice or nozzle

Scale ratios using Froude's law

$\frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}}$

$\frac{V_r}{\sqrt{L_r}} = 1$ where subscript r represents respective scale ratio. ($g_m = g_p$)

$V_r = \sqrt{L_r}$

Time scale ratio $(T_r) = \frac{L_r}{V_r} = \frac{L_r}{\sqrt{L_r}} = \sqrt{L_r}$

Acceleration scale ratio $(a_r) = \frac{V_r}{T_r} = \frac{\sqrt{L_r}}{\sqrt{L_r}} = 1$

Force scale ratio $(F_r) = m_r a_r = \rho_r L_r^3 x1 = \rho_r L_r^3$

Discharge scale ratio $(Q_r) = A_r V_r = L_r^2 V_r = L_r^2 \sqrt{L_r} = L_r^{2.5}$

Work scale ratio $(W_r) = F_r L_r = \rho_r L_r^3 L_r = \rho_r L_r^4$

Power scale ratio $(P_r) = \frac{F_r L_r}{T_r} = \frac{\rho_r L_r^4}{\sqrt{L_r}} = \rho_r L_r^{3.5}$

c. Euler model law

In addition to inertia force, if pressure force is the only predominant force, then the similarity of flow in the model and its prototype can be established if the Eu is same for both systems. This is known as Euler model law.

(Eu) model = (Eu) prototype

$$\frac{V_m}{\sqrt{P_m/\rho_m}} = \frac{V_p}{\sqrt{P_p/\rho_p}}$$

Application: pressure rise due to sudden closure and opening of valve

d. Mach model law

In addition to inertia force, if elastic force is the only predominant force, then the similarity of flow in the model and its prototype can be established if the Ma is same for both systems. This is known as Mach model law.

(Ma) model = (Ma) prototype

$$\frac{V_m}{\sqrt{K_m/\rho_m}} = \frac{V_p}{\sqrt{K_p/\rho_p}}$$

Application: water hammer problems, aerodynamic testing

e. Weber model law

In addition to inertia force, if surface tension force is the only predominant force, then the similarity of flow in the model and its prototype can be established if the Wb is same for both systems. This is known as Weber model law.

(Wb) model = (Wb) prototype

$$\frac{V_m}{\sqrt{\sigma_m/\rho_m L_m}} = \frac{V_p}{\sqrt{\sigma_p/\rho_p L_p}}$$

Application: Flow over a weir involving very low heads, very thin sheet of liquid flowing over a surface

10.6 Types of models

a. Undistorted model

An undistorted model is that model which is geometrically similar to its prototype. That means the scale ratios for corresponding linear dimensions of the model and its prototype are same. Result from such model can be directly applied to the prototype.

b. Distorted model

A distorted model is that model in which one or more terms of the model are not identical with their counter-parts in the prototype. A distorted model may have either geometrical distortion (e.g. different scale for vertical and horizontal dimension), or material distortion (use of different material for model and prototype), or distortion of hydraulic quantities (e.g. velocity, discharge) or a combination of these. Results obtained from the distorted models cannot be directly applied. These models can be applied to modeling of rivers, dams etc.

Reasons for adopting distorted models

- Maintaining accuracy in vertical dimensions
- Maintaining turbulent flow
- Accommodating available facilities
- Obtaining suitable bed material

Demerits of distorted model

- Variables such as pressure, velocity, slope of river beds may not be truly reproduced.
- Difficult to extrapolate and interpolate results

Scale ratios for distorted models

$(L_r)_H$ = horizontal scale ratio, $(L_r)_V$ = Vertical scale ratio

A_p, V_p, B_p, h_p, Q_p = C/s Area of flow, velocity, width, depth of flow and discharge for prototype

A_m, V_m, B_m, h_m, Q_m = C/s Area of flow, velocity, width, depth of flow and discharge for model

$$(L_r)_H = \frac{L_p}{L_m} = \frac{B_p}{B_m}$$

$$(L_r)_V = \frac{h_p}{h_m}$$

a. Scale ratio for area

$$\frac{A_p}{A_m} = \frac{B_p h_p}{B_m h_m} = (L_r)_H (L_r)_V \quad \text{(a)}$$

b. Scale ratio for velocity

From Froude's model law

$$\frac{V_m}{\sqrt{g_m h_m}} = \frac{V_p}{\sqrt{g_p h_p}}$$

$$\frac{V_p}{V_m} = (L_r)_V^{1/2} \qquad \text{(b)}$$

c. Scale ratio for time

$$\frac{T_p}{T_m} = \frac{L_p/V_p}{L_m/V_m} = \frac{L_p}{L_m}\frac{V_m}{V_p}$$

Substituting value of $\frac{V_p}{V_m}$ from b

$$\frac{T_p}{T_m} = (L_r)_H (L_r)_V^{1/2}$$

d. Scale ratio for discharge

$$\frac{Q_p}{Q_m} = \frac{A_p V_p}{A_m V_m}$$

Substituting the values of $\frac{A_p}{A_m}$ from a and $\frac{V_p}{V_m}$ from b

$$\frac{Q_p}{Q_m} = (L_r)_H (L_r)_V (L_r)_V^{1/2} = (L_r)_H x (L_r)_V^{3/2}$$

10.7 Scale effects in models, advantages and limitations of model

If complete similarity does not exist between a model and its prototype, there will be some discrepancy between the results obtained from the model tests and those which will be indicated by the prototype after construction. This discrepancy is called scale effect.

Advantages of model

I. The behavior and working of a structure or machine can be predicted.

II. A number of alternatives can be worked out.

III. Safety and reliability can be analyzed for a particular problem for which analytical method cannot be used.

IV. Defects can be detected and rectified in case when existing structure is not functioning properly.

Limitations of model

I. No any model is perfectly similar to its prototype. Therefore, experience and judgment is required to analyze the results based on partial similarity.

II. Bigger model is expensive.

III. It is more difficult to predict the performance of actual body from its model of distorted type.

IV. In general the model results are qualitative but not quantitative.